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PROCEEDINGS

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ROYAL SOCIETY OF EDINBURGH

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I.—On a Problem concerning Matrices with Variable Diagonal Elements. By Walter Ledermann. *Communicated by* Professor GODFREY H. THOMSON, D.Sc.

(MS. received August 18, 1939. Revised MS. received November 6, 1939. Read November 6, 1939.)

§ 1. INTRODUCTION.

SUPPOSE that in a given *non-negative definite* symmetric matrix $R=[r_{ij}]$ of order n the diagonal entries are replaced by arbitrary quantities x_1, $x_2, \ldots, x_n$ so that the matrix assumes the form

$$R_x = \begin{bmatrix} x_1 & r_{12} & r_{13} & \cdots & r_{1n} \\ r_{21} & x_2 & r_{23} & \cdots & r_{2n} \\ r_{31} & r_{32} & x_3 & \cdots & r_{3n} \\ \cdots & \cdots & \cdots & & \cdots \\ r_{n1} & r_{n2} & r_{n3} & \cdots & x_n \end{bmatrix} \quad (r_{ij}=r_{ji}).$$

Matrices of this type are met with in the statistical technique known as Factorial Analysis (Thurstone, 1935; Thomson, 1939); there the non-diagonal elements $r_{ij}(i \neq j)$ are the correlation coefficients of certain tests and are given by observation. The diagonal entries of the "correlational matrix," which is always non-negative definite, are originally all equal to unity, but, on the hypothesis which underlies the process of Factorial Analysis, it is permissible to diminish the diagonal entries arbitrarily *provided the modified matrix is still non-negative definite*, it being assumed that the amount which has been deducted from the diagonal cells is due to the variance of the specific factors, the investigation of which is not the primary aim of the theory.

It is obvious that there is considerable latitude in the choice of the new diagonal elements, and L. L. Thurstone (1935), generalising the original theory of C. Spearman, has proposed to make the solution unique by selecting that set of values $x_1, x_2, \ldots, x_n$ which reduces the matrix R_x to its lowest possible rank, or to its ***minimum rank***, as it is briefly called. Professor Godfrey H. Thomson (1938) pointed out that in all practical cases in which such a set of diagonal entries had been used, the matrix R_x was not only of minimum rank but also had the property that its trace, *i.e.* the expression

$$x_1 + x_2 + \ldots + x_n,$$

was less than for any other admissible set of diagonal entries. It was realised that exceptions to this rule were mathematically possible, *i.e.* that in some artificially constructed cases the minimum rank and minimum trace did not coincide, and it is the object of this paper * to find conditions under which such exceptions may occur.

In psychological data the minimum rank to which R_x can be reduced is as a rule very low; indeed in Spearman's Theory of Two Factors (Spearman, 1932, Appendix, p. ii) the correlational matrices considered are always capable of being reduced to rank unity. In § 2 we shall present a complete solution for this elementary case, and in the last section we shall give a sufficient condition for a matrix of minimum rank $\geqslant 2$ to be an "exceptional" matrix.

§ 2. SPEARMAN MATRICES.

A Spearman matrix is a matrix of the form

$$R_x = \begin{bmatrix} x_1 & l_1 l_2 & l_1 l_3 & \ldots & l_1 l_n \\ l_2 l_1 & x_2 & l_2 l_3 & \ldots & l_2 l_n \\ l_3 l_1 & l_3 l_2 & x_3 & \ldots & l_3 l_n \\ \ldots & \ldots & \ldots & & \ldots \\ l_n l_1 & l_n l_2 & l_n l_3 & \ldots & x_n \end{bmatrix},$$

where $l_1, l_2, \ldots, l_n$ are given *positive* quantities. If we put

$$x_1 = l_1^2, \quad x_2 = l_2^2, \quad \ldots, \quad x_n = l_n^2, \qquad (1)$$

the matrix R_x reduces to

$$R = \begin{bmatrix} l_1^2 & l_1 l_2 & \ldots & l_1 l_n \\ l_2 l_1 & l_2^2 & \ldots & l_2 l_n \\ \ldots & \ldots & & \ldots \\ l_n l_1 & l_n l_2 & \ldots & l_n^2 \end{bmatrix} = ll', \qquad (2)$$

---

* The author wishes to express his thanks to Professor Godfrey H. Thomson for suggesting this problem to him.

where

$$l = \begin{bmatrix} l_1 \\ l_2 \\ \cdot \\ \cdot \\ \cdot \\ l_n \end{bmatrix}$$

is a column vector. The matrix (2) is evidently of rank unity, and the set of values (1) represents the minimum rank solution which is adopted by Spearman and Thurstone. The trace of the matrix (2) is

$$t_0 = l_1^2 + l_2^2 + \ . \ . \ . \ + l_n^2.$$

We now put

$$x_i = l_i^2 + u_i, \quad (i = 1, 2, \ . \ . \ ., n)$$

and write

$$\mathrm{R}_x \equiv \mathrm{R}(u) = \begin{bmatrix} l_1^2 + u_1 & l_1 l_2 & . \ . \ . & l_1 l_n \\ l_2 l_1 & l_2^2 + u_2 & . \ . \ . & l_2 l_n \\ . \ . \ . & . \ . \ . & & . \ . \ . \\ l_n l_1 & l_n l_2 & . \ . \ . & l_n^2 + u_n \end{bmatrix}, \qquad (3)$$

or more briefly,

$$\mathrm{R}(u) = ll' + \mathrm{U}, \qquad (3a)$$

where

$$\mathrm{U} = \begin{bmatrix} u_1 & & & & \\ & u_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & u_n \end{bmatrix}$$

is an unknown diagonal matrix. Spearman's solution is then given by

$$u_1 = u_2 = \ . \ . \ . \ = u_n = 0. \qquad (4)$$

Generally the trace of $R(u)$ is

$$t_u = l_1^2 + l_2^2 + \ . \ . \ . \ + l_n^2 + u_1 + u_2 + \ . \ . \ . \ + u_n,$$

and differs from the trace of (2) by an amount

$$e = t_u - t_0 = u_1 + u_2 + \ . \ . \ . \ + u_n. \qquad (5)$$

Our problem is then to make the function e a minimum under the condition that the matrix $R(u)$ shall be non-negative definite. We shall prove the

THEOREM I.—*In general, the minimum trace of the non-negative definite matrix* R(u) *is attained for the values*

$$u_1 = u_2 = \ . \ . \ . \ = u_n = 0,$$

i.e. *the minimum trace and minimum rank solutions coincide. An* *exception to this rule occurs if, and only if, one of the* l's *exceeds the sum of all the others,* e.g. *if, with a suitable arrangement of suffixes,*

$$l_1 > l_2 + l_3 + \ldots + l_n. \qquad (6)$$

The minimum trace of R(u) *is then obtained for the unique set of values*

$$u_1 = -l_1(l_1 - l_2 - l_3 - \ldots - l_n), \qquad (7)$$
$$u_j = l_j(l_1 - l_2 - l_3 - \ldots - l_n), \quad (j = 2, 3, \ldots, n)$$

so that

$$e = -(l_1 - l_2 - \ldots - l_n)^2. \qquad (7a)$$

The corresponding matrix R(u) *is of rank* n − l.

Proof.—1. *The condition* (6) *is necessary.* Suppose that the trace of the non-negative matrix $R(u)$ reaches its minimum for a certain set u *other than* $u_1 = u_2 = \ldots = u_n = 0$. The trace of the corresponding matrix $R(u)$ must then be less than, or equal to, that of (2), *i.e.* we must have

$$e = u_1 + u_2 + \ldots + u_n \leqslant 0. \qquad (8)$$

This is possible only if at least one of the u's is negative. Let

$$u_1 < 0. \qquad (9)$$

We shall now show that all the other u's are then necessarily positive. Suppose, *e.g.*, that

$$u_2 \leqslant 0. \qquad (10)$$

It is known (Turnbull and Aitken, 1932, p. 91) that in a non-negative definite matrix all principal minors, including the diagonal elements themselves, are non-negative. In particular, by (9) and (10)

$$l_1^2 > l_1^2 + u_1 \geqslant 0, \qquad (11)$$

$$l_2^2 \geqslant l_2^2 + u^2 \geqslant 0, \qquad (12)$$

$$\begin{vmatrix} l_1^2 + u_1 & l_1 l_2 \\ l_2 l_1 & l_2^2 + u_2 \end{vmatrix} \geqslant 0. \qquad (13)$$

On forming the product of (11) and (12) we obtain

$$u_1 u_2 + u_1 l_2^2 + u_2 l_1^2 < 0.$$

But this contradicts (13), which on expansion takes the form

$$u_1 u_2 + u_1 l_2^2 + u_2 l_1^2 \geqslant 0.$$

Hence we must have $u_2 > 0$, and generally

$$u_j > 0. \quad (j = 2, 3, \ldots, n) \qquad (14)$$

It is easy to see that if the trace of $R(u)$ is a minimum, the determinant of $R(u)$ must be zero. For since $R(u)$ is non-negative definite, all its latent roots are either positive or zero. Suppose all its latent roots were actually positive and let them be arranged in their order of magnitude, thus

$$\omega_1 \geqslant \omega_2 \geqslant \ldots \geqslant \omega_n > 0.$$

Then the latent roots of

$$R(u) - \omega_n I, \qquad (15)$$

where I is the n-dimensional unit matrix, would be

$$\omega_1 - \omega_n, \quad \omega_2 - \omega_n, \quad \ldots, \quad \omega_{n-1} - \omega_n, \quad 0,$$

and would all be positive or zero so that (15) would be a non-negative definite matrix which differs from $R(u)$ only in the diagonal cells. On the other hand, the trace of (15) is less by $n\omega_n$ than that of $R(u)$, which contradicts our assumption that the trace of $R(u)$ is a minimum. Hence we must have $\omega_n = 0$, and consequently the determinant, being the product of all the latent roots, is equal to zero.

Since $R(u)$ is a singular matrix, there exists a non-zero column vector

$$q = \{q_1, q_2, \ldots, q_n\}$$

such that

$$R(u)q = 0, \qquad (16)$$

or in the notation (3*a*),

$$(U + ll')q = 0,$$
$$Uq = -l(l'q),$$
$$Uq = -\phi l, \qquad (17)$$

where

$$\phi = l'q = l_1q_1 + l_2q_2 + \ldots + l_nq_n$$

is a scalar. Writing (17) in components we obtain

$$u_iq_i = -\phi l_i. \quad (i = 1, 2, \ldots, n) \qquad (18)$$

Since the u's and the l's are all non-zero, it follows that none of the q's can be zero, and therefore

$$u_i = -\frac{\phi l_i}{q_i}. \quad (i = 1, 2, \ldots, n) \qquad (19)$$

It is evident from (16) that the vector q is determined only apart from a non-zero scalar factor, and there is therefore no loss of generality in assuming that

$$q_1 = 1. \qquad (20)$$

Hence writing (18) for $i = 1$ we find that

$$\phi = -\frac{u_1}{l_1}.$$

But $u_1 < 0$ by (9) and $l_1 > 0$, whence

$$\phi = \sum_{i=1}^{n} l_i q_i > 0. \qquad (21)$$

On the other hand, since for any $j > 1$, $u_j > 0$ (by (14)), we have

$$q_j < 0. \quad (j = 2, 3, \ldots, n)$$

It is convenient to put

$$q_j = -z_j, \quad (j = 2, 3, \ldots, n) \qquad (22)$$

where the z_j's are positive quantities. Thus by (20) and (22) we can write (21) in the form

$$\phi \equiv l_1 - z_2 l_2 - z_3 l_3 - \ldots - z_n l_n > 0. \qquad (23)$$

Also, in virtue of (19) the inequality (8) now becomes

$$e = -\phi\left[l_1 - \frac{l_2}{z_2} - \frac{l_3}{z_3} - \ldots - \frac{l_n}{z_n}\right] < 0, \qquad (24)$$

whence by (23)

$$\psi \equiv l_1 - \frac{l_2}{z_2} - \frac{l_3}{z_3} - \ldots - \frac{l_n}{z_n} > 0. \qquad (25)$$

We shall now show that the two inequalities (23) and (25) are compatible only if

$$\delta \equiv l_1 - l_2 - l_3 - \ldots - l_n > 0. \qquad (26)$$

For

$$2\delta - \phi - \psi = \sum_{j=2}^{n}\left(z_j - 2 + \frac{1}{z_j}\right)l_j$$

$$= \sum_{j=2}^{n}\left(\sqrt{z_j} - \frac{1}{\sqrt{z_j}}\right)^2 l_j > 0.$$

Hence

$$\delta > \tfrac{1}{2}(\phi + \psi) > 0. \qquad (27)$$

This proves (26), which is identical with (6).

Again, by (24) and (25) we have

$$e = -\phi\psi > -\left(\frac{\phi+\psi}{2}\right)^2 > -\delta^2.$$

Therefore

$$e > -(l_1 - l_2 - \ldots - l_n)^2,$$

i.e. the quantity e can never be less than the value given in (7*a*).

2. *The condition* (6) *is sufficient.* Suppose now that

$$\delta > 0, \qquad (28)$$

where as before

$$\delta = l_1 - l_2 - \ldots - l_n, \qquad (29)$$

and put

$$u_1 = -l_1\delta, \quad . \quad . \quad . \quad . \quad . \qquad (30)$$

$$u_j = l_j\delta. \quad (j = 2,\ 3,\ .\ .\ .,\ n)$$

Hence

$$e = u_1 + u_2 + \ .\ .\ .\ + u_n = (-l_1 + l_2 + \ .\ .\ .\ + l_n)\delta = -\delta^2,$$

which is the least possible value that e can take under any circumstances. To complete our proof we have to show that the values (30) are admissible, *i.e.* that the corresponding matrix $R(u)$ is non-negative definite. It is known (Turnbull and Aitken, 1932, p. 91) that a real symmetrical matrix is non-negative definite if all its leading principal minors are positive, possibly with the exception of the determinant itself, which may be either positive or zero. In our case these conditions become:

$$\Delta_1 = l_1^2 + u_1 > 0,$$

$$\Delta_2 = \begin{vmatrix} l_1^2 + u_1 & l_1 l_2 \\ l_2 l_1 & l_2^2 + u_2 \end{vmatrix} > 0,$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$\Delta_k = \begin{vmatrix} l_1^2 + u_1 & l_1 l_2 & l_1 l_k \\ l_2 l_1 & l_2^2 + u_2 & l_2 l_k \\ .\ .\ . & .\ .\ . & .\ .\ . \\ l_k l_1 & l_k l_2 & l_k^2 + u_k \end{vmatrix} > 0, \quad (1 < k < n-1)$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$\Delta_n = |\ \mathrm{R}(u)\ | \geqslant 0.$$

Expanding Δ_k with respect to the main diagonal we find that

$$\Delta_k = u_1 u_2 \ .\ .\ .\ u_k \left[1 + \frac{l_1^2}{u_1} + \frac{l_2^2}{u_2} + \ .\ .\ .\ + \frac{l_k^2}{u_k} \right], \quad . \quad . \qquad (31)$$

and on substituting for the u's from (30) we obtain

$$\Delta_k = -(l_1 l_2 \ .\ .\ .\ l_k)\delta^k \left[1 - \frac{l_1}{\delta} + \frac{l_2}{\delta} + \ .\ .\ .\ + \frac{l_k}{\delta} \right],$$

$$\Delta_k = -(l_1 l_2 \ .\ .\ .\ l_k)\delta^{k-1}[\delta - l_1 + l_2 + \ .\ .\ .\ + l_k], \quad . \quad . \qquad (32)$$

whence on using the definition of δ (29) we get

$$\Delta_k = (l_1 l_2 \ .\ .\ .\ l_k)\delta_{k-1}(l_{k+1} + l_{k+2} + \ .\ .\ .\ + l_n).$$

Hence for $1 < k < n-1$ we have

$$\Delta_k > 0.$$

The formula (32) is evidently still valid for $k = n$; thus

$$\Delta_n = |\ \mathrm{R}(u)\ | = -(l_1 l_2 \ .\ .\ .\ l_n)\delta^{n-1}[\delta - l_1 + l_2 + \ .\ .\ .\ + l_n],$$

i.e. by (26),

$$\Delta_n=0. \qquad (33)$$

Hence $R(u)$ is non-negative definite. From (33) we see that the determinant of $R(u)$ vanishes, as we expected (see p. 5); but since $\Delta_{n-1}>0$, the matrix $R(u)$ is of rank $n-1$. This completes the proof of the theorem.

In psychological data the numbers l_i are proper fractions, *i.e.* we have

$$0<l_i<1, \qquad (i=1,\ 2,\ \dots,\ n) \qquad (34)$$

and the matrix $R(u)$ is meaningful only if none of its diagonal elements exceeds unity. This condition is always fulfilled by our solution (30) provided this solution is available at all. For we have

$$l_1^2+u_1=l_1^2-l_1\delta<l_1^2<1,$$

and for any suffix $j\neq 1$,

$$l_j^2+u_j=l_j^2+l_j\delta.$$

But

$$\delta=l_1-l_2-\dots-l_j-\dots-l_n<l_1-l_j.$$

Hence

$$l_j^2+u_j<l_j^2+l_j(l_1-l_j)=l_1l_j<1$$

by (34). Thus all diagonal elements of $R(u)$ are less than unity.

A necessary criterion whether a given Spearman matrix is an "exceptional" matrix is obtained by computing the average

$$\bar{r}=\frac{2}{n(n-1)}\sum_{i<k}{}_{1}^{n}l_il_k \qquad (35)$$

of its non-diagonal elements, when by using Schwarz's inequality it can be shown that, if the matrix is exceptional,

$$\bar{r}<\frac{3n-4}{n(n-1)}l_1^2. \qquad (36)$$

When $0<l_1<1$, this tends to zero as n^{-2}, and thus a large Spearman matrix cannot be exceptional unless the average of its elements is small.

§ 3. MATRICES OF MINIMUM RANK GREATER THAN 2.

We now turn to the discussion of the general case of a matrix

$$R_x=\begin{bmatrix} x_1 & r_{12} & \dots & r_{1n} \\ r_{21} & x_2 & \dots & r_{2n} \\ \\ r_{n1} & r_{n2} & \dots & x_n \end{bmatrix}.$$

Suppose that the set of values

$$x_1=r_{11},\quad x_2=r_{22},\quad \dots,\quad x_n=r_{nn} \qquad (37)$$

reduce the matrix to its minimum rank $r(\geqslant 2)$. Assuming that the values (37) are given we consider the matrix

$$\mathrm{R}(u)=\begin{bmatrix} r_{11}+u_1 & r_{12} & \dots & r_{1n} \\ r_{21} & r_{22}+u_2 & \dots & r_{2n} \\ \\ r_{n1} & r_{n2} & \dots & r_{nn}+u_n \end{bmatrix}, \quad . \quad . \quad . \quad (38)$$

where $u_1, u_2, \dots, u_n$ are arbitrary quantities. In the minimum rank solution we have

$$u_1=u_2=\dots=u_n=0, \quad . \quad . \quad . \quad . \quad . \quad (39)$$

and the trace of the corresponding matrix is

$$t_0=r_{11}+r_{22}+\dots+r_{nn}. \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Our problem is to ascertain if it is possible *to depress the trace of* R(u) *below the value* (41), i.e. *to make*

$$e=u_1+u_2+\dots+u_n<0 \quad . \quad . \quad . \quad . \quad . \quad (41)$$

without violating the condition that R(u) *shall be non-negative definite.* It is, of course, understood that this can only be accomplished at the price of raising the rank. A matrix for which (40) is *not* the minimum trace will be termed an *exceptional* matrix. A complete treatment of this problem along the lines of § 2 appears to be difficult, and we shall content ourselves with finding a *sufficient condition* for the existence of such exceptional matrices.

Our argument will be based on the following two lemmas.

LEMMA I (MacDuffee, 1933, p. 23; Turnbull and Aitken, 1932, p. 181, Ex. 14).—*Let* A *and* B *be any matrices of orders* p × q *and* q × p *respectively. Then the non-zero latent roots of* AB *are the same as those of* BA.

Proof.—Let $\lambda(\neq 0)$ be a latent root of AB. Then there exists a non-zero vector x (i.e. $\bar{x}'x\neq 0$, where the bar denotes conjugate complex) such that

$$\mathrm{AB}x=\lambda x. \quad . \quad . \quad . \quad . \quad . \quad (42)$$

Premultiplying this matrix equation by B we obtain

$$\mathrm{BAB}x=\lambda \mathrm{B}x$$

or

$$\mathrm{BA}y=\lambda y,$$

where

$$y=\mathrm{B}x. \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Hence λ is a latent root of BA provided it can be shown that the corresponding latent vector (43) is non-zero. On premultiplying (42) by $\bar{x}'$ we find that

$$\bar{x}'ABx = y\bar{x}'x \neq 0,$$

i.e.

$$(\bar{x}'A)y \neq 0,$$

whence

$$y \neq 0.$$

In the same way it can be shown that every non-zero latent root of BA is also a latent root of AB.

LEMMA II.—*Let a real symmetric matrix* Q *be partitioned in the form*

$$Q = \begin{bmatrix} A & B \\ B' & C \end{bmatrix}, \qquad . \quad . \quad . \quad . \quad . \quad (44)$$

where A *and* C *are square matrices (which are then necessarily symmetric). If the least and the greatest latent roots of any real symmetrical matrix* S *be denoted by* λ_S *and* γ_S *respectively, we have*

$$\lambda_Q > \tfrac{1}{2}\{\lambda_A + \lambda_C - \sqrt{(\lambda_A - \lambda_C)^2 + 4\gamma_{BB'}}\}. \qquad . \quad . \quad . \quad (45)$$

Proof.—It is well known (Turnbull and Aitken, 1932, p. 170) that if S be a real symmetric matrix, λ_S and γ_S are respectively equal to the minimum and maximum of the quadratic form $z'Sz$, where z is a variable real *unit* vector; more generally, it can be shown that the extreme values of $x'Sx/x'x$ are respectively λ_S and γ_S, where x is *any* non-zero real vector. Hence we have

$$\gamma_S x'x > x'Sx > \lambda_S x'x, \qquad . \quad . \quad . \quad . \quad (46)$$

and there are certain vectors (latent vectors) for which either of the two equality signs has to be taken. Let

$$w = \begin{bmatrix} x \\ y \end{bmatrix}$$

be a real non-zero vector in partitioned form so that

$$x'x + y'y = 1,$$

where the dimensions of the partial vectors x and y correspond to the orders of A and C in (45). It can readily be verified that

$$w'Qw = x'Ax + 2x'By + y'Cy. \quad . \qquad . \quad . \quad . \quad (47)$$

Putting

$$\left.\begin{aligned} x'x &= \xi^2 \quad (\xi > 0), \\ y'y &= \eta^2 \quad (\eta > 0), \end{aligned}\right\} \qquad . \quad . \quad . \quad . \quad (48)$$

we deduce from (46) that

$$\begin{aligned} x'Ax &> \lambda_A \xi^2, \\ y'Cy &> \lambda_C \eta^2. \end{aligned}$$

Again, by Schwarz's inequality for the scalar product of two vectors, namely,

$$(u'v)^2 \leqslant (u'u)(v'v),$$

we find

$$\{x'(\mathrm{B}y)\}^2 \leqslant (x'x)(y'\mathrm{B}'\mathrm{B}y),$$
$$\{x'(\mathrm{B}y)\}^2 \leqslant (x'x)(y'y)\gamma_{\mathrm{BB}'}$$

by (46), *i.e.*

$$|\, x'\mathrm{B}y \,| \leqslant \gamma^{\frac{1}{2}}_{\mathrm{B}'\mathrm{B}}\xi\eta.$$

Thus from (47) we infer that

$$w'\mathrm{Q}w \geqslant \lambda_{\mathrm{A}}\xi^2 - 2\gamma^{\frac{1}{2}}_{\mathrm{BB}'}\xi\eta + \lambda_{\mathrm{C}}\eta^2.$$

The right-hand side can be written in matrix notation

$$[\xi \quad \eta]\begin{bmatrix} \lambda_{\mathrm{A}} & -\gamma^{\frac{1}{2}}_{\mathrm{BB}'} \\ -\gamma^{\frac{1}{2}}_{\mathrm{BB}'} & \lambda_{\mathrm{C}} \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix}.$$

By (46) this expression is greater than, or equal to, $\xi^2+\eta^2$ multiplied by the least latent root of the two-rowed matrix

$$\begin{bmatrix} \lambda_{\mathrm{A}} & -\gamma^{\frac{1}{2}}_{\mathrm{BB}'} \\ -\gamma^{\frac{1}{2}}_{\mathrm{BB}'} & \lambda_{\mathrm{C}} \end{bmatrix}.$$

By solving the characteristic equation this latent root is easily found to be equal to

$$\tau = \tfrac{1}{2}\left\{\lambda_{\mathrm{A}} + \lambda_{\mathrm{C}} - \sqrt{(\lambda_{\mathrm{A}} - \lambda_{\mathrm{C}})^2 + 4\gamma_{\mathrm{BB}'}}\right\}.$$

Hence

$$w'\mathrm{Q}w \geqslant \tau(\xi^2+\eta^2).$$

But by (48)

$$\xi^2+\eta^2 = x'x + y'y = w'w,$$

and therefore

$$\frac{w'\mathrm{Q}w}{w'w} \geqslant \tau \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

for every vector $w \neq 0$.

But since there exists a vector w_0 for which the left-hand side of (49) is equal to λ_{Q}, it follows that

$$\lambda_{\mathrm{Q}} \geqslant \tau.$$

This proves (45).

We now turn to the discussion of the problem stated on p. 9. In order to simplify the formulæ we consider a matrix of minimum rank 2, *i.e.*

$$r = 2,$$

but the argument is quite general. Since

$$\mathrm{R} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ & & & \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad (50)$$

is non-negative and of rank 2, it can be factorised thus (Thurstone, 1935, chap. ii):

$$R = FF', \qquad (51)$$

where

$$F = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ a_n & b_n \end{bmatrix}$$

is a real $n \times 2$ matrix; hence

$$r_{11} = a_1^2 + b_1^2, \qquad r_{12} = a_1 a_2 + b_1 b_2. \qquad (52)$$

By Lemma I the non-zero roots of $R = FF'$ are the same as those of the two-rowed matrix

$$A = F'F = \begin{bmatrix} \alpha & \chi \\ \chi & \beta \end{bmatrix},$$

where

$$\begin{aligned} \alpha &= a_1^2 + a_2^2 + \ldots + a_n^2, \\ \beta &= b_1^2 + b_2^2 + \ldots + b_n^2, \\ \chi &= a_1 b_1 + a_2 b_2 + \ldots + a_n b_n. \end{aligned}$$

We now introduce an arbitrary quantity θ and consider the matrix

$$R^* = \begin{bmatrix} r_{11} & r_{12} & \cdot & r_{13} & \ldots & r_{1n} \\ r_{21} & r_{22} & \cdot & r_{23} & \ldots & r_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ r_{31} & r_{32} & \cdot & r_{33} + \theta^2 & \ldots & r_{3n} \\ & & \cdot & & & \\ r_{n1} & r_{n2} & \cdot & r_{n3} & \ldots & r_{nn} + \theta^2 \end{bmatrix}.$$

The latent roots of R* will be functions of θ^2; suppose that its least latent root is

$$\omega = \omega(\theta^2).$$

It is easy to verify that

$$R^* = F^* F^{*\prime},$$

where F* is the $n \times n$ matrix

$$F^* = \begin{bmatrix} a_1 & b_1 & 0 \ldots 0 \\ a_2 & b_2 & 0 \ldots 0 \\ a_3 & b_3 & \theta \ldots 0 \\ \cdot & \cdot & \\ \cdot & \cdot & \\ \cdot & \cdot & \\ a_n & b_n & 0 \ldots \theta \end{bmatrix}.$$

If F* be non-singular, as it will turn out to be in our case, the least latent root of $R = F^* F^{*\prime}$ is equal to the least latent root of

$$Q = F^{*\prime}F^{*} = \begin{bmatrix} \alpha & \gamma & \theta a_3 & \cdots & \theta a_n \\ \gamma & \beta & \theta b_3 & \cdots & \theta b_n \\ \theta a_3 & \theta b_3 & \theta^2 & \cdots & 0 \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ \theta a_n & \theta b_n & 0 & \cdots & \theta^2 \end{bmatrix},$$

i.e. we should have

$$\omega = \lambda_Q. \qquad (53)$$

Using the notation of Lemma II we find

$$\lambda_Q > \tfrac{1}{2}\{\lambda_A + \lambda_C - \sqrt{(\lambda_A - \lambda_C)^2 + 4\gamma_{BB'}}\},$$

where

$$A = \begin{bmatrix} \alpha & \chi \\ \chi & \beta \end{bmatrix}, \qquad C = \begin{bmatrix} \theta^2 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \theta^2 \end{bmatrix}, \qquad B = \theta\begin{bmatrix} a_3 & \cdots & a_n \\ b_3 & \cdots & b_n \end{bmatrix}.$$

But, as we have seen above, the least latent root of *A* is equal to the least non-zero latent root of *R*,

$$\lambda_A = \lambda_R = \lambda, \quad \text{say}.$$

Evidently,

$$\lambda_C = \theta^2.$$

Again, if we put

$$\gamma_{BB'} = \gamma_{B'B} = \theta^2\gamma',$$

it follows by (52) that γ' is the greatest latent root of

$$\frac{1}{\theta^2}B'B = \begin{bmatrix} r_{33} & \cdots & r_{3n} \\ & & \\ r_{n3} & \cdots & r_{nn} \end{bmatrix}.$$

Substituting this in (45) we get

$$\lambda_Q > \tfrac{1}{2}\{\lambda + \theta^2 - \sqrt{(\lambda - \theta^2)^2 + 4\theta^2\gamma'}\}. \qquad (54)$$

Now for any real quantity x

$$(1 + x)^{\frac{1}{2}} < 1 + \tfrac{1}{2}x.$$

Hence

$$\begin{aligned} \sqrt{(\lambda - \theta^2)^2 + 4\theta^2\gamma'} &= \lambda\left(1 - \frac{2\theta^2}{\lambda} + \frac{\theta^4}{\lambda^2} + \frac{4\theta^2\gamma'}{\lambda^2}\right)^{\frac{1}{2}} \\ &< \lambda\left(1 - \frac{\theta^2}{\lambda} + \frac{\theta^4}{2\lambda^2} + \frac{2\theta^2\gamma'}{\lambda^2}\right) \\ &= \lambda - \theta^2 + \frac{\theta^4}{2\lambda} + \frac{2\theta^2\gamma'}{\lambda}. \end{aligned}$$

Therefore, assuming the truth of (53), we get

$$\omega = \lambda_Q > \theta^2\left(1 - \frac{\gamma'}{\lambda} - \frac{\theta^2}{4\lambda}\right) = \omega_1, \text{ say.} \quad . \quad . \quad (55)$$

We now consider the matrix $R(u)$ (equ. (38)), where

$$u_1 = u_2 = \ . \ . \ . \ = u_r = -\omega_1$$

($r = 2$ in our example) and

$$u_{r+1} = u_{r+2} = \ . \ . \ . \ = u_n = \theta^2 - \omega_1,$$

i.e.

$$\mathrm{R}(u) = \mathrm{R}^* - \omega_1 \mathrm{I}.$$

This matrix is certainly non-negative, as can be seen by an argument similar to that on p. 5, and the excess of its trace over that of R (equ. 50) is

$$e = -r\omega_1 + (n - r)(\theta^2 - \omega_1),$$
$$e = (n - r)\theta^2 - n\omega_1.$$

Hence by (55)

$$e < (n - r)\theta^2 - n\theta^2\left(1 - \frac{\gamma'}{\lambda} - \frac{\theta^2}{4\lambda}\right),$$
$$e < -\theta^2 n\left(\frac{r}{n} - \frac{\gamma'}{\lambda} - \frac{\theta^2}{4\lambda}\right).$$

Thus we shall have $e < 0$ provided

$$\frac{r}{n} - \frac{\gamma'}{\lambda} - \frac{\theta^2}{4\lambda} > 0.$$

This condition can be fulfilled if, and only if,

$$\frac{\gamma'}{\lambda} < \frac{r}{n}. \quad . \quad . \quad . \quad . \quad . \quad (56)$$

For θ^2 may then take any value subject to

$$0 < \theta^2 < 4\lambda\left(\frac{r}{n} - \frac{\gamma'}{\lambda}\right). \quad . \quad . \quad . \quad . \quad . \quad (57)$$

The condition (56) also implies that $\lambda_Q > 0$, *i.e.* that Q and consequently F* are non-singular, for

$$\lambda_Q > \omega_1 = \theta^2\left(1 - \frac{\gamma'}{\lambda} - \frac{\theta^2}{4\lambda}\right), \quad \text{by (55)},$$
$$\frac{\theta^2}{4\lambda} < \frac{r}{n} - \frac{\gamma'}{\lambda}, \qquad \text{by (57)}.$$

Hence
$$\lambda_Q > \theta^2\left(1 - \frac{r}{n}\right) > 0,$$
so that (53) is justified.

Our results can be summarised in the following

THEOREM II.—*Suppose that the matrix*
$$R_x = \begin{bmatrix} x_1 & r_{12} & \cdots & r_{1n} \\ r_{21} & x_2 & \cdots & r_{2n} \\ & & & \\ r_{n1} & r_{n2} & \cdots & x_n \end{bmatrix}$$
is reduced to its minimum rank when the diagonal elements have the values
$$x_1 = r_{11}, \quad x_2 = r_{22}, \quad \ldots, \quad x_n = r_{nn}.$$
Let λ *be the least non-zero latent root of the non-negative definite matrix*
$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ & & & \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix},$$
and let γ' *be the greatest latent root of the minor*
$$\begin{bmatrix} r_{r+1,\, r+1} & \cdots & r_{r+1,\, n} \\ & & \\ r_{n,\, r+1} & \cdots & r_{n,\, n} \end{bmatrix}.$$
Then if
$$\frac{\gamma'}{\lambda} < \frac{r}{n},$$
the trace of R_x *can be made smaller than the trace of* R *in such a way that the matrix* R_x *still remains non-negative definite;* e.g. *this is accomplished by putting*
$$\begin{aligned} x_k &= r_{kk} - \omega_1, & (k &= 1, 2, \ldots, r) \\ x_j &= r_{jj} + \theta^2 - \omega_1, & (j &= r+1, r+2, \ldots, n) \end{aligned}$$
where θ^2 *is any number satisfying*
$$0 < \theta^2 < 4\lambda\left(\frac{r}{n} - \frac{\gamma'}{\lambda}\right),$$
and where
$$\omega_1 = \theta^2\left(1 - \frac{\gamma'}{\lambda} - \frac{\theta^2}{4\lambda}\right).$$

It is not difficult to construct matrices *R* which have the property (56); generally speaking, the elements in the last $n-r$ rows of *F* must be small compared with those in the first r rows. The following matrix, in which $n=5$ and $r=2$, will serve as an example.

The minimum rank of

$$\mathrm{R}_x = \begin{bmatrix} x_1 & \cdot 54 & \cdot 15 & \cdot 03 & \cdot 18 \\ \cdot 54 & x_2 & \cdot 21 & \cdot 09 & \cdot 06 \\ \cdot 15 & \cdot 21 & x_3 & \cdot 02 & \cdot 02 \\ \cdot 03 & \cdot 09 & \cdot 02 & x_4 & \cdot 00 \\ \cdot 18 & \cdot 06 & \cdot 02 & \cdot 00 & x_5 \end{bmatrix}$$

is two; for if we put

$$x_1 = \cdot 90, \quad x_2 = \cdot 90, \quad x_3 = \cdot 05, \quad x_4 = \cdot 01, \quad x_5 = \cdot 04,$$

the matrix reduces to

$$\mathrm{R} = \begin{bmatrix} \cdot 90 & \cdot 54 & \cdot 15 & \cdot 03 & \cdot 18 \\ \cdot 54 & \cdot 90 & \cdot 21 & \cdot 09 & \cdot 06 \\ \cdot 15 & \cdot 21 & \cdot 05 & \cdot 02 & \cdot 02 \\ \cdot 03 & \cdot 09 & \cdot 02 & \cdot 01 & \cdot 00 \\ \cdot 18 & \cdot 06 & \cdot 02 & \cdot 00 & \cdot 04 \end{bmatrix} = \mathrm{FF}',$$

where

$$\mathrm{F} = \begin{bmatrix} \cdot 9 & \cdot 3 \\ \cdot 3 & \cdot 9 \\ \cdot 1 & \cdot 2 \\ 0 & \cdot 1 \\ \cdot 2 & 0 \end{bmatrix}.$$

The trace of R is

$$t_0 = 1 \cdot 90.$$

Using the notation of Theorem II we find that

$$\lambda = \text{least latent root of} \quad \begin{bmatrix} \cdot 95 & \cdot 56 \\ \cdot 56 & \cdot 95 \end{bmatrix} \quad = 0 \cdot 39$$

and

$$\gamma' = \text{greatest latent root of} \begin{bmatrix} \cdot 05 & \cdot 02 & \cdot 02 \\ \cdot 02 & \cdot 01 & \cdot 00 \\ \cdot 02 & \cdot 00 & \cdot 04 \end{bmatrix} = 0 \cdot 07.$$

Thus the condition (56), namely

$$\frac{\gamma'}{\lambda} = \frac{7}{39} < \frac{r}{n} = \frac{2}{5},$$

is satisfied, and by (57) we must have

$$\theta^2 < 4\lambda\left(\frac{r}{n} - \frac{\gamma'}{\lambda}\right) = 0 \cdot 344.$$

Choosing

$$\theta^2 = \tfrac{1}{2} \times 0 \cdot 344 = 0 \cdot 172,$$

which, as can easily be seen, will bring out the effect most clearly, we find

$$\omega_1 = 0 \cdot 122 \text{ (approximately).}$$

Thus according to Theorem II the matrix

$$\begin{bmatrix} \cdot778 & \cdot540 & \cdot150 & \cdot030 & \cdot180 \\ \cdot540 & \cdot778 & \cdot210 & \cdot090 & \cdot060 \\ \cdot150 & \cdot210 & \cdot100 & \cdot020 & \cdot020 \\ \cdot030 & \cdot090 & \cdot020 & \cdot060 & \cdot000 \\ \cdot180 & \cdot060 & \cdot020 & \cdot000 & \cdot090 \end{bmatrix}$$

is positive definite (as can best be proved by subjecting it to Aitken's (Aitken, 1937) process of pivotal condensation, and verifying that all "pivotal elements" turn out to be positive); yet the trace of this matrix is

$$t = 1{\cdot}806,$$

i.e. less than $t_0 = 1{\cdot}90$, which is the trace corresponding to the minimum rank solution.

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II.—The Maximal Range of Error in Gross Reproduction Rates.

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I. INTRODUCTION.

IN the past two methods have been applied to the study of differential fertility. One may be called the *absolute* method, recently applied to Scottish data by the writer. The other may be called the *comparative* method, subsequently applied to the same statistics by Mr Barclay and Dr Kermack. Since the conclusions obtained by the two methods are not always consonant, it is important to evaluate the scope and reliability of each.

The comparative method is essentially due to Newsholme and Stevenson, who adapted to the study of fertility a technique previously used for comparing mortality differences. Its limitations can best be seen by contrasting the issues raised in comparing mortality and fertility data. For any given disease the incidence of mortality generally varies in different age-groups of the population. In all probability this variation is mainly due to the changing biological make-up of the individual arising from normal processes of age and decay. It must always be taken into account, if we wish to assess the results of a public health policy by comparing the fall and rise of mortality rates in populations with different age-compositions, or to explore ætiological agencies by comparing local mortality differences and populations in which the expectation of life is different. That people more usually contract cancer in the decade whose mid-year is 45 than in the decade whose mid-year is 15, is a fact about the history of their tissues rather than a fact about the social circumstances of their lives. So a comparison of total incidence of cancer from two communities with widely different proportions of people in these two age-groups would therefore tell us nothing about whether external circumstances were, or were not, propitious to liability to this disease.

In practice, mortality figures for different age-groups are not always available, and an indirect method of standardisation is used. Where the numbers of a community in each age- and sex-group are known, but only the crude death-rate for the whole population is available, we can still obtain a rough estimate of what the death-rate would be in the standard population, if its age-composition were that of the population to be standardised. This is called the *index* rate of the latter. High or low values of the index rate indicate an age-composition favouring or diminishing liability to death, and the ratio of the death-rate of the standard population to the index rate can be used as a correction factor to reduce or raise the crude mortality rate in the unstandardised population to a standardised value. The validity of this procedure depends on the assumption that age variation in mortality in different communities is approximately comparable.

During the fertile period of a woman's life, whether married or unmarried, the incidence of child-bearing also varies considerably; and for this reason we cannot assess the reproductive capacities of communities with different age-composition by comparing figures for general fertility over the entire reproductive period. Owing to defective statistics, a direct method of age standardisation analogous to that described in the last paragraph is rarely applicable. An indirect method, analogous to the indirect method for standardising mortality, advocated by Newsholme and Stevenson and later adopted by Beveridge, consists of applying to the crude fertility rate a correction factor. This is the ratio of the fertility rate for the standard population to the fertility rate the standard population would have if it had the same age-composition as the population to be standardised. The fruitfulness of this procedure depends on the closeness of the analogy between mortality and fertility age-variations.

Problems which arise in handling comparative mortality and comparative fertility are essentially different both with respect to the nature of the material and to the end in view. As regards the nature of the material, the important difference is this. The age variation with respect to mortality from a given source is generally inherent in the biological make-up of the individual, and we need not assume that its character differs greatly from one community to another. Thus it is generally true that aged people are more likely to get cancer than younger ones in any community for which such statistics are available. The incidence of child-bearing at different ages is not simply related to changes in the biological make-up of the individual. The falling-off in the incidence of child-bearing with age during the fertile period depends on a complex nexus of social tradition and institutions which differ widely in different communities. These differences are reflected in wide differences between the observed statistics of declining fertility as age advances. This fact is of special significance in connection with the use for which the peculiar merit of the comparative method is claimed.

To assess relative fertility irrespective of the incidence of, and age at, marriage, some statisticians follow Newsholme and Stevenson in calculating fertility rates for married women only. The validity of the comparative method as applied to the fertility of married women depends on *whether marriage is a social agency which has no effect on the incidence of child-bearing at a given age.* In fact, this is not so. If a woman marries for the first time at 39, the probability of child-bearing at the age of 40 is much greater than if she marries at the age of 18. This consideration is both a matter of statistical information and a matter of common experience.

Let us suppose that in one community (*a*) every woman marries at 18, has three children at two-yearly intervals before the age of 25, thereafter has none and remains married till 50. Let us also suppose that in a neighbouring community (*b*) each woman marries at age 38, and has one child and one only before the age of 40. In practice, any standard population selected to get a correction factor for the standardised fertility of married women in either (*a*) or (*b*) would be one in which some children would be born to married women at all ages throughout the period of child-bearing, *i.e.* the standard population would have more children born to women over 24 than in (*a*), and more children born to women under 38 than in (*b*). Since effective fertility falls off with age, an older standard population would yield a correction factor which would reduce the crude fertility rate to a lower standardised value. If the standard population had the same age composition as (*b*), its total fertility would certainly be less. So the standardised rate for (*b*) would necessarily be higher. Similarly, the standardised value for (*a*) would be lower than the crude rate.

Thus total fertility of (*b*) would probably be greater if the mean age at marriage in (*b*) were lower; and the fertility of (*a*) might well be lower if the mean age at marriage were raised. Such conclusions require no elaborate manipulation of fertility data to demonstrate their plausibility. They could be directly inferred from a comparison of the fertility rates, and they imply nothing concerning the *magnitude* of relative fertility in the two groups, if circumstances were so changed. How *much* net fertility would be increased by a redistribution of the age-incidence of marriage, does not depend on any universal biological constant. It depends on *social agencies which do not lie within the scope of arithmetical treatment.* It is therefore difficult to see why a tabulation of standardised fertility rates for married women compiled in this way has any advantage over parallel columns of ordinary fertility rates and the mean age at marriage in different communities. Since there is every reason to believe that *the*

age-specific fertility rates of the standard population would be different if the age distribution of marriage were also different, the specific fertility rates of the standard population provide no basis for assessing the effect of early or late marriage on the fertility of another group.

The index of the *absolute* method, known as the *gross reproduction rate*, represents the mean number of female children born to a woman during her reproductive period. It can be obtained *directly* by adding the specific fertility rates in each age-group of the reproductive period. The great merit of this method is that the gross reproduction rate has a very precise meaning. It is, in fact, the maximum replacement value of a community. No community with a gross reproduction rate less than unity is replacing itself, even when the death-rate under 50 is zero. As a measure of differential fertility, the gross reproduction rate therefore tells us whether different communities are, or are not, potentially above the survival minimum.

Unfortunately a direct value for the gross reproduction rate is not always obtainable. As with the comparative method, it is necessary to proceed by a more devious route. It has been customary to estimate it approximately by fitting age-specific fertility rates from other countries to the total births recorded. In practice this procedure can be checked by applying different sets of specific fertility rates to find which would yield a total fertility rate corresponding to that of the estimated population when weighted by its own age-composition.

In contradistinction to the *maximum* replacement value of the community, the *current* replacement value, or *net reproduction rate*, is obtained by weighting the age-specific fertility rates by the incidence of women in the corresponding groups of the life table. It thus corresponds to the replacement of mothers by future mothers when fertility is not greatly changing. An analogous index, the *nuptial reproduction rate*, is obtained by combining the data of the nuptiality table of the current year with those used in computing gross reproduction rates. The nuptial reproduction rate never differs widely from the gross reproduction rate. If marriage habits had been constant in the past, they would be identical. Since they change, differences between the two indicate how marriage rates are tending to raise or lower future replacement values. A fourth index, the *gross reproduction rate of married women*, also has a concrete meaning. It gives the mean number of children born to a married woman in the course of her life at current rates. This does not yield any direct information about the replacement status of a community, because it involves a combination of two variables, age at marriage, and number of children born per married year. To analyse its significance we have to combine it with other data. If we neglect differential illegitimacy, we can say that when two communities have the same reproduction rates for *married* women, but different *gross* reproduction rates, the replacement value difference is due to difference of marriage incidence.

Thus we can distinguish the effect of four variables which are known to play a part in determining total fertility:

(*a*) Age-composition of the female population.
(*b*) Incidence of marriage.
(*c*) Age at marriage.
(*d*) Rate at which children are born during marriage.

Gross reproduction rates eliminate effects of (*a*), and reproduction rates for married women eliminate effects of both (*a*) and (*b*). Standardised fertility rates attempt to assess the magnitude of (*d*) only, since by reference to a standard population of married women, they claim to eliminate the effect of age at marriage. Whatever method we use the unknown data are age-specific fertility rates, and the known data are total numbers of births. Thus the first step with all methods is to assume some arbitrary set of age-specific fertility rates, and to apply them to the age-composition of the different populations. This gives a figure for the ratio of actual births to births so calculated.

When total fertility is the end in view, the difference between the two methods lies in the presentation of the results. The absolute method yields a rate which eliminates age-composition. Its advantage is that the figure it yields has a concrete meaning. The comparative method yields a rate based on an arbitrary standard age-composition. If one is concerned only with comparative values, the ratios of actual births to births estimated by applying the standard age-specific fertility rates to the known age-composition are, as Mr Barclay and Dr Kermack point out, sufficient in themselves. Both methods make the assumption that fertility of women can be studied independently of the numbers in each age-group in a given population.

When we apply the two methods for the study of the fertility of *married* women, a fundamental difference emerges. We then have two new variables to consider. One is the proportion of women who marry during the course of their lives. The other is the age at which they marry. Here the two methods part company. The *gross reproduction rate of married women* incorporates the known data of age at marriage, and presents the total fertility of women who marry at given ages. Like standardised legitimate fertility rates, this figure is independent of the proportion who marry, and it is an admitted limitation of *both* methods that the proportion of women who marry is a relevant feature of the social *mores* which influences prevalent fertility; but by referring the fertility of married women to a standard *married* population, standardised legitimate fertility rates deal with their fertility *as if it were also independent of the age at which women marry.* So its usefulness depends on how far neglect of another variable is likely to lead to reliable conclusions.

Both methods involve an assumption about the specific fertility of married women in the absence of the required data, and theoretically it should be possible to estimate the range of error involved in *both* methods. As far as the writer is aware, this has not been done for standardised fertility rates. Most workers who have used the gross reproduction rate technique have satisfied themselves that they have chosen appropriate fertility rates and have made some attempt to estimate the maximum range of error involved. Since such estimates have not been extensive, those who are not familiar with sources and possible magnitude of error are unable to evaluate the order of reliability claimed. It is therefore desirable to define more precisely the range of error involved and conditions under which maximum errors occur with this technique. Since there is no empirical law connecting observed distribution of age-specific fertility rates with gross reproduction rates, the issue stated is not amenable to formal analysis. The best we can do is to explore the order of error, when we apply to a population age-specific fertility rates known to differ as widely as possible from its own.

2. EMPIRICAL ERRORS ARISING IN THE USE OF THE ABSOLUTE METHOD.

According to Kuczynski (1935), the maximum error likely to arise from using inappropriate fertility rates is about 6%, but an error as large as this is unlikely except in quite exceptional circumstances. Glass (1938) applied two extreme sets of fertility rates to the births of England and Wales for 1860–62, and obtained a difference of 5·77%. He further states that, when using appropriate fertility rates, "the range of error which may be taken as maximal is plus or minus 3%." The criterion of *appropriateness* given is that the fertility rates used, when applied to a population whose rate is being estimated, will yield total births differing from the recorded number by less than 10%. This criterion is used throughout the present communication. Glass takes the view that errors are smaller (*a*) when appropriate rates are used, and (*b*) when the different age-groups approach equality in size. Previous attempts to determine the range of error have been based on the application of different sets of fertility rates to populations whose true rates are *unknown*. An attempt will now be made to define the range and conditions of maximum and minimum error by applying the technique of estimated rates to populations whose true age-specific fertility rates are ***known.***

In a study of differential fertility rates in Scotland (1938), the present writer estimated 130 gross reproduction rates for different regions (*e.g.* counties, burghs, landward areas, etc.) twice, using two different sets of fertility rates, mainly Swedish and Danish. The differences obtained are summarised in Table I. In 89% of the cases, the difference obtained was less than 3%, and there was no difference greater than 5%. When the second gross reproduction rate was substituted for the first in the table of rates, there were very few changes in rank. In no case was there a change in rank of more than two places. Table I does not, however, show the differences between either of the estimated rates and the actual but unknown rates. Conceivably, both of the estimated rates may diverge in the same direction from the actual rates. This difficulty does not arise in the later tables in this paper where estimates of known rates are made.

TABLE I.—DIFFERENCES IN GROSS REPRODUCTION RATES OF SCOTTISH REGIONS, CALCULATED TWICE.

Below 1%	54
1%–2%	37
2%–3%	25
3%–4%	12
4%–5%	2
	130

Table II gives the results of twenty estimations each of two Australian gross reproduction rates. The first ten of each were estimated with ***appropriate*** fertility rates, and the second ten with ***inappropriate*** rates. In both Tables II and III, the first are designated by (*a*), and the second by (*b*). Table III gives the result of a further set of 42 determinations of 7 gross reproduction rates. These 7 rates were selected to show as great a variety as possible in the amount of inequality of age-groups. The rates used are taken from the three following sources: from Kuczynski, from Charles, and from the *League of Nations Statistical Year Book*, 1937–38. The rates for Italy (U.S.A.), and Germany (U.S.A.), relate to foreign-born mothers in the United States. Owing to the almost complete cessation of immigration, the age-composition is so abnormal that nothing resembling it is likely to be found anywhere else. The results for these populations can therefore be disregarded, except as a warning of what might occur in small populations subject to the same conditions, if any such exist. It is, moreover, doubtful whether the gross

TABLE II.

AUSTRALIA 1909. G.R.R. = 1·662. Ratio 15–19/45–49 Age-Groups = 2·17.

	Specific Fertility Rates of:		Estimated G.R.R.	% Error in Est. G.R.R.	% Difference in Total Births.
(*a*)	Denmark	1911–15	1·651	−0·66	+1·9
	Canada	1926–27	1·640	−1·32	+1·0
	Sweden	1911–15	1·692	+1·81	−4·7
	Czechoslovakia	1920–21	1·629	−1·99	+3·2
	France	1892–97	1·625	−2·23	−7·0
	Sweden	1906–10	1·715	+3·19	+6·6
	Norway	1916–20	1·725	+3·79	−2·9
	Saxony	1911–14	1·597	−3·91	−4·4
	Finland	1911–20	1·739	+4·63	+0·3
	Bulgaria	1934–35	1·568	−5·66	+8·8
				M = 2·92%	
(*b*)	Germany	1881–90	1·672	+0·60	+ 52·8
	Ukraine	1896–97	1·677	+0·90	+117·9
	Germany	1901–10	1·638	−1·44	+ 33·9
	Sweden	1931	1·638	−1·44	− 44·1
	Bulgaria	1926–27	1·630	−1·93	+ 37·4
	Denmark	1926–30	1·620	−2·53	− 27·4
	Denmark	1878–84	1·714	+3·13	+ 31·7
	Germany	1924–26	1·596	−3·97	− 29·6
	Sweden	1871–75	1·763	+6·08	+ 24·0
	Norway	1871–76	1·785	+7·40	+ 24·3
				M = 2·94%	

AUSTRALIA 1933. G.R.R. = 1·052. Ratio 15–19/45–49 Age-Groups = 1·52.

	Specific Fertility Rates of:		Estimated G.R.R.	% Error in Est. G.R.R.	% Difference in Total Births.
(*a*)	Germany	1924–26	1·051	−0·01	+6·6
	Germany	1929	1·053	+0·10	−7·4
	Denmark	1936	1·048	−0·38	−0·8
	New Zealand	1933	1·057	+0·48	−1·1
	Sweden	1926–30	1·066	+1·33	−6·0
	France	1928–31	1·038	−1·33	+4·9
	Esthonia	1927–30	1·069	+1·62	−4·8
	Latvia	1929	1·072	+1·90	+0·3
	U.S.A. (col.)	1929–31	1·015	−2·52	+7·3
	Norway	1930–31	1·094	+3·99	−4·8
				M = 1·37%	
(*b*)	Sweden	1935	1·057	−0·48	− 22·0
	Austria	1933–34	1·046	−0·57	− 23·1
	Japan	1925	1·062	+0·95	+133·6
	Bulgaria	1934	1·039	−1·24	+ 63·9
	Chile	1930–32	1·068	+1·52	+112·6
	Italy	1930–32	1·070	+1·71	+ 47·7
	Saxony	1931	1·032	−1·90	− 34·3
	Ukraine	1896–97	1·075	+2·19	+239·1
	Finland	1871–75	1·088	+3·42	+122·3
	Norway	1871–75	1·106	+5·13	+100·2
				M = 1·91%	

TABLE III.

Country.	G.R.R.	Ratio 15–19/45–49 Age-Groups.	Ratio Max./Min. Age-Groups.	Specific Fertility Rates of:		% Error on G.R.R.	% Error on Total Births.
France, 1931	1·103	1·08	1·21	(*a*) Germany	1924	0	+ 2·4
				Czechoslovakia	1934	+0·18	− 5·1
				Australia	1932	+0·36	− 2·9
				Denmark	1926	+0·54	+ 6·6
				Sweden	1926	+1·00	− 9·4
						M = 0·22%	
				(*b*) Ukraine	1896	+1·36	+228·7
				Norway	1871	+2·36	+ 97·45
						M = 1·86%	
Latvia, 1937	1·056	1·02	1·38	(*a*) Norway	1930	+0·09	− 1·2
				Germany	1924	+1·23	+ 5·3
				Australia	1932	+1·80	− 0·5
				Czechoslovakia	1934	+2·75	− 3·8
				France	1928	+2·94	+ 0·4
						M = 1·66%	
Australia, 1912	1·781	2·04	2·04	(*a*) Poland	1931–32	−0·06	− 3·2
				Denmark	1906–10	+0·39	+ 4·7
				Sweden	1906–10	+3·03	− 0·1
				Hungary	1920–21	−3·71	+ 8·1
				Norway	1910–11	+5·39	− 0·8
						M = 2·52%	
Bulgaria, 1926–27	2·217	2·51	2·51	(*a*) Germany	1901–10	+0·99	− 3·0
				Japan	1930	+1·44	+ 1·1
				Chile	1930–32	+2·03	− 0·4
				Denmark	1878	+6·45	− 5·3
				Finland	1891	+6·72	− 3·3
						M = 3·53%	
				(*b*) Norway	1871	+11·05	− 10·8
Lithuania, 1928	1·850	2·89	2·89	(*a*) Finland	1911	+ 1·14	− 6·9
				Sweden	1906	− 1·24	− 0·2
				Norway	1910	− 3·19	+ 3·5
				Portugal	1930	− 4·49	+ 5·9
				Denmark	1906	− 5·24	+ 6·2
						M = 3·06%	

TABLE III—*continued.*

Country.	G.R.R.	Ratio 15–19/45–49 Age-Groups.	Ratio Max./Min. Age-Groups.	Specific Fertility Rates of:		% Error on G.R.R.	% Error on Total Births.
Italy (U.S.A.), 1931	1·864	0·29	4·78	(*a*) Bulgaria	1934	+ 1·07	− 9·3
				Canada	1921	− 5·36	− 3·7
				Poland	1931	− 8·80	+ 1·1
				Sweden	1911	− 12·61	− 0·9
				Finland	1911	− 12·71	+ 7·3
						M = 8·11%	
				(*b*) Denmark	1906	− 8·96	+ 10·1
				Lithuania	1928	− 13·68	+ 15·2
						M = 11·32%	
Germany (U.S.A.), 1931	1·003	0·16	6·24	(*a*) Saxony	1924	− 3·69	− 7·5
				Germany	1929	− 6·38	+ 4·0
				Sweden	1930–32	− 8·97	+ 0·8
				Esthonia	1931–34	− 9·47	+ 5·7
				Switzerland	1934–36	− 9·77	− 0·5
						M = 7·66%	
				(*b*) Sweden	1926	− 10·27	+ 11·7
				Latvia	1929	− 11·57	+ 21·7
						M = 10·92%	

reproduction rate itself has much significance when applied to such abnormal populations. Omitting these two countries, there remain 45 estimations of seven different gross reproduction rates, in which appropriate fertility rates were used. The mean error in these is 2·18%, and the maximum error found is 6·72%. In the 23 estimations where inappropriate rates were used, the mean error was 2·53%, and the maximum error, 11·05%.

The conditions set forth by Glass can be stated in more formal terms as follows:—

If l_1, l_2, etc., are the true quinquennial rates, m_1, m_2, etc., the rates used for estimation after adjustment, and if the age-groups are all equal to p,

Then

$$pl_1 + pl_2 \dots pl_7 = \text{Total Births}$$
$$= pm_1 + pm_2 + pm_3 \dots pm_7,$$

i.e.

$$l_1 + l_2 \dots l_7 = m_1 + m_2 \dots m_7.$$

If K is the proportion of girl births to total births, then the true gross reproduction rate

$$
\begin{aligned}
&= 5\mathrm{K}(l_1 + l_2 \ . \ . \ . \ l_7) \\
&= 5\mathrm{K}(m_1 + m_2 \ . \ . \ . \ m_7) \\
&= \text{Estimated gross reproduction rate.}
\end{aligned}
$$

Thus when the age-groups are all equal, no error is possible, whatever set of fertility rates are used.

Further, if γ is the error in the estimated rate; $p+\delta_1$, $p+\delta_2$, etc., the age-groups; $\pm\,\theta_1$, $\pm\,\theta_2$, etc., the differences between the actual and adjusted fertility rates, then

$$\gamma = \frac{(m_1 \pm \theta_1) + (m_2 \pm \theta_2) \ . \ . \ . \ (m_7 \pm \theta_7) - (m_1 + m_2 \ . \ . \ . \ + m_7)}{m_1 + m_2 \ . \ . \ . \ m_7},$$

i.e.

$$\gamma = \frac{\pm\,\theta_1 \pm \theta_2 \ . \ . \ . \ \pm\,\theta_7}{m_1 + m_2 \ . \ . \ . \ m_7}.$$

But

$$
\begin{aligned}
(m_1 \pm \theta_1)(p+\delta_1) + (m_2 \pm \theta_2)(p+\delta_2) \ . \ . \ . \ (m_7 \pm \theta_7)(p+\delta_7) \\
= m_1(p+\delta_2) + m_2(p+\delta_2) \ . \ . \ . \ m_7(p+\delta_7), \\
\therefore \ \pm\,\theta_1(p+\delta_1) \pm \theta_2(p+\delta_2) \ . \ . \ . \ \pm\theta_7(p+\delta_7) = 0. \\
\therefore \ p(\pm\,\theta_1 \pm \theta_2 \pm \theta_3 \ . \ . \ . \ \pm\,\theta_7) = \pm\,\theta_1\delta_1 \pm \theta_2\delta_2 \ . \ . \ . \ \pm\theta_7\delta_7, \\
\therefore \ \gamma = \frac{\pm\,\theta_1\delta_1 \pm \theta_2\delta_2 \ . \ . \ . \ \pm\,\theta_7\delta_7}{p(m_1 + m_2 \ . \ . \ . \ m_7)}.
\end{aligned}
$$

We thus see that γ can be very small under three conditions:

(i) θ_1, θ_2, etc., can all be very small.
(ii) The positive and negative errors can balance one another.
(iii) δ_1, δ_2, can all be very small, *i.e.* the age-groups can be very nearly equal.

Since in practice the true rates are unknown, it is impossible to say whether the first two conditions are fulfilled or not. The first is exemplified in the figures obtained when the Danish rate is applied to Australia, 1909 (Table II). The differences between the specific rates were all less than 10%, except in those for the 45–50 age-group. As the rate for this group was so low, a big difference in the rate was immaterial. The second condition is seen in the rates for Norway, 1916, applied to Australia, 1909. The differences were large, *e.g.* 56% in the 15–20 age-group and 44% in the 40–45 age-group, but negative differences in the young age-groups were balanced by positive differences in the older groups. When the true age-specific fertility rates are not known, it is not possible by any direct means to ensure that the rates used for estimation will correspond individually to the true ones and thus control conditions (i) and (ii). The tables show that the error in estimation is slightly smaller when appropriate rates are used. A fall in the gross reproduction rate may be associated with similar changes in the relative fertility of different age-groups. One can therefore say that the use of appropriate fertility rates increases somewhat the probability of obtaining a good fit between the estimated and the actual rates. The elimination of large errors is possibly even better achieved by avoiding the use of rates from countries such as Norway and Finland which are known to have unusual age-specific fertility rates.

The differences in size of age-groups are known. The rates for France, 1931 (Table III), show that when they are nearly equal, the error is negligible. The relationship between inequality in age-groups and magnitude of error in estimation is summarised in Table IV.

TABLE IV.

Country.		Ratio 15–19/45–49 Age-Groups.	Ratio Maximum to Minimum Age-Groups.	Mean % Error in Estimated G.R.R.	Maximum % Error in Estimated G.R.R.
France .	1931	1·08	1·21	0·22	1·00
Latvia .	1937	1·02	1·38	1·66	2·94
Australia	1933	1·52	1·52	1·37	3·99
Australia	1912	2·04	2·04	2·52	5·39
Australia	1909	2·17	2·17	2·92	5·66
Bulgaria.	1926–27	2·51	2·51	3·53	6·72
Lithuania	1928	2·89	2·89	3·06	5·24

Increase in size both of the mean error and of the maximum error corresponds closely to inequality in size of age-groups. It might safely be concluded that when the largest age-group is not more than one-and-a-half times the smallest, in the majority of cases the error in estimation will be negligible, and in any event is not likely to be more than 4%. Usually inequality of age-groups is associated with high gross reproduction rates, and *vice versa.* So one may say that in practice high rates are likely to be more inaccurate than low ones.

If we consider that the data presented in Table IV provide materials for assessing the size of the maximum error, then we can go further and say that for populations having a ratio of maximum to minimum age-groups of not more than 1·2, the maximum error is probably not more than 2%. If the ratio is not more than 1·5, the maximum error is about 4%. If the ratio is not more than 2·2, the error is about 6%, and if the ratio is over 2·2, the error may be as much as 7%. In Table V these criteria are applied to the writer's estimates of gross reproduction rates for Scottish regions (1938).

TABLE V.

Ratio Maximum/Minimum Age-Groups.

Year.	Category.	1·0–1·2.	1·21–1·5.	1·51–2·2.	Over 2·2.
				Maximum Error.	
		2%.	4%.	6%.	7%.
1911	Aggregates	..	..	Scotland and all Aggregates	..
	Counties	..	Orkney Perth Berwick Roxburgh Selkirk Bute	26 Counties	Lanark
	Small Burghs (by Counties)	..	Perth	22 Small Burghs (by Counties)	West Lothian
	Landward Areas (by Counties)	..	Orkney Dumfries Perth Argyll Roxburgh Zetland	20 Landward Areas (by Counties)	Lanark Clackmannan
	Large Burghs	..	..	17 Large Burghs	Hamilton Airdrie Rutherglen
	Cities	..	..	All Cities	
1931	Aggregates	..	Scotland Counties Landward Areas Small Burghs	Large Burghs Cities	..
	Counties	Berwick Selkirk Zetland Bute	20 Counties	9 Counties	..
	Small Burghs (by Counties)	Perth Selkirk	14 Small Burghs (by Counties)	8 Small Burghs (by Counties)	..
	Landward Areas (by Counties)	Zetland Argyll Berwick	17 Landward Areas (by Counties)	8 Landward Areas (by Counties)	..
	Large Burghs	..	8 Large Burghs	12 Large Burghs	..
	Cities	..	Dundee Edinburgh	Glasgow Aberdeen	..

This table shows that the error would probably not exceed 6% for the great majority of the rates in 1911. The most important exceptions are the county of Lanark, its small burghs, and three of its large burghs, all of which might have a larger error To see what would be the effect of the maximum error, suppose Lanark to be 7% in excess, and West Lothian 6%, while other rates remained unaltered. Any change would obviously be in the same direction for both. In 1911 West Lothian had the highest gross reproduction rate, viz. 2·362, while Lanark came next with 2·276. If these alterations were made, then West Lothian would be 2·228 and Lanark 2·127, and both are still higher than the next county, Stirling, 1·985. Turning to the rates for the year 1931, there were now no rates in the highest category. There were still many in the 6% category, but the majority of the rates fell in the 4% category, while there were some still lower. Thus any possible error in the estimated Scottish rates would not be likely to affect any of the conclusions which were drawn from them.

3. SUMMARY.

1. The implicit assumptions made in using standardised fertility rates and gross reproduction rates, when specific age-fertility rates are not available, are essentially similar when either method is used to assess total fertility.

2. The standardised fertility rate of married women implies the arbitrary and highly questionable assumption that fertility and age of marriage are independent variables. The gross reproduction rate of married women does not involve this assumption.

3. When age-specific fertility rates are used for estimations which give a total of births within 10% of the recorded births, the mean error in estimated gross reproduction rates is probably between 2% and 3%, and is not likely to exceed about 7%.

4. The magnitude of the error increases with increasing inequality in the age-groups, being negligible when they are nearly equal, and rising to a possible maximum of about 13% in the most extreme cases of inequality.

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III.—Observations on the Œstrous Cycle of the Guinea-Pig.

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THE pioneer work of Stockard and Papanicolaou (1917, 1919) forms the basis of most subsequent observations on the œstrous cycle of the guinea-pig, and their results are, in the main, accepted by later workers—Ishii (1920), Selle (1922), Nicol (1932), etc.

Hitherto it has been agreed that the cytology of the vaginal smears furnishes a reliable index of the sexual rhythm of the animal. In a recent series of publications Young *et al.* (1935–37–38) aver that copulatory response is the only valid criterion of heat, which they state cannot be detected by a mere examination of vaginal smears. In spite of these statements we are still of the opinion that the vaginal smear technique provides the only practical, and for most purposes, satisfactory working formula.

Allen (1922), working on the œstrous cycle of the mouse, reported variations in the length, not only of the total cycle, but also of its separate phases. Hemmongsen and Krarup (1937) have demonstrated the effect of light on the duration of œstrus in rats, and E. H. Bohanan (1939) has produced positive evidence of the influence of environment on the œstrous cycle of the rat. She determines variation in the length of the cycle by examination of vaginal smears, but no detailed analysis was made of the separate phases of œstrus. The sexual cycle of the cat has also been the subject of recent investigation by H. Liche (1939) who found that change of habitation, for example climate, has a marked influence in determining the appearance or otherwise of œstrus. Bohanan's paper was published in January 1939, and that of Liche in May 1939. Our observations on the guinea-pig were completed during 1938, and we were interested to note that somewhat parallel investigation on other animals has produced results in some measure confirming our conclusions.

This work is concerned with the duration of the period of sexual activity, *i.e.* œstrus, and its separate stages, and an endeavour to account for the very marked prolongation of œstrus in our animals as compared to those of Stockard and Papanicolaou. There is considerable literature on the subject of variation in length of the total sexual cycle of the guinea-pig, but the time occupied by œstrus and its separate phases as originally given by Stockard and Papanicolaou has not thus far been challenged. It is perhaps of some significance that much of the work on this subject has been conducted in America, and it is possible that the greatly increased length of œstrus in our animals compared to the "normal" is a reflection of climatic, domestic or genetic differences.

Thirty-five virgin female guinea-pigs were used, and a hundred œstrous cycles examined by the vaginal smear method. Our animals were common white, brown or mixed colours, obtained from reliable dealers. They were kept in a commodious, suitably ventilated and thermostatically controlled animal house at a temperature of 55° F., and were fed on a well-rationed diet of green vegetables, carrots, hay and bran. The animals were examined as a routine, morning and evening, as it was thought that too frequent interference might influence the normal course of the cycle. However, on several occasions, when deemed necessary, additional examinations were made throughout the twenty-four hours. The technique of Selle was adopted: a small glass pipette containing about 1 c.c. of warm normal saline is inserted into the vagina, and by compression of the rubber bulb attached to the pipette the vagina is washed out. On releasing the bulb, a few minims of the vaginal contents are sucked into the pipette. This fluid is placed on a slide, fixed with heat, and stained with hæmalum and eosin.

There is a considerable choice of terms to denote the sexual cycle and its various stages in the guinea-pig, but throughout we shall adhere to the following: the œstrous or sexual cycle is divided into the diœstrous or quiescent period during which the vagina is closed, and the œstrus or period of sexual activity when the vagina is open. There are four stages of œstrus, determined by an examination of the vaginal smear.

1st stage—vaginal smears show large, irregular, superficial, nucleated epithelial cells.
2nd stage—vaginal smears show a preponderance of cornified cells.
3rd stage—vaginal smears show a decreasing number of cornified cells; small, rounded, deep, epithelial cells; and some leucocytes.
4th stage—vaginal smears show predominantly leucocytes.

The above stages differ in some respects from those given by Stockard and Papanicolaou and also from those of Selle, but adhere more closely to the findings of the latter with whom we agree that the presence of red blood cells in the 4th stage of Stockard and Papanicolaou was in all probability the result of their method of obtaining smears.

The deep epithelial cells appear towards the end of the 2nd stage, and it is of some interest to observe that there is a gradual decrease in the size of these cells in subsequent stages. This is in accordance with the general opinion that the catabolic process involves successive layers of the vaginal epithelium until finally the deepest stratum is affected. We consider that the majority of these cells are vaginal in origin, and that uterine elements are few in number and of no particular significance.

As it is generally accepted that ovulation occurs at the time of, or just before, the appearance of the leucocytes, we regard this as the critical point of œstrus, and for our purposes have divided the period of sexual activity into non-leucocytic (stages I and II) and leucocytic (stages III and IV) phases. A few animals were killed in the late second stage, and the occurrence of ovulation verified by examination of the ovaries.

RESULTS.

In our analysis we include only those cycles where we could with some degree of accuracy determine the beginning of œstrus.

Fig. 1 illustrates in graphic form the length of the complete sexual cycles. There is a range of from 13–20 days (*cf.* Stockard and Papanicolaou (1917), 15–18; Selle (1922), 14–17; Loeb (1918), 13–19; Nicol (1932), 10–24) with a mean length of 16 days. There was but a single instance of a cycle of 13 days and only two of 20 days. There is some variation, although within smaller limits (1–2 days), in the length of consecutive cycles in the same animal. The majority of the cycles are within the recognised mode, and even those at the limits of the graph exhibit a normal vaginal cytology during œstrus.

Fig. 2 illustrates the length of the œstrous period during which the vagina remains open. There is a considerable range of difference, from 24 to 84 hours, with a mean length of 50 hours. The mode of the graph is 48 hours. These figures show more than 100 per cent. increase over those given by Stockard and Papanicolaou, who are the only authors, as far as can be ascertained, who have dealt with the time-factor of the various phases of the sexual cycle in the guinea-pig. Again, it should be noted that while the length of œstrus in the same animal may vary in different sexual cycles, this variation is within fairly narrow limits, and very generally the œstrous time of each animal was remarkably constant.

Fig. 3 presents graphically the duration of the non-leucocytic phase of œstrus (stages I and II). There is a range of from 12–48 hours with a mean length of 23 hours. The mode of the graph is 24 hours. The corresponding figures given by Stockard and Papanicolaou are 8–16 hours, and again we have to show a very considerable increase. It is of interest, however, that the proportion of the total time of œstrus occupied by the non-leucocytic phase remains fairly constant. Even in our longest œstrus—over 72 hours—more than half of this period is taken up by stages I and II.

In 20 per cent. of our cases the smears taken *immediately following the opening of the vagina showed typical diœstrous elements, i.e.* leucocytes and a varying amount of mucous secretion, and only after a lapse of from 6–36 hours did we find the epithelial cells characteristic of the first stage of œstrus. We consider such cases to be due to a premature disappearance of the vaginal closure membrane, and have designated as "Pre-œstrus" the stage indicated by those early smears exhibiting the vaginal cytology of the diœstrus. In most instances the pre-œstrous picture changed abruptly into that of œstrus. At other times the change was more gradual, but only seldom did the leucocytosis of the pre-œstrus persist in a minor degree throughout stages I and II.

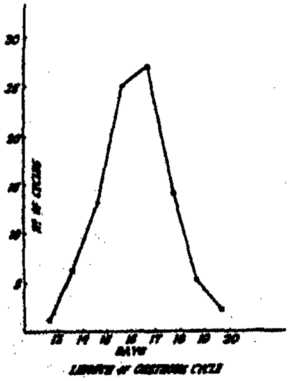

FIG. 1.—Frequency polygon of length of œstrous cycles.

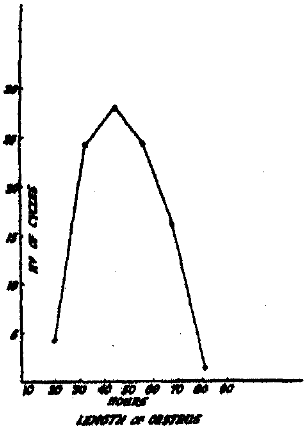

FIG. 2.—Frequency polygon of length of œstrus.

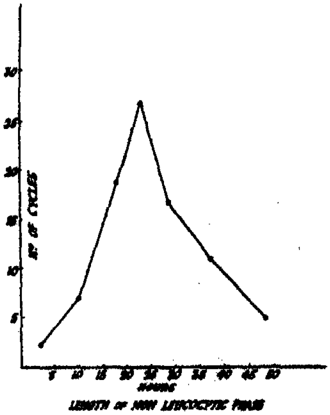

FIG. 3.—Frequency polygon of length of non-leucocytic phase of œstrus.

We would lay particular emphasis on the fact that in most of our cases this pre-œstrous phase was followed by a normal œstrous period. The time occupied by the pre-œstrus was not included in the length of œstrus.

CONCLUSIONS.

The series examined is perhaps rather small, but the marked prolongation of œstrus in our animals as compared to those of Stockard and

Papanicolaou is so consistent throughout that any question of occasional divergence attributed to random sampling can be eliminated. The increase in length affects all four phases of the cycle, but as the appearance of the leucocytes heralds the final stages of the catabolic period, more importance perhaps attaches to the prolongation of the non-leucocytic phase. Moreover, in many endocrinological experiments attention is focussed on stages I and II, and it is necessary to recognise that considerable range of differences may occur within the normal mode. In some of our animals, smears showing only cornified cells persisted throughout almost two days.

We would again point out that the length of the complete cycle was within accepted limits, and, indeed, had a mean of 16 days. Nor does a prolonged œstrus appear either to delay or accelerate the appearance of the succeeding cycle. Interest, of course, centres on the explanation of the difference between the length of œstrus in our animals and those of other workers.

Comment has frequently been made on the delicacy of the œstrous mechanism—how the cyclic rhythm may be altered, may even become atypical or irregular from such causes as unsuitable temperature or an ill-balanced diet. Most of the work in this field has been carried out in America, and we suggest that this fact, implying as it does climatic, environmental and genetic differences, should be considered in any endeavour to account for the greater length of the œstrus and its phases which was so prominent and consistent a feature in our results.

Although he declines to relate "heat" to any definite phase of the sexual cycle as determined by examination of the vaginal smears, Young (1938) suggests that the length of heat is related to the time required for the completion of ovulation plus the sensitivity of the individual to hormonal action. Mayhap variation of these two factors could be regarded as an expression of genetic deviation.

It might be said in criticism of the validity of our results that a more frequent vaginal examination would give more exact information regarding the various phases, but, on the other hand, as previously stated, we believe that a minimum interference will conduce to a more normal sequence of events.

We have been unable to find in the literature on this subject any reference to what we have described as a pre-œstrous phase. Young, indeed, mentions an overlap of leucocytes in the first stage and sometimes persisting throughout the second, but makes no reference to early smears showing the cytology of the pre-œstrus.

We feel that the occurrence of this pre-œstrous phase should be recognised as a possible prelude to a cycle normal in every other respect.

Many whose work necessitates an investigation of the sexual cycle of the guinea-pig by the method of examination of vaginal smears, may have experienced some difficulty in reconciling their results with those of Stockard and Papanicolaou, and may even, as did we, in the first instance, incline to view with suspicion animals with a prolonged œstrus; but we feel assured that such results should not be considered as in any way irregular, but regarded as falling within a legitimate range of difference whose limits are determined by climatic, domestic, and more particularly genetic, variations.

SUMMARY.

1. 100 œstrous cycles from 35 virgin female guinea-pigs were examined by the vaginal smear method.

2. The length of the œstrous cycles varied from 13–20 days with a mean of 16 days.

3. Duration of œstrus varied from 24–84 hours with a mean of 50 hours, while the non-leucocytic period (stages I and II) had a range of from 12–48 hours with a mean of 23 hours.

4. In 20 per cent. of cases a normal œstrus was preceded by a pre-œstrous period (leucocytes and mucous secretion).

5. It is suggested that the marked increase in the duration of œstrus and its non-leucocytic phase compared to the corresponding results of American workers is an expression of climatic, domestic and, more particularly, genetic variations.

We are indebted to Professor D. M. Blair for guidance and supervision.

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IV.—On the Independence of Linear and Quadratic Forms in Samples of Normally Distributed Variates.

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1. A GENERAL CRITERION OF INDEPENDENCE.

IT is well known, and fundamentally important in the justification of the current mode of deriving Student's t-distribution, that the mean m of n values x_i all obeying the same normal law of probability, and the associated estimate of variance, namely

$$s^2 = \Sigma(x_i - m)^2/(n-1), \quad . \quad . \quad . \quad . \quad (1)$$

are statistically independent. We obtain below a general criterion for the independence of a linear and a quadratic form in n sample values x_i, or, still more generally, in the $h(n_1+n_2+ \ . \ . \ . \ +n_k)$ elements of k samples each of h normally correlated variates, the numbers of h-ads of values in the respective samples being $n_1, n_2, \ . \ . \ ., n_k$.

The normal law for a single variate x is characterized by the probability differential $dp = \phi(x)dx$, where $\phi(x)$ is the probability density

$$\phi(x) = (2\pi\sigma^2)^{-\frac{1}{2}} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}, \quad . \quad . \quad . \quad . \quad (2)$$

the mean being μ, the variance σ^2. Without essential loss of generality we may take μ as the origin, so that $\mu = 0$.

The compound probability differential for ν normally correlated variates x_i may be written analogously in matrix notation as $dp = \phi(x)dx$, where

$$\phi(x) = (2\pi)^{-\frac{1}{2}\nu} \mid V \mid^{-\frac{1}{2}} \exp\left(-\tfrac{1}{2}x' V^{-1} x\right), \quad . \quad . \quad . \quad (3)$$

where x now indicates the vector $x = \{x_1 x_2 \ . \ . \ . \ x_\nu\}$, dx represents $dx_1 dx_2 \ . \ . \ . \ dx_\nu$, V is the variance matrix of the ν variates, $\mid V \mid$ is its determinant, and $x' V^{-1} x$ is a quadratic form.

In the present case let us write out the $h(n_1+n_2+ \ . \ . \ . \ +n_k)$ elements of the k samples, each single constituent of a sample being a vector of h simultaneously observed values such as $\{x_i y_i \ . \ . \ .\}$, in the form of a single vector

$$x = \{x_1 x_2 \ . \ . \ . \ x_n y_1 y_2 \ . \ . \ . \ y_n \ . \ . \ .\}, \quad . \quad . \quad . \quad (4)$$

where $n = n_1 + n_2 + \ . \ . \ . \ + n_k$. Then any linear form such as $a_1 x_1 + a_2 x_2 + \ . \ . \ . \ + a_n x_n + b_1 y_1 + b_2 y_2 + \ . \ . \ . \ + b_n y_n + \ . \ . \ .$ can be written as the vector product (row vector into column vector)

$$[a_1 a_2 \ldots a_n b_1 b_2 \ldots b_n \ldots]\{x_1 x_2 \ldots x_n y_1 y_2 \ldots y_n \ldots\} = a'x. \qquad (5)$$

Any quadratic form in the nh values can be written in matrix notation as $x'Bx$, where $B \equiv [b_{ij}]$ is the symmetric matrix of the form. For convenience we shall introduce a factor $\frac{1}{2}$, and shall consider under what condition $a'x$ and $\frac{1}{2}x'Bx$ are independent.

The joint moment generating function (m.g.f.) or exponential transform of $a'x$ and $\frac{1}{2}x'Bx$ is

$$G(\alpha, \beta) = (2\pi)^{-\frac{1}{2}nh} \mid V \mid^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}(x'V^{-1}x - 2\alpha a'x - \beta x'Bx)\}dx, \qquad (6)$$

where the integral is nh-fold, and α, β are arbitrary variables. Since $x'V^{-1}x$ is a positive definite quadratic form, the integral exists over some continuous region in both α and β.

The integral may be evaluated by "completing the quadratic form" and changing the origin of all variates by the simple linear transformation so derived. Thus

$$\begin{aligned} &x'(V^{-1} - \beta B)x - 2\alpha a'x \\ &= \{x - \alpha(V^{-1} - \beta B)^{-1}a\}'(V^{-1} - \beta B)\{x - \alpha(V^{-1} - \beta B)^{-1}a\} - \alpha^2 a'(V^{-1} - \beta B)^{-1}a \\ &= y'(V^{-1} - \beta B)y - \alpha^2 a'(V^{-1} - \beta B)^{-1}a, \qquad (7) \end{aligned}$$

where
$$y = x - \alpha(V^{-1} - \beta B)^{-1}a. \qquad (8)$$

Hence
$$\begin{aligned} G(\alpha, \beta) &= \mid V \mid^{-\frac{1}{2}} \mid V^{-1} - \beta B \mid^{-\frac{1}{2}} \exp\{\tfrac{1}{2}\alpha^2 a'(V^{-1} - \beta B)^{-1}a\} \\ &= \mid I - \beta VB \mid^{-\frac{1}{2}} \exp\{\tfrac{1}{2}\alpha^2 a'(V^{-1} - \beta B)^{-1}a\}. \qquad (9) \end{aligned}$$

The first factor in (9) involves β alone, and is indeed the m.g.f. $G(0, \beta)$ of $\frac{1}{2}x'Bx$. The second factor would be the m.g.f. $G(\alpha, 0)$ of $a'x$, if and only if it were independent of β. But the necessary and sufficient condition for the independence of any two functions (under any law of probability) with m.g.f. $G(\alpha, \beta)$ is

$$G(\alpha, \beta) = G(\alpha, 0)G(0, \beta), \qquad (10)$$

and so we deduce the desired criterion of independence:

The quadratic form $a'(V^{-1} - \beta B)^{-1}a$ must be independent of β.

If the variates are expressed in standard scale, so that their variances are equal to unity, V is the matrix $R \equiv [\rho_{ij}]$ of total correlations, and if, still more specially, the variates are uncorrelated, $V = I$, the unit matrix.

2. APPLICATION TO MEAN AND VARIANCE ESTIMATE.

Let the above criterion be applied to the estimates m and s^2 of mean and variance in a sample of n independent single values of one variate,

$$m = \Sigma x_i / n, \qquad s^2 = \Sigma(x_i - m)^2/(n - 1). \qquad (1)$$

To avoid unnecessary complexity we may disregard the factors $1/n$ and $1/(n-1)$ and consider the independence of the sum Σx_i and half the sum of squared deviations from the mean, namely $\frac{1}{2}\Sigma(x_i-m)^2$. Thus we have

$$a' = [1 \quad 1 \quad \ldots \quad 1], \qquad B = \begin{bmatrix} 1-\frac{1}{n} & -\frac{1}{n} & \ldots & -\frac{1}{n} \\ -\frac{1}{n} & 1-\frac{1}{n} & \ldots & -\frac{1}{n} \\ \cdot & \cdot & \cdot & \cdot \\ -\frac{1}{n} & -\frac{1}{n} & \ldots & 1-\frac{1}{n} \end{bmatrix} . \qquad (2)$$

Evidently $a'(I-\beta B)^{-1}a$ is in this case the sum of all the n^2 elements in $(I-\beta B)^{-1}$. We note that

$$I-\beta B = (1-\beta)I+\beta M, \quad . \quad . \quad . \quad . \quad (3)$$

where M is the matrix having all elements equal to $1/n$. Thus $M^2=M$, that is, M is idempotent. Here we introduce a Lemma useful in cases of this kind.

Lemma.—If Q is an idempotent matrix, then

$$(\rho I-\sigma Q)^{-1} = \rho^{-1}\left(I+\frac{\sigma}{\rho-\sigma}Q\right), \qquad \rho \neq \sigma. \quad . \quad . \quad . \quad (4)$$

The Lemma is arrived at heuristically by expanding $\rho^{-1}(I-\sigma\rho^{-1}Q)^{-1}$ in powers of Q, substituting Q for each power of Q above the first, and summing the resulting geometric progression in $\sigma\rho^{-1}$. This derivation requires for convergence the restriction $|\sigma| < |\rho|$. However, the result once derived, we justify it, under the much less stringent condition $\rho \neq \sigma$, by observing that in virtue of $Q^2=Q$ the product of the matrix on the right of (4) by $\rho I-\sigma Q$ in either pre- or postmultiplication yields I, the unit matrix.

Now in our present application $\rho = 1-\beta$, $\sigma = -\beta$, and so

$$(I-\beta B)^{-1} = (1-\beta)^{-1}(I-\beta M). \quad . \quad . \quad . \quad . \quad (5)$$

Also

$$I-\beta M = \begin{bmatrix} 1-\frac{\beta}{n} & -\frac{\beta}{n} & \ldots & -\frac{\beta}{n} \\ -\frac{\beta}{n} & 1-\frac{\beta}{n} & \ldots & -\frac{\beta}{n} \\ \cdot & \cdot & \cdot & \cdot \\ -\frac{\beta}{n} & -\frac{\beta}{n} & \ldots & 1-\frac{\beta}{n} \end{bmatrix}, \quad . \quad . \quad . \quad (6)$$

and the sum of all its elements is $n(1-\beta)$. Hence the sum of all elements in $(I-\beta B)^{-1}$ is n, which is independent of β. It thus follows from the criterion that m and s^2 are independent.

3. THE DIFFERENCE OF MEANS OF TWO SAMPLES.

A further important application is to this question: given two samples, one of n_1 values x_i, the other of n_2 values which we may distinguish by y_j, to decide whether the x_i and y_j conform to the same law of normal probability, as judged by the difference of the means, m and m' say, of the respective samples. In the test the absolute value of the difference, $|m-m'|$, is compared with an estimate, from the sample, of its variance

$$(n_1+n_2)\sigma^2/n_1n_2, \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

the comparison being conducted by reference to the appropriate t-distribution.

Let the n_1+n_2 sample values, the x_i first, the y_j following, be taken as elements of the partitioned vector

$$x=\{x_1x_2\ldots x_{n_1} \,\vdots\, y_1y_2\ldots y_{n_2}\}. \quad . \quad . \quad . \qquad (2)$$

Then the difference $m-m'$ is the linear form $a'x$, where

$$\begin{aligned} a' &= [1/n_1\ \ 1/n_1\ \ldots\ 1/n_1 \,\vdots\, -1/n_2\ \ -1/n_2\ \ldots\ -1/n_2] \\ &= [a_1 \,\vdots\, a_2]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)\end{aligned}$$

The estimate of σ^2 used for assessing $m-m'$ is a weighted sum of estimates derived from the separate samples, the weights being as n_1-1 and n_2-1. It is in fact

$$s^2=\{\Sigma(x_i-m)^2+\Sigma(y_j-m')^2\}/(n_1+n_2-2), \quad . \quad . \quad . \qquad (4)$$

and so, disregarding the unessential factor $1/(n_1+n_2-2)$, we consider $\frac{1}{2}x'Bx$, where

$$B=\left[\begin{array}{cccc|cccc}
1-\frac{1}{n_1} & -\frac{1}{n_1} & \cdots & -\frac{1}{n_1} & & & & \\
-\frac{1}{n_1} & 1-\frac{1}{n_1} & \cdots & -\frac{1}{n_1} & & \text{(zeros)} & & \\
\cdots & \cdots & \cdots & \cdots & & & & \\
-\frac{1}{n_1} & -\frac{1}{n_1} & \cdots & 1-\frac{1}{n_1} & & & & \\
\hdashline
 & & & & 1-\frac{1}{n_2} & -\frac{1}{n_2} & \cdots & -\frac{1}{n_2} \\
 & \text{(zeros)} & & & -\frac{1}{n_2} & 1-\frac{1}{n_2} & \cdots & -\frac{1}{n_2} \\
 & & & & \cdots & \cdots & \cdots & \cdots \\
 & & & & -\frac{1}{n_2} & -\frac{1}{n_2} & \cdots & 1-\frac{1}{n_2}
\end{array}\right]. \qquad (5)$$

Thus B is the "direct sum" (that is, constructed from matrices juxtaposed diagonally) of two matrices, let us say B_1 and B_2, each of the form $I-M$, of respective orders $n_1\times n_1$ and $n_2\times n_2$.

Since B_1 and B_2 are isolated in this way, the reciprocal $(I-\beta B)^{-1}$ is the direct sum of the two reciprocals $(I-\beta B_1)^{-1}$ and $(I-\beta B_2)^{-1}$, whence it follows from (3) that the quadratic form

$$a'(I-\beta B)^{-1}a \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

is the sum of the elements in $(I-\beta B_1)^{-1}$, multiplied by $1/n_1^2$, plus the sum of those in $(I-\beta B_2)^{-1}$, multiplied by $1/n_2^2$. But by the result of § 2 the sums of the elements in $(I-\beta B_1)^{-1}$ and $(I-\beta B_2)^{-1}$ in neither case contain β; in fact they are n_1 and n_2 respectively. Hence the quadratic form $a'(I-\beta B)^{-1}a$ takes the value $(n_1+n_2)/n_1n_2$, which is independent of β; and thus the independence of $m-m'$, the difference of means, and the estimate s^2 of variance is established.

Evidently the factors $1/n_1$ and $1/n_2$ in the vector a' might have been replaced by any others without prejudice to the above result, so that not only the difference $m-m'$, but any linear combination $\rho m-\sigma m'$ is independent of this particular estimate s^2. The fact is, however, of no more than academic interest, because the only linear combination that does not involve the unknown mean μ of the underlying normal law is the difference $m-m'$, or some multiple of the difference.

An exactly similar proof, based on the direct sum of k matrices of the type $I-M$, shows that any linear combination of the k means of respective samples of $n_1, n_2, \dots, n_k$ values is statistically independent of the estimate of variance s^2 obtained by weighting the k separate estimates from sample with weights proportional to $n_1-1, n_2-1, \dots, n_k-1$. Thus

$$s^2=\{\Sigma(x_i-m)^2+\Sigma(y_j-m')^2+\dots\}/(n_1+n_2+\dots+n_k-k). \quad . \qquad (7)$$

Again, however, the only cases of practical value are those in which the linear combinations of $m, m', \dots$ do not involve the unknown mean μ of the underlying normal law. For example any linear combination for which the sum of the coefficients is zero, such as the $(k-1)^{\text{th}}$ difference of the k means, is free from μ and amenable to a test by the t-distribution.

4. A Case of Dependence.

In judging the difference of the means of two samples it is important to estimate σ^2 as in § 3 (4), and not by the estimate involving the mean $\hat{m}$ of the *pooled* sample of n_1+n_2 values, namely

$$\hat{s}^2=\Sigma\{(x_i-\hat{m})^2+(y_j-\hat{m})^2\}/(n_1+n_2-1), \quad . \quad . \quad . \qquad (1)$$

which, as being of weight n_1+n_2-1 instead of n_1+n_2-2, might plausibly suggest itself. In point of fact this quadratic form is not in general independent of $m-m'$; for, neglecting the unessential $1/(n_1+n_2-1)$, we have

$$(I-\beta B)^{-1}=(1-\beta)^{-1}\begin{bmatrix} 1-\dfrac{\beta}{n_1+n_2} & -\dfrac{\beta}{n_1+n_2} & \cdots & -\dfrac{\beta}{n_1+n_2} \\ -\dfrac{\beta}{n_1+n_2} & 1-\dfrac{\beta}{n_1+n_2} & \cdots & -\dfrac{\beta}{n_1+n_2} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ -\dfrac{\beta}{n_1+n_2} & -\dfrac{\beta}{n_1+n_2} & \cdots & 1-\dfrac{\beta}{n_1+n_2} \end{bmatrix}, \qquad (2)$$

which gives for $a'(I-\beta B)^{-1}a$ the value

$$(1-\beta)^{-1}\left\{\left(n_1-\frac{n_1^2\beta}{n_1+n_2}\right)\Big/n_1^2-2\left(\frac{n_1n_2\beta}{n_1+n_2}\right)\Big/n_1n_2+\left(n_2-\frac{n_2^2\beta}{n_1+n_2}\right)\Big/n_2^2\right\}$$

$$=(1-\beta)^{-1}\left(\frac{1}{n_1}+\frac{1}{n_2}-\frac{4\beta}{n_1+n_2}\right). \qquad (3)$$

This is not independent of β unless $n_1=n_2$, that is, unless the samples are of equal size. It then takes the value $2/n_1$.

The conclusion is that when $n_1=n_2$ a variant of the ordinary t-test is possible, based on the different estimate of σ^2 given at (1) above, but involving $2n_1-1$ instead of $2n_1-2$ degrees of freedom in entering the tables of the t-integral. This alternative is very slightly more accurate than the customary test.

5. CRITERION FOR INDEPENDENCE OF QUADRATIC FORMS.

Suppose we have two quadratic forms $\frac{1}{2}x'Bx$ and $\frac{1}{2}x'Cx$, where the vector x has for its elements nk sample values, of probability density

$$(2\pi)^{-\frac{1}{2}nk}\,|\,V\,|^{-\frac{1}{2}}\exp\left(-\tfrac{1}{2}x'V^{-1}x\right), \quad . \quad . \quad . \quad (1)$$

as in § 1 (3). The joint m.g.f. is then

$$G(\beta,\gamma)=(2\pi)^{-\frac{1}{2}nk}\,|\,V\,|^{-\frac{1}{2}}\int_{-\infty}^{\infty}\exp\left\{-\tfrac{1}{2}x'(V^{-1}-\beta B-\gamma C)x\right\}dx,$$

$$=|\,I-\beta VB-\gamma VC\,|^{-\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Thus the condition

$$G(\beta,\gamma)=G(\beta,0)G(0,\gamma) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

yields the criterion (to be satisfied identically in β and γ)

$$|\,I-\beta VB-\gamma VC\,|=|\,I-\beta VB\,|\;\;|\,I-\gamma VC\,| \quad . \quad . \quad (4)$$

as the necessary and sufficient condition for the independence of $x'Bx$ and $x'Cx$.

Most practical cases, however, do not require a deep application of this criterion, since the quadratic forms are usually estimates of variance derived from independent samples, and so are *ipso facto* independent.

The extension of the criterion to the case of any number of quadratic forms is evident.

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V.—The Periodic Lamé Functions. By E. L. Ince, D.Sc.,

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1. INTRODUCTION.

ANY solution of the Lamé equation

$$\frac{d^2y}{du^2}=\{n(n+1)k^2\text{sn}^2\,u-h\}y, \qquad (1.1)$$

in which sn u has the modulus k, may be termed a Lamé function, but this paper is devoted mainly to functions of real periods $2K$ or $4K$. We suppose that $k^2<1$; the number n is real, but not necessarily an integer, and it is sufficient to take $n\geqslant-\frac{1}{2}$. These periodic solutions exist for certain *characteristic* values of h; a table of such values is included in this paper.

For *integral* values of n, the periodic Lamé functions may be termed *Lamé polynomials*, for they are then polynomials of degree n in one of the Jacobian elliptic functions sn u, cn u, dn u, or in two or more of them jointly.

From the point of view of elliptic harmonics and their applications in applied mathematics, these polynomials are of paramount importance. They are usually divided into eight classes (Whittaker and Watson, *Modern Analysis*, 1927, § 23.5), four of which have, and the rest have not, the factor dn u. But when our survey is not restricted to polynomial forms this system breaks down, for every periodic Lamé function with the factor dn u is found (as will be proved) to be identical with a form without that factor, and therefore there are, from the analytic point of view, no more than four distinct types of Lamé functions of periods $2K$ and $4K$.

Most of the attempts at devising a precise notation have so far been based upon the assumption of eight distinct classes (*cf.* P. Humbert, *Fonctions de Lamé et Fonctions de Mathieu*, 1926, p. 12). The best of such notations, that of Poincaré, is marred by the presence of an index whose value (1, 2, 3, or 4) is determined by an arbitrary rule, together with two suffixes, namely n and a number to be determined by considering the degenerate case $k^2=1$. Furthermore, all existing notations collapse when n ceases to be an integer. Here we shall suggest a simple notation involving a single suffix and an index (the order n and a number m to be specified), which suffices for all even periodic Lamé functions; an obvious modification of it embodies all the odd periodic functions.

2. FUNCTIONS OF REAL PERIOD $2K$ EXPRESSED AS SERIES OF EVEN POWERS OF sn u.

The expression

$$y=A_0+\sum_r A_{2r}\,\mathrm{sn}^{2r} u \qquad . \quad . \quad . \quad . \quad (2.1)$$

will be a formal solution of (1.1) if the coefficients are such as to satisfy the recurrence-relations

$$1.2A_2+hA_0=0,$$

$$(2r+1)(2r+2)A_{2r+2}+\{h-4r^2(1+k^2)\}A_{2r}=(n-2r+2)(n+2r-1)k^2A_{2r-2} \quad (r \geqslant 1).$$

First of all, let n be an *even* integer, say $n=2N$. Then the set of recurrence-relations can be satisfied in two distinct ways:

(i) Let $A_{2N+2}=0$. Taking the relation in which $r=N+1$, we see that the coefficient of A_{2N} is zero, and conclude that $A_{2N+4}=0$. Then taking in turn relations in which $r=N+2$, $N+3$, . . ., we see that $A_{2r}=0$ whenever $r \geqslant N+1$; thus (2.1) reduces to a polynomial of degree $2N$ (or n) in sn u. There remain $N+1$ homogeneous equations between the $N+1$ coefficients A_0, . . ., A_{2N}, whose elimination leads to a determinantal equation or to the equivalent continued fraction relation

$$h=\frac{\frac{1}{2}n(n+1)k^2}{1+k^2-\frac{1}{4}h+}\;\frac{3(n-2)(n+3)k^2/2^2.4}{1+k^2-\frac{1}{16}h}\;+\;\frac{5(n-4)(n+5)k^2/4^2.6}{1+k^2-\frac{1}{36}h}\;+\;.\;.\;.\;, \qquad (2.2)$$

in which the right-hand member terminates at the Nth component fraction. To the roots of this equation of degree $N+1$ in h correspond $N+1$ Lamé polynomials of type (2.1) which have respectively 0, 1, . . ., N zeros in the open interval (0, K).

(ii) Let $A_{2N+2}\neq 0$. Then the first $N+1$ relations determine A_{2N}, A_{2N-2}, . . ., A_0 linearly in terms of A_{2N+2}. In the other direction, the set of relations for $r \geqslant N+1$ does not terminate, but we have

$$\frac{A_{2r}}{A_{2r-2}}=-\frac{(n-2r+2)(n+2r-1)k^2/4r^2}{1+k^2-h/4r^2}$$
$$+\;\frac{(2r+1)(2r+2)(n-2r)(n+2r+1)k^2/16r^2(r+1)^2}{1+k^2-h/4(r+1)^2}\;+\;.\;.\;.\;, \qquad (2.3)$$

where the right-hand member is an infinite continued fraction which converges when $k^2 < 1$, and in that case

$$A_{2r}/A_{2r-2} \to k^2 \quad \text{as} \quad r \to \infty$$

(Perron, *Die Lehre von den Kettenbrüchen*, §§ 56, 57).

The series (2.1) is therefore absolutely and uniformly convergent throughout any region in which $| k \operatorname{sn} u | < 1$, and in particular in a band of finite breadth spread along the entire real axis in the u-plane. By equating the continued fraction for A_{2N+4}/A_{2N+2} to the value of the latter given by the recurrence-relation with $r = N+1$, we obtain the equation for h_1:

$$h = 4(N+1)^2(1+k^2) - \frac{(2N+3)(4N+3)k^2/(N+2)}{1+k^2-h/(2N+4)^2} - \dots$$

The roots of this equation provide the remainder of the infinite set of characteristic values of h, whose existence is a consequence of the Sturmian theory of equations with periodic boundary-conditions.

On the other hand, when n is other than an even integer, the recurrence-relations still hold, but no coefficient of any A will vanish. The characteristic numbers are all determined by the one equation (2.2), in which the continued fraction does not terminate. The corresponding periodic solutions are of the form (2.1), convergent when $| k \operatorname{sn} u | < 1$.

When $n \to \infty$, $k \to 0$, this type of periodic Lamé function degenerates into the even elliptic cylinder (Mathieu) function of period π denoted by $\operatorname{ce}_{2m}(u, \theta)$, where, in the notation of the author's paper in these *Proceedings* (vol. lii, 1931–32, p. 355), $\lim n(n+1)k^2 = -4\theta$. To preserve the analogy, the Lamé function that reduces to $\operatorname{ce}_{2m}(u, \theta)$ will be denoted by $\operatorname{Ec}_n^{2m}(u, k^2)$, where $2m$ is the number of zeros in the interval $(0, 2K)$. The corresponding characteristic value of h will be written $a_n^{2m}(k^2)$. Usually k has a pre-determined value and may be omitted from the symbols.

3. SOLUTION OF REAL PERIOD $2K$ EXPRESSED AS A SERIES OF EVEN POWERS OF sn u WITH THE FACTOR dn u.

The series

$$y = \operatorname{dn} u \{ C_0 + \sum_r C_{2r} \operatorname{sn}^{2r} u \} \quad . \quad . \quad . \quad . \quad (3.1)$$

formally satisfies (1.1) subject to the observance of the recurrence-relations

$$1.2C_2 + (h - k^2)C_0 = 0,$$

$$(2r+1)(2r+2)C_{2r+2} + \{h - 4r^2 - (2r+1)^2k^2\}C_{2r} = (n-2r+1)(n+2r)k^2C_{2r-2} \quad (r \geqslant 1).$$

Polynomial solutions are obtained by taking n to be an *odd* integer $2N+1$, and assuming $C_{2N+2}=0$. The corresponding characteristic equation is

$$h=k^2+\frac{\frac{1}{2}(n-1)(n+2)k^2}{1+\frac{9}{4}k^2-\frac{1}{4}h}\;+\;\frac{3(n-3)(n+4)k^2/2^2.4}{1+\frac{25}{16}k^2-\frac{1}{16}h}\;+\;\frac{5(n-5)(n+6)k^2/4^2.6}{1+\frac{49}{36}k^2-\frac{1}{36}h}\;+\cdots \tag{3.2}$$

The other cases ($C_{2N+2}\neq 0$, and n not an odd integer) may be treated as in § 2, but, in view of what is to follow, such a discussion is of no interest or value.

Any solution of the form (3.1) is identical with one of the form (2.1), for let

$$\operatorname{dn} u=\sqrt{(1-k^2\operatorname{sn}^2 u)}=1+\sum_r a_r\operatorname{sn}^{2r} u, \quad . \quad . \quad . \tag{3.3}$$

where $a_2=-\frac{1}{2}k^2$, $a_4=-\frac{1}{8}k^4$, etc. Then since both the series (3.1) and (3.3) are absolutely convergent when $|k\operatorname{sn} u|<1$, the right-hand member of (3.1), when multiplied out and arranged in the form

$$A_0+\sum_r A_{2r}\operatorname{sn}^{2r} u, \quad . \quad . \quad . \quad . \quad . \tag{3.4}$$

where

$$A_{2r}=C_{2r}+\sum_{s=1}^{r} a_{2s}C_{2r-2s},$$

is likewise absolutely convergent. But (3.4) is a solution of (1.1); its coefficients must therefore satisfy the recurrence-relations of § 2, and it itself is a Lamé function of the set discussed in that section. In other words, the sets of functions (2.1) and (3.1) are identical, apart from a multiplying constant, and the characteristic equations (2.2) and (3.2) have the same roots, whatever n may be. Thus all functions of type (3.1), and in particular the Lamé polynomials of that type, are included in the symbol $\mathrm{Ec}_n^{2m}(u)$.

As illustrations of the notation, applied to the polynomials, we may quote

$$\mathrm{Ec}_0^0(u)=1, \qquad \mathrm{Ec}_1^0(u)=\operatorname{dn} u,$$
$$\mathrm{Ec}_2^0(u)=1-\{1+k^2-\sqrt{(1-k^2+k^4)}\}\operatorname{sn}^2 u,$$
$$\mathrm{Ec}_2^2(u)=1-\{1+k^2+\sqrt{(1-k^2+k^4)}\}\operatorname{sn}^2 u,$$
$$\mathrm{Ec}_3^0(u)=\operatorname{dn} u[1-\{1+2k^2-\sqrt{(1-k^2+4k^4)}\}\operatorname{sn}^2 u],$$
$$\mathrm{Ec}_3^2(u)=\operatorname{dn} u[1-\{1+2k^2+\sqrt{(1-k^2+4k^4)}\}\operatorname{sn}^2 u].$$

In these and later examples the initial constants have been chosen so as to simplify the expressions. These are not to be regarded as standard forms, but as constant multiples of them. The question of standardisation will be deferred to another occasion.

4. EVEN SOLUTIONS OF REAL PERIOD $4K$.

Two forms are admissible. In the first place, if

$$y = \operatorname{cn} u\{A_0 + \sum_r A_{2r} \operatorname{sn}^{2r} u\}, \quad . \quad . \quad . \quad (4.1)$$

the recurrence-relations are

$$1.2A_2 + (h-1)A_0 = 0,$$

$$(2r+1)(2r+2)A_{2r+2} + \{h - (2r+1)^2 - 4r^2k^2\}A_{2r} = (n-2r+1)(n+2r)k^2A_{2r-2} \quad (r \geqslant 1).$$

These lead to the characteristic equation

$$h = 1 + \frac{\tfrac{1}{2}(n-1)(n+2)k^2}{\tfrac{9}{4} + k^2 - \tfrac{1}{4}h} + \frac{3(n-3)(n+4)k^2/2^2.4}{\tfrac{25}{16} + k^2 - \tfrac{1}{16}h} + \frac{5(n-5)(n+6)k^2/4^2.6}{\tfrac{49}{36} + k^2 - \tfrac{1}{36}h} + \ldots \quad (4.2)$$

The continued fraction terminates, and (4.1) reduces to a polynomial, when n is an odd integer $2N+1$ and A_{2N+2} is taken to be zero. Otherwise the solution remains as an infinite series, convergent when $|k \operatorname{sn} u| < 1$; a complete discussion would follow the lines of § 2.

The zeros of (4.1) within the interval $(0, 2K)$ consist of the zero of $\operatorname{cn} u$ at $u = K$ and an even number of zeros belonging to the second factor. If there are $2m+1$ zeros in all, the function will be denoted by $\mathrm{Ec}_n^{2m+1}(u)$, and the corresponding value of h by a_n^{2m+1}.

In the second place,

$$y = \operatorname{cn} u \operatorname{dn} u\{C_0 + \sum_r C_{2r} \operatorname{sn}^{2r} u\} \quad . \quad . \quad . \quad (4.3)$$

will be a solution of (1.1) if

$$1.2C_2 + (h - 1 - k^2)C_0 = 0,$$

$$(2r+1)(2r+2)C_{2r+2} + \{h - (2r+1)^2(1+k^2)\}C_{2r} = (n-2r)(n+2r+1)k^2C_{2r-2} \quad (r \geqslant 1).$$

The characteristic equation now is

$$h = 1 + k^2 + \frac{\tfrac{1}{2}(n-2)(n+3)k^2}{\tfrac{9}{4}(1+k^2) - \tfrac{1}{4}h} + \frac{3(n-4)(n+5)k^2/2^2.4}{\tfrac{25}{16}(1+k^2) - \tfrac{1}{16}h}$$
$$+ \frac{5(n-6)(n+7)k^2/4^2.6}{\tfrac{49}{36}(1+k^2) - \tfrac{1}{36}h} + \ldots \quad (4.4)$$

Thus (4.3) can terminate only when n is an even integer. It can, however, be proved, as in § 3, that (4.1) and (4.3) are expressions of the same function for the same value of h, and therefore functions of the form (4.3) can be included in the symbol $\mathrm{Ec}_n^{2m+1}(u)$. For instance,

$$\mathrm{Ec}_1^1(u) = \operatorname{cn} u, \qquad \mathrm{Ec}_2^1(u) = \operatorname{cn} u \operatorname{dn} u,$$
$$\mathrm{Ec}_3^1(u) = \operatorname{cn} u[1 - \{2 + k^2 - \sqrt{(4 - k^2 + k^4)}\} \operatorname{sn}^2 u],$$
$$\mathrm{Ec}_3^3(u) = \operatorname{cn} u[1 - \{2 + k^2 + \sqrt{(4 - k^2 + k^4)}\} \operatorname{sn}^2 u],$$
$$\mathrm{Ec}_4^1(u) = \operatorname{cn} u \operatorname{dn} u[1 - \{2 + 2k^2 - \sqrt{(4 + k^2 + 4k^4)}\} \operatorname{sn}^2 u],$$
$$\mathrm{Ec}_4^3(u) = \operatorname{cn} u \operatorname{dn} u[1 - \{2 + 2k^2 + \sqrt{(4 + k^2 + 4k^4)}\} \operatorname{sn}^2 u].$$

5. ODD SOLUTIONS OF REAL PERIOD $4K$.

The first form that a solution of this type may assume is

$$y = \sum_r B_{2r+1} \operatorname{sn}^{2r+1} u, \qquad (5.1)$$

subject to the recurrence-relations

$$2.3B_3 + (h - 1 - k^2)B_1 = 0,$$

$$(2r+2)(2r+3)B_{2r+3} + \{h - (2r+1)^2(1+k^2)\}B_{2r+1} = (n - 2r + 1)(n + 2r)k^2 B_{2r-1} \quad (r \geqslant 1),$$

which, in turn, lead to the characteristic equation

$$h = 1 + k^2 + \frac{\frac{3}{8}(n-1)(n+2)k^2}{\frac{9}{16}(1+k^2) - \frac{1}{16}h +} \; \frac{5(n-3)(n+4)k^2/4.6^2}{\frac{25}{36}(1+k^2) - \frac{1}{36}h} \; + \frac{7(n-5)(n+6)k^2/6.8^2}{\frac{49}{64}(1+k^2) - \frac{1}{64}h} + \dots \qquad (5.2)$$

The second form is

$$y = \operatorname{dn} u \sum_r D_{2r+1} \operatorname{sn}^{2r+1} u, \qquad (5.3)$$

in which case the recurrence-relations and characteristic equation are

$$2.3D_3 + (h - 1 - 4k^2)D_1 = 0,$$

$$(2r+2)(2r+3)D_{2r+3} + \{h - (2r+1)^2 - (2r+2)^2k^2\}D_{2r+1} = (n - 2r)(n + 2r + 1)k^2 D_{2r-1} \quad (r \geqslant 1),$$

$$h = 1 + 4k^2 + \frac{\frac{3}{8}(n-2)(n+3)k^2}{\frac{9}{16} + k^2 - \frac{1}{16}h} + \; \frac{5(n-4)(n+5)k^2/4.6^2}{\frac{25}{36} + k^2 - \frac{1}{36}h} \; + \frac{7(n-6)(n+7)k^2/6.8^2}{\frac{49}{64} + k^2 - \frac{1}{64}h} + \dots \qquad (5.4)$$

The two forms are equivalent, and the function they represent will be denoted by $\operatorname{Es}_n^{2m+1}(u)$, where $2m+1$ is the number of zeros in the interval $0 \leqslant u < 2K$. The corresponding value of h will be written b_n^{2m+1}. Polynomials of form (5.1) occur when n is odd, and of form (5.3) when n is even. Thus

$$\operatorname{Es}_1^1(u) = \operatorname{sn} u, \qquad \operatorname{Es}_2^1(u) = \operatorname{sn} u \operatorname{dn} u,$$

$$\operatorname{Es}_3^1(u) = 3 \operatorname{sn} u - \{2 + 2k^2 - \sqrt{(4 - 7k^2 + 4k^4)}\} \operatorname{sn}^3 u,$$

$$\operatorname{Es}_3^3(u) = 3 \operatorname{sn} u - \{2 + 2k^2 + \sqrt{(4 - 7k^2 + 4k^4)}\} \operatorname{sn}^3 u,$$

$$\operatorname{Es}_4^1(u) = \operatorname{dn} u[3 \operatorname{sn} u - \{2 + 3k^2 - \sqrt{(4 - 9k^2 + 9k^4)}\} \operatorname{sn}^3 u],$$

$$\operatorname{Es}_4^3(u) = \operatorname{dn} u[3 \operatorname{sn} u - \{2 + 3k^2 + \sqrt{(4 - 9k^2 + 9k^4)}\} \operatorname{sn}^3 u].$$

6. ODD SOLUTIONS OF REAL PERIOD $2K$.

In this case the first form is

$$y = \operatorname{cn} u \sum_r B_{2r+1} \operatorname{sn}^{2r+1} u, \quad . \quad . \quad . \quad . \quad . \qquad (6.1)$$

and the corresponding recurrence-relations and characteristic equation are

$$2.3B_3 + (h - 4 - k^2)B_1 = 0,$$

$$(2r+2)(2r+3)B_{2r+3} + \{h - (2r+2)^2 - (2r+1)^2k^2\}B_{2r+1}$$
$$= (n-2r)(n+2r+1)k^2B_{2r-1} \quad (r \geqslant 1),$$

$$h = 4 + k^2 + \frac{\frac{3}{8}(n-2)(n+3)k^2}{1 + \frac{9}{16}k^2 - \frac{1}{16}h\ +}\ \frac{5(n-4)(n+5)k^2/4.6^2}{1 + \frac{25}{36}k^2 - \frac{1}{36}h}$$
$$+\ \frac{7(n-6)(n+7)k^2/6.8^2}{1 + \frac{49}{64}k^2 - \frac{1}{64}h}\ + \ldots \qquad (6.2)$$

The second form with its recurrence-relations and characteristic equation are

$$y = \operatorname{cn} u \operatorname{dn} u \sum_r D_{2r+1} \operatorname{sn}^{2r+1} u, \quad . \quad . \quad . \quad . \quad . \qquad (6.3)$$

$$2.3D_3 + (h - 4 - 4k^2)D_1 = 0,$$

$$(2r+2)(2r+3)D_{2r+3} + \{h - (2r+2)^2(1+k^2)\}D_{2r+1}$$
$$= (n-2r-1)(n+2r+2)k^2D_{2r-1} \quad (r \geqslant 1),$$

$$h = 4 + 4k^2 + \frac{\frac{3}{8}(n-3)(n+4)k^2}{1 + k^2 - \frac{1}{16}h\ +}\ \frac{5(n-5)(n+6)k^2/4.6^2}{1 + k^2 - \frac{1}{36}h}$$
$$+\ \frac{7(n-7)(n+8)k^2/6.8^2}{1 + k^2 - \frac{1}{64}h}\ + \ldots \qquad (6.4)$$

The function represented by these two equivalent forms will be denoted by $\mathrm{Es}_n^{2m+2}(u)$, where $2m+2$ is the number of zeros in the interval $0 \leqslant u < 2K$. The corresponding value of h will be written b_n^{2m+2}. Polynomials of types (6.1) and (6.2) occur when n is even and odd respectively. Thus

$$\mathrm{Es}_2^2(u) = \operatorname{sn} u \operatorname{cn} u, \qquad \mathrm{Es}_3^2(u) = \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u.$$
$$\mathrm{Es}_4^2(u) = \operatorname{cn} u[3 \operatorname{sn} u - \{3 + 2k^2 - \sqrt{(9 - 9k^2 + 4k^4)}\} \operatorname{sn}^3 u],$$
$$\mathrm{Es}_4^4(u) = \operatorname{cn} u[3 \operatorname{sn} u - \{3 + 2k^2 + \sqrt{(9 - 9k^2 + 4k^4)}\} \operatorname{sn}^3 u],$$
$$\mathrm{Es}_5^2(u) = \operatorname{cn} u \operatorname{dn} u[\operatorname{sn} u - \{1 + k^2 - \sqrt{(1 - k^2 + k^4)}\} \operatorname{sn}^3 u],$$
$$\mathrm{Es}_5^4(u) = \operatorname{cn} u \operatorname{dn} u[\operatorname{sn} u - \{1 + k^2 + \sqrt{(1 - k^2 + k^4)}\} \operatorname{sn}^3 u].$$

To sum up, *Lamé functions of periods 2K and 4K exist for every real value of n. They may be represented by the symbol* $\mathrm{Ec}_n^{\nu}(u)$ *when even, and by* $\mathrm{Es}_n^{\nu}(u)$ *when odd, where ν, which is necessarily a positive integer, denotes the number of zeros in the interval* $0 \leqslant u < 2K$.

The functions are expressible in polynomial form when n is a positive integer, and ν has any of the values 0 to n when the function is even, or 1 to n when the function is odd.

The continued fractions that form one member of the characteristic equations may be thrown into other forms, but those given in this and the preceding sections were the ones generally used in actual computation.

7. TABLES OF CHARACTERISTIC VALUES.

The tables that follow contain values to four decimal places of the characteristic functions $a_n^\nu(k^2)$ and $b_n^{\nu+1}(k^2)$, for $\nu=0, 1, 2, 3$; for all integral values of n from 0 to 25, together with the minimum values of the functions

TABLE I.—CHARACTERISTIC NUMBERS FOR $k^2=0·1$.

n	a_n^0	a_n^1	b_n^1	b_n^2	a_n^2	a_n^3	b_n^3	b_n^4
$-\frac{1}{2}$	−0·0127	0·9426	0·9301	3·7833	3·7835	8·5285	8·5285	15·1715
0	0·0000	0·9490	0·9490	3·7960	3·7960	8·5411	8·5411	15·1842
1	0·1000	1·0000	1·1000	3·8978	3·8978	8·6426	8·6426	15·2856
2	0·2921	1·1000	1·4000	4·1000	4·1079	8·8465	8·8465	15·4889
3	0·5609	1·2453	1·8449	4·4000	4·4391	9·1547	9·1551	15·7953
4	0·8849	1·4306	2·4279	4·7939	4·9089	9·5694	9·5721	16·2061
5	1·2402	1·6500	3·1392	5·2764	5·5360	10·0926	10·1034	16·7236
6	1·6060	1·8970	3·9650	5·8411	6·3344	10·7259	10·7579	17·3502
7	1·9684	2·1654	4·8870	6·4804	7·3094	11·4694	11·5477	18·0888
8	2·3216	2·4499	5·8825	7·1861	8·4568	12·3217	12·4890	18·9420
9	2·6646	2·7457	6·9254	7·9490	9·7646	13·2797	13·5994	19·9127
10	2·9993	3·0494	7·9896	8·7599	11·2153	14·3379	14·8960	21·0031
11	3·3278	3·3582	9·0537	9·6099	12·7875	15·4897	16·3896	22·2138
12	3·6521	3·6703	10·1037	10·4906	14·4560	16·7264	18·0820	23·5449
13	3·9737	3·9844	11·1339	11·3944	16·1924	18·0385	19·9649	24·9940
14	4·2934	4·2997	12·1444	12·3152	17·9667	19·4158	22·0215	26·5574
15	4·6121	4·6157	13·1382	13·2479	19·7512	20·8480	24·2291	28·2290
16	4·9300	4·9321	14·1194	14·1886	21·5236	22·3247	26·5606	30·0008
17	5·2475	5·2487	15·0915	15·1345	23·2702	23·8367	28·9856	31·8638
18	5·5647	5·5654	16·0572	16·0836	24·9862	25·3755	31·4712	33·8076
19	5·8817	5·8821	17·0185	17·0346	26·6729	26·9340	33·9850	35·8210
20	6·1985	6·1988	17·9768	17·9863	28·3346	28·5064	36·4967	37·8931
21	6·5153	6·5154	18·9330	18·9386	29·9770	30·0880	38·9842	40·0128
22	6·8320	6·8321	19·8878	19·8912	31·6049	31·6756	41·4347	42·1700
23	7·1486	7·1487	20·8415	20·8436	33·2221	33·2666	43·8434	44·3560
24	7·4652	7·4653	21·7945	21·7957	34·8319	34·8594	46·2136	46·5623
25	7·7818	7·7818	22·7470	22·7477	36·4357	36·4530	48·5504	48·7842

at $n=-\frac{1}{2}$; and for $k^2=0{\cdot}1$, 0·5, and 0·9. The method of computation is similar to that for the characteristic numbers of the elliptic-cylinder functions previously detailed in these *Proceedings* (vol. lii, 1932, p. 359). When the continued fraction in a characteristic equation terminates, the process is straightforward; but in other cases, owing to the fact that the convergence of A_{2r}/A_{2r-2} in (2.3), and similar ratios, to the limiting value k^2 is very slow when k is in the upper part of the range (0, 1), the work is very much more laborious than in dealing with the elliptic-cylinder functions. In fact, it would have been impracticable when $k^2=0{\cdot}9$ without the use of a second approximation to the tail of the continued fraction in (2.2), *i.e.* to that part which begins with the rth component fraction. Since the tail converges to $-k^2$, let the second approximation

TABLE II.—CHARACTERISTIC NUMBERS FOR $k^2=0{\cdot}5$.

n	a_n^0	a_n^1	b_n^1	b_n^2	a_n^2	a_n^3	b_n^3	b_n^4
$-\frac{1}{2}$	−0·0686	0·6808	0·6187	2·8004	2·8072	6·3923	6·3920	11·4167
0	0·0000	0·7178	0·7178	2·8711	2·8711	6·4599	6·4599	11·4843
1	0·5000	1·0000	1·5000	3·4284	3·4284	7·0084	7·0084	12·0302
2	1·2679	1·5000	3·0000	4·5000	4·7321	8·1373	8·1373	13·1392
3	2·0505	2·1270	5·0505	6·0000	6·9495	9·8730	9·9495	14·8455
4	2·7889	2·8096	7·3542	7·8096	10·0000	12·1904	12·6458	17·1904
5	3·5059	3·5109	9·6370	9·8038	13·6312	15·0000	16·3688	20·1962
6	4·2161	4·2172	11·8348	11·8857	17·4928	18·1684	21·0000	23·8316
7	4·9244	4·9247	13·9861	14·0000	21·2920	21·5557	26·2302	28·0000
8	5·6322		16·1200	16·1235	24·9640	25·0507	31·6656	32·5594
9	6·3397		18·2479	18·2487	28·5615	28·5871	36·9954	37·3602
10	7·0471		20·3732	20·3734	32·1278	32·1348	42·1552	42·2820
11	7·7545		22·4976		35·6815	35·6833	47·2109	47·2505
12	8·4618		24·6212		39·2296	39·2300	52·2190	52·2304
13	9·1691		26·7444		42·7746	42·7747	57·2060	57·2091
14	9·8763		28·8673		46·3177	46·3178	62·1826	62·1834
15	10·5836		30·9900		49·8596		67·1534	67·1536
16	11·2908		33·1124		53·4003		72·1202	
17	11·9980		35·2347		56·9403		77·0841	
18	12·7052		37·3569		60·4797		82·0459	
19	13·4123		39·4789		64·0186		87·0061	
20	14·1195		41·6009		67·5571		91·9648	
21	14·8267		43·7228		71·0952		96·9225	
22	15·5338		45·8447		74·6331		101·8793	
23	16·2410		47·9665		78·1708		106·8354	
24	16·9481		50·0882		81·7082		111·7907	
25	17·6553		52·2100		85·2455		116·7454	

be $-k^2+a_r$. Then

$$-k^2+a_r=\frac{(2r-1)(n-2r+2)(n+2r-1)k^2/(2r-2)^2.2r}{1+k^2-h/4r^2-k^2+a_{r+1}}.$$

Then, neglecting 3rd and higher powers of $1/r$,

$$-k^2+a_r=-\frac{\{1+(1-n-n^2)/4r^2\}k^2}{1-h/4r^2+a_{r+1}}$$

or

$$a_r=k^2a_{r+1}-(h+1-n-n^2)k^2/4r^2, \qquad . \quad . \quad . \quad (7.1)$$

and thus

$$a_r=-(h+1-n-n^2)\left\{\frac{k^2}{4r^2}+\frac{k^4}{4(r+1)^2}+\frac{k^6}{4(r+2)^2}+\ .\ .\ .\right\}.$$

TABLE III.—CHARACTERISTIC NUMBERS FOR $k^2=0·9$.

n	a_n^0	a_n^1	b_n^1	b_n^2	a_n^2	a_n^3	b_n^3	b_n^4
$-\frac{1}{2}$	−0·1467	0·2798	0·1748	1·3269	1·3596	3·2023	3·1951	5·7964
0	0·0000	0·3712	0·3712	1·4849	1·4849	3·3411	3·3411	5·9397
1	0·9000	1·0000	1·9000	2·7061	2·7061	4·5156	4·5156	7·0998
2	1·8921	1·9000	4·6000	4·9000	5·7079	7·0321	7·0321	9·5135
3	2·8449	2·8453	7·5609	7·6000	10·1551	10·7547	11·4391	13·3546
4	3·7938	3·7939	10·4279	10·4306	15·0911	15·2061	17·5721	18·5694
5	4·7426		13·2762	13·2764	19·8966	19·9074	24·4640	24·7236
6	5·6913		16·1228		24·6490	24·6498	31·2422	31·2741
7	6·6400		18·9692		29·3948		37·9086	37·9112
8	7·5887		21·8154		34·1394		44·5560	44·5562
9	8·5374		24·6616		38·8836		51·2001	
10	9·4861		27·5078		43·6276		57·8430	
11	10·4348		30·3539		48·3714		64·4852	
12	11·3835		33·2000		53·1151		71·1271	
13	12·3322		36·0461		57·8588		77·7688	
14	13·2809		38·8922		62·6024		84·4103	
15	14·2295		41·7383		67·3460		91·0516	
16	15·1782		44·5844		72·0896		97·6928	
17	16·1269		47·4305		76·8331		104·3340	
18	17·0756		50·2766		81·5766		110·9751	
19	18·0243		53·1226		86·3202		117·6162	
20	18·9730		55·9687		91·0637		124·2572	
21	19·9217		58·8148		95·8072		130·8983	
22	20·8703		61·6608		100·5508		137·5393	
23	21·8190		64·5069		105·2943		144·1802	
24	22·7677		67·3529		110·0378		150·8212	
25	23·7164		70·1990		114·7813		157·4621	

Upper and lower bounds to the above series are obtained in the usual way, and finally

$$a_r = -\frac{(h+1-n-n^2)k^2}{4r^2}\left\{\frac{\theta_r}{1-k^2}+\frac{1-\theta_r}{1-r^2k^2/(r+1)^2}\right\}, \quad . \quad . \quad (7.2)$$

where $0 < \theta_r < 1$.

By substituting (7.2) in (7.1) an equation for θ_r is found; for sufficiently large values of r, θ_r may be taken equal to $\frac{k^2}{(1-k^2)r}$. Similar approximations were used in the other cases.

Even with these refinements it was found necessary, for any reliance to be put on the fourth decimal place, to go as far as the 36th component fraction when $k^2=0{\cdot}9$.

When the continued fraction terminates, the characteristic equation may be transformed into an algebraic equation in h, whose coefficients are polynomials in k^2. In general such equations appear to be irreducible, but for particular values of k they may be reducible, and when $k^2=0{\cdot}5$ there are four remarkable cases in which the characteristic numbers are integers.

Thus the equation whose roots are a_4^0, a_4^2, and a_4^4 is, for general values of k,

$$h^3-20(1+k^2)h^2+16(4+21k^2+4k^4)h-640k^2(1+k^2)=0.$$

When $k^2=0{\cdot}5$ it becomes

$$(h-10)(h^2-20h+48)=0,$$

whence $a_4^2(0{\cdot}5)=10$. Two similar instances occur in $a_5^3(0{\cdot}5)=15$, $b_6^2(0{\cdot}5)=21$. More remarkable still is the equation for b_7^ν ($\nu=2, 4, 6$), which has three integral roots when $k^2=0{\cdot}5$, namely

$$b_7^2(0{\cdot}5)=14, \qquad b_7^4(0{\cdot}5)=28, \qquad b_7^6(0{\cdot}5)=42.$$

The values for $n=0$ were obtained from the formulæ

$$a_0^p=b_0^p=\frac{p^2\pi^2}{4K^2}.$$

The most striking feature of the characteristic numbers is that, with increasing n, they tend to equality in pairs, and that the first difference in each column tends to become constant. The nearer k^2 approaches the upper end of the interval (0, 1) the more pronounced does this feature become. Thus interpolation for non-integral values of n is simple when n is large, but is practically impossible in the lower ranges.

8. Asymptotic Expression for h.

In (1.1) write

$$\kappa^2 = n(n+1)k^2,$$

then it may be proved by an application of the First Comparison Theorem of Sturm (cf. *Journ. Lond. Math. Soc.*, vol. ii, 1926, pp. 48–50) that, for large positive κ,

$$a_n^{2m} \sim a_n^{2m+1} \sim (4m+1)\kappa; \qquad b_n^{2m+1} \sim b_n^{2m+2} \sim (4m+3)\kappa,$$

and thus, in either case,

$$h \sim (2p+1)\kappa,$$

where p is a positive integer.

If (1.1) is transformed by the substitution

$$y = ve^{\alpha\int \operatorname{sn} u\, du} = v\left(\frac{d-kc}{d+kc}\right)^{\alpha/2k}, \quad . \quad . \quad . \quad . \quad (8.1)$$

it becomes

$$v'' + 2\alpha s v' + \{h + \alpha c d + (\alpha^2 - \kappa^2)s^2\}v = 0, \quad . \quad . \quad . \quad (8.2)$$

where s, c, d are written for sn u, cn u, dn u respectively.

This equation may be formally satisfied by

$$\begin{aligned} h &= \kappa(2p + 1 + a_1\kappa^{-1} + a_2\kappa^{-2} + \ . \ . \ .), \\ v &= v_0 + v_1\kappa^{-1} + v_2\kappa^{-2} + \ . \ . \ ., \\ \alpha &= \kappa(1 + b_2\kappa^{-2} + \ . \ . \ .). \end{aligned}$$

In fact v_0, v_1, v_2, . . . satisfy the succession of linear equations of the first order:

$$\begin{aligned} &2sv_0' + (2p+1+cd)v_0 = 0, \\ &2sv_1' + (2p+1+cd)v_1 = -(v_0'' + a_1v_0 + 2b_2s^2v_0), \\ &2sv_2' + (2p+1+cd)v_2 = -(v_1'' + a_1v_1 + 2b_2s^2v_1 + 2b_2sv_0' + a_2v_0 + b_2cdv_0), \end{aligned}$$

which have the common integrating factor

$$s^{-\frac{1}{2}}e^{(p+\frac{1}{2})\int s^{-1}du} = s^{-\frac{1}{2}}\left(\frac{d-c}{d+c}\right)^{\frac{2p+1}{4}}.$$

The first of these equations has the solution

$$v_0 = s^{-\frac{1}{2}}\left(\frac{d-c}{d+c}\right)^{-\frac{2p+1}{4}};$$

when this value is substituted in the equation for v_1, and the integrating factor is introduced, that equation becomes

$$\frac{d}{du}\left\{2s^{\frac{1}{2}}\left(\frac{d-c}{d+c}\right)^{\frac{2p+1}{4}} v_1\right\} = -\left\{\frac{p^2+p+1}{s^3} + \frac{(2p+1)cd}{s^2} + \frac{a_1 - \frac{1}{2}(1+k^2)}{s} + (2b_2 - \tfrac{1}{2}k^2)s\right\}.$$

Using the formulæ

$$\int\frac{du}{s^3}=\frac{1+k^2}{4}\log\left(\frac{d-c}{d+c}\right)-\frac{cd}{2s^2},$$

$$\int\frac{du}{s}=\tfrac{1}{2}\log\left(\frac{d-c}{d+c}\right);\qquad \int s\,du=\frac{1}{2k}\log\left(\frac{d-kc}{d+kc}\right),$$

we choose the constants a_1, b_2, so that no logarithmic terms remain on integrating the right-hand member of this equation. Thus

$$a_1=-\tfrac{1}{4}(2p^2+2p+1)(1+k^2),\qquad b_2=\tfrac{1}{8}k^2,$$

$$v_1=v_0\left\{\frac{2p+1}{4s^2}+\frac{(p^2+p+1)cd}{4s^2}\right\}.$$

When these values are introduced into the equation for v_2, and the integrating factor supplied, that equation becomes

$$\frac{d}{du}\left\{2s^{\frac{1}{2}}\left(\frac{d-c}{d+c}\right)^{\frac{2p+1}{4}}v_2\right\}=-\left[\frac{(2p+1)(p^2+p+3)}{s^5}+\frac{\frac{3}{4}(7p^2+7p+4)cd}{s^5}\right.$$
$$-\frac{\frac{1}{8}(2p+1)(5p^2+5p+17)(1+k^2)}{s^3}-\frac{\frac{1}{8}(p^2+p+1)(p^2+p+5)(1+k^2)cd}{s^3}$$
$$\left.+\frac{\frac{1}{8}(2p+1)(2p^2+2p+9)k^2+a_2}{s}\right].$$

Integrating, with the aid of the relation

$$\int\frac{du}{s^5}=-\frac{cd}{4s^4}-\frac{3(1+k^2)cd}{8s^2}+\left\{\frac{3(1+k^2)^2}{16}-\frac{k^2}{4}\right\}\log\left(\frac{d-c}{d+c}\right),$$

and choosing a_2 so that the logarithmic term disappears, we obtain

$$a_2=\tfrac{1}{8}(2p+1)(2p^2+2p+3)k^2-\tfrac{1}{16}(2p+1)(p^2+p+1)(1+k^2)^2.$$

Thus, for large values of κ,

$$h\sim(2p+1)\kappa-\tfrac{1}{4}(2p^2+2p+1)(1+k^2)$$
$$+\tfrac{1}{16}(2p+1)\{2(2p^2+2p+3)k^2-(p^2+p+1)(1+k^2)^2\}\kappa^{-1}+\;.\;.\;.\qquad(8.3)$$

This formula gave $b^1_{25}(0{\cdot}9)$ correctly to within two units in the fifth decimal place (seventh significant figure). A consequence of (8.3) is the formula for the first difference,

$$\Delta h_n=h_{n+1}-h_n=(2p+1)k+O(n^{-2}),$$

for large n.

9. THE DEGENERATE CASE $k^2=1$.

When $k^2=1$, the real periods of the elliptic functions are infinite, and the functions themselves become hyperbolic functions, thus:

$$\operatorname{sn}(u, 1)=\tanh u, \qquad \operatorname{cn}(u, 1)=\operatorname{dn}(u, 1)=\operatorname{sech} u,$$

and (1.1) is replaced by

$$\frac{d^2y}{du^2}=\{n(n+1)\tanh^2 u-h\}y. \qquad . \qquad . \qquad . \qquad . \qquad (9.1)$$

It is evident from the recurrence-relations that (3.1) becomes identical with (4.1), and (5.3) with (6.1). Therefore

$$\mathrm{Ec}_n^{2m}(u, 1)=\mathrm{Ec}_n^{2m+1}(u, 1); \qquad \mathrm{Es}_n^{2m+1}(u, 1)=\mathrm{Es}_n^{2m+2}(u, 1),$$

and, in consequence,

$$a_n^{2m}(1)=a_n^{2m+1}(1); \qquad b_n^{2m+1}(1)=b_n^{2m+2}(1).$$

If (9.1) is transformed by the substitutions

$$y=w\operatorname{sech}^\nu u, \qquad z=\tanh^2 u,$$

where $\nu^2=n(n+1)-h$, the resulting equation is

$$z(1-z)\frac{d^2w}{dz^2}+\{\tfrac{1}{2}-(\nu+\tfrac{3}{2})z\}\frac{dw}{dz}-\tfrac{1}{4}(\nu-n)(\nu+n+1)w=0.$$

Two distinct solutions of this hypergeometric equation are

$$\begin{aligned} w_1&=F\{\tfrac{1}{2}(\nu-n),\ \tfrac{1}{2}(\nu+n+1);\ \tfrac{1}{2};\ z\},\\ w_2&=z^{\frac{1}{2}}F\{\tfrac{1}{2}(\nu-n+1),\ \tfrac{1}{2}(\nu+n+2);\ \tfrac{3}{2};\ z\}.\end{aligned}$$

Of these, w_1 terminates when $n-\nu$ is an even positive integer $2m$; it then reduces to a polynomial of degree m in z which, by a theorem of van Vleck (*Trans. Amer. Math. Soc.*, vol. iii, 1902, p. 128), has precisely m zeros in the interval $0<z<1$. Likewise w_2 terminates when $n-\nu$ is an odd positive integer $2m+1$, and then is the product of $z^{\frac{1}{2}}$ and a polynomial of degree m which has m zeros in that interval.

Thus (9.1) has a terminating solution

$$y=\operatorname{sech}^{n-2m} u\, F(-m,\ n-m+\tfrac{1}{2};\ \tfrac{1}{2};\ \tanh^2 u)$$

when $h=(4m+1)n-4m^2$, or

$$y=\operatorname{sech}^{n-2m-1}u \tanh u\, F(-m,\ n-m+\tfrac{1}{2};\ \tfrac{3}{2};\ \tanh^2 u)$$

when $h=(4m+3)n-(2m+1)^2$. These solutions are both constant multiples of associated Legendre functions of $\tanh u$.

In brief, the Lamé functions of periods $2K$ and $4K$ degenerate in the limiting case $k^2=1$ as follows:—

$\mathrm{Ec}_n^{2m}(u, 1)$ and $\mathrm{Ec}_n^{2m+1}(u, 1)$ are constant multiples of $P_n^{n-2m}(\tanh u)$, with $a_n^{2m}(1)=a_n^{2m+1}(1)=(4m+1)n-4m^2$;

$\mathrm{Es}_n^{2m+1}(u, 1)$ and $\mathrm{Es}_n^{2m+2}(u, 1)$ are constant multiples of $P_n^{n-2m-1}(\tanh u)$, with $b_n^{2m+1}(1)=b_n^{2m+2}(1)=(4m+3)n-(2m+1)^2$.

The fact that the characteristic numbers are linear in n when $k^2=1$ supplements the result (8.3) that they are asymptotically linear when $k^2<1$.

10. INTEGRAL EQUATIONS SATISFIED BY PERIODIC LAMÉ FUNCTIONS.

It is a well-known fact that every Lamé polynomial is a solution of a set of homogeneous integral equations of the form

$$E(u)=\lambda\int_{-2K}^{2K} N(u, v)E(v)dv.$$

The following scheme specifies the Lamé polynomials that are the characteristic functions of each of four particular nuclei, in which P_n symbolises the ordinary Legendre polynomial, and $\mu=k\,\mathrm{sn}\,u\,\mathrm{sn}\,v$:—

	$N(u, v)$.	n even.	n odd.
I	$P_n(\mu)$	$\mathrm{Ec}_n^{2m}(u)$	$\mathrm{Es}_n^{2m+1}(u)$
II	$\mathrm{dn}\,u\,\mathrm{dn}\,v\,P_n'(\mu)$	$\mathrm{Es}_n^{2m+1}(u)$	$\mathrm{Ec}_n^{2m}(u)$
III	$\mathrm{cn}\,u\,\mathrm{cn}\,v\,P_n'(\mu)$	$\mathrm{Es}_n^{2m+2}(u)$	$\mathrm{Ec}_n^{2m+1}(u)$
IV	$\mathrm{cn}\,u\,\mathrm{dn}\,u\,\mathrm{cn}\,v\,\mathrm{dn}\,v\,P_n''(\mu)$	$\mathrm{Ec}_n^{2m+1}(u)$	$\mathrm{Es}_n^{2m+2}(u)$

I and IV are given, together with a proof of the main theorem, in Whittaker and Watson, *Modern Analysis* (1927), §§ 23.6, 23.61; II and III do not appear to have been published hitherto. Other nuclei are recorded in that work, but the above set is clearly complete in itself.

The proof depends firstly upon the fact that the nucleus considered is annihilated by the operator

$$\frac{\partial^2}{\partial u^2}-\frac{\partial^2}{\partial v^2}-n(n+1)k^2(\mathrm{sn}^2 u-\mathrm{sn}^2 v),$$

which is the case without restriction upon n, and secondly upon the fact that E, N, and their derivatives are one-valued functions, with the period $4K$ (or possibly $2K$). But when u, v are real and $k^2<1$, this second condition is satisfied for all values of n such that $n > -\frac{1}{2}$. Therefore when $n > -\frac{1}{2}$, but is not an integer, both $\mathrm{Ec}_n^{2m}(u)$ and $\mathrm{Es}_n^{2m+1}(u)$ satisfy the integral equation with $N(u, v)=P_n(k\,\mathrm{sn}\,u\,\mathrm{sn}\,v)$. A similar conclusion will hold in the other three cases.

It is, however, somewhat of a disadvantage to have two distinct types of function, the one even and the other odd, satisfying one and the same form of integral equation. We therefore proceed to modify the nuclei so as to construct new integral equations whose solutions are, for general values of n, either even or odd. We write

$$\mathbf{P}_n(\mu)=\tfrac{1}{2}\{P_n(\mu)+P_n(-\mu)\},$$
$$\mathbf{Q}_n(\mu)=\tfrac{1}{2}\{P_n(\mu)-P_n(-\mu)\},$$

and obtain the four pairs of nuclei which, with their characteristic functions, appear in the following scheme:—

$$\left.\begin{array}{c}\mathbf{P}_n(\mu)\\ \operatorname{dn} u \operatorname{dn} v\, \mathbf{Q}_n'(\mu)\end{array}\right\}\mathrm{Ec}_n^{2m}(u)$$
$$\left.\begin{array}{c}\mathbf{Q}_n(\mu)\\ \operatorname{dn} u \operatorname{dn} v\, \mathbf{P}_n'(\mu)\end{array}\right\}\mathrm{Es}_n^{2m+1}(u)$$
$$\left.\begin{array}{c}\operatorname{cn} u \operatorname{cn} v\, \mathbf{Q}_n'(\mu)\\ \operatorname{cn} u \operatorname{dn} u \operatorname{cn} v \operatorname{dn} v\, \mathbf{P}_n''(\mu)\end{array}\right\}\mathrm{Ec}_n^{2m+1}(u)$$
$$\left.\begin{array}{c}\operatorname{cn} u \operatorname{cn} v\, \mathbf{P}_n'(\mu)\\ \operatorname{cn} u \operatorname{dn} u \operatorname{cn} v \operatorname{dn} v\, \mathbf{Q}_n''(\mu)\end{array}\right\}\mathrm{Es}_n^{2m+2}(u)$$

In the limiting case in which $k\to 0$, $n\to\infty$ in such a way that $\lim kn=\kappa$, the Lamé functions become Mathieu functions, and $\mathbf{P}_n(\mu)$, $\operatorname{dn} u \operatorname{dn} v\, \mathbf{Q}_n'(\mu)$ both become constant multiples of $\cosh(\kappa \sin u \sin v)$, which is the nucleus of an integral equation whose solutions are the complete set of functions $\mathrm{ce}_{2m}(u)$. In fact, every nucleus whose characteristic functions are periodic Mathieu functions is the limiting case of each of a pair of nuclei associated with periodic Lamé functions.

11. SUMMARY

This paper proposes a basis for the theory of Lamé functions whose degree n is not necessarily an integer. It is shown that there exist Lamé functions of four types, namely (i) even, period $2K$, (ii) even, period $4K$, (iii) odd, period $4K$, (iv) odd, period $2K$, corresponding exactly to the four types of periodic Mathieu (elliptic-cylinder) functions. These functions are expressible as polynomials in the three Jacobian elliptic functions for integral values of n, but exist for all real values of n. Functions of these four types are denoted respectively by

$$\mathrm{Ec}_n^{2m}(u),\qquad \mathrm{Ec}_n^{2m+1}(u),\qquad \mathrm{Es}_n^{2m+1}(u),\qquad \mathrm{Es}_n^{2m+2}(u).$$

In the limiting (Mathieu) case, they become

$$\mathrm{ce}_{2m}(u),\qquad \mathrm{ce}_{2m+1}(u),\qquad \mathrm{se}_{2m+1}(u),\qquad \mathrm{se}_{2m+2}(u).$$

A periodic function exists when the parameter h has a characteristic value determined as the root of an equation, transcendental in general, but having an algebraic factor when n is integral. Tables of the eight lowest characteristic numbers are given when $k^2=0{\cdot}1$, $0{\cdot}5$, and $0{\cdot}9$, for $n=-\frac{1}{2}$, 0, and integral values up to 25. These suggest an asymptotic relation, for large n, between consecutive characteristic numbers; this relation is obtained. In the case $k^2=1$ the characteristic numbers are linear functions of n. Finally, it is shown that the integral equations, hitherto considered only when n is an integer, are valid also when n is any real number.

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VI.—The Estimation of Factor Loadings by the Method of Maximum Likelihood.

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1. WHEN a battery of intelligence tests is administered to a set of persons it is a common practice among psychologists to "explain" the scores obtained in terms of a number of "factors." Thus if we suppose that there are altogether t tests and that x_i denotes the score of any one person in the ith test, then it is assumed that

$$x_i = \lambda_i f + \mu_i g + \ldots + \nu_i h + \tau_i s_i, \qquad (i = 1, 2, \ldots, t),$$

where $f, g, \ldots, h$ represent the person's measures in one or more general or group factors, and s_i is the person's specific ability in the ith test. It is further assumed that for a given and supposed infinite population of persons all the factors, specific and otherwise, are distributed normally and independently and that they are standardised, *i.e.* that their standard deviations are unity. The numbers $\lambda_i, \mu_i, \ldots, \nu_i, \tau_i$ are the "factor loadings," and for any given test they are fixed quantities independent of the particular person tested.

The population variances and covariances $\{c_{ij}\}$ of the t tests are defined by the relation

$$c_{ij} = E\{(x_i - \xi_i)(x_j - \xi_j)\}, \qquad (i, j = 1, 2, \ldots, t),$$

where ξ_i is the population mean of x_i and the symbol E denotes the average taken over the whole population.

Since $f, g, \ldots, h, s_1, s_2, \ldots, s_t$ are independent normal variates, it follows that

$$c_{ij} = \lambda_i\lambda_j + \mu_i\mu_j + \ldots + \nu_i\nu_j, \qquad (i \neq j),$$
$$c_{ii} = \lambda_i^2 + \mu_i^2 + \ldots + \nu_i^2 + \tau_i^2.$$

The quantities $\lambda_i, \mu_i, \ldots, \nu_i, \tau_i$, and also the variances and covariances $\{c_{ij}\}$, which are functions of them, are termed the population parameters, since for a given population they are assumed to have definite fixed values.

Now in any practical case these population parameters are unknown, and must therefore be estimated from the data available. Various methods have been employed for doing this, many of which proceed from different theoretical bases. The object of this paper is to apply to the problem the method of estimation developed by Fisher (1921, 1925), and to attempt a comparison of the results obtained with those of some other methods.

2. Let us suppose that we have a sample of n persons to whom the t tests have been given. The scores obtained from these form a matrix

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ x_{t1} & x_{t2} & \cdots & x_{tn} \end{bmatrix},$$

where x_{ir} is the score obtained by the rth person in the ith test, and the quantities $x_{i1}, x_{i2}, \ldots, x_{in}$ may be regarded as a set of n independent observations of the normal variate x_i.

We shall define the sample variances and covariances $\{a_{ij}\}$ by the relation

$$a_{ij}=\frac{1}{(n-1)}\sum_{r=1}^{n}(x_{ir}-\bar{x}_i)(x_{jr}-\bar{x}_j), \qquad (i, j=1, 2, \ldots, t),$$

where

$$\bar{x}_i=\frac{1}{n}\sum_{r=1}^{n}(x_{ir}),$$

these values having been chosen so that $E(a_{ij})=c_{ij}$.

We begin by making some hypothesis as to the number of factors required. In what follows we shall for convenience assume the existence of only two general factors in addition to the specifics; the population covariance matrix C is then given by

$$C=\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1t} \\ c_{12} & c_{22} & \cdots & c_{2t} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ c_{1t} & c_{2t} & \cdots & c_{tt} \end{bmatrix}=\begin{bmatrix} \lambda_1^2+\mu_1^2+\tau_1^2 & \lambda_1\lambda_2+\mu_1\mu_2 & \cdots & \lambda_1\lambda_t+\mu_1\mu_t \\ \lambda_1\lambda_2+\mu_1\mu_2 & \lambda_2^2+\mu_2^2+\tau_2^2 & \cdots & \lambda_2\lambda_t+\mu_2\mu_t \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \lambda_1\lambda_t+\mu_1\mu_t & \lambda_2\lambda_t+\mu_2\mu_t & \cdots & \lambda_t^2+\mu_t^2+\tau_t^2 \end{bmatrix}.$$

The joint probability distribution of the sample variances and covariances, which was first found by Wishart (1928), and later by Wishart and Bartlett (1933), takes the form [the notation $\exp(x)=e^x$]

$$\frac{\left(\frac{n-1}{2}\right)^{\frac{1}{2}t(n-1)}}{\pi^{\frac{1}{4}t(t-1)}\prod_{v=1}^{t}\Gamma\{\frac{1}{2}(n-v)\}}\cdot\frac{|\,a_{ij}\,|^{\frac{1}{2}(n-t-2)}}{\Delta^{\frac{1}{2}(n-1)}}\exp\left[-\frac{(n-1)}{2\Delta}\sum_{i,j}(\Delta_{ij}a_{ij})\right]\prod_{i\leqslant j}(da_{ij})=\mathrm{L}\prod_{i\leqslant j}(da_{ij}),$$

where

$$L \propto \frac{|a_{ij}|^{\frac{1}{2}(n-t-2)}}{\Delta^{\frac{1}{2}(n-1)}} \exp\left[-\frac{(n-1)}{2\Delta}\sum_{i,j}(\Delta_{ij}a_{ij})\right],$$

Δ denotes the determinant of C and Δ_{ij} is the cofactor of c_{ij} in Δ.

The expression L, regarded as a function of the population parameters, is termed the likelihood function. Fisher has shown that, where one exists, an efficient set of estimates of the unknown parameters may be obtained by choosing that set which makes L a maximum, this method of estimation using the largest possible amount of information available from the data. The conditions required to make L a maximum are found by differentiating L, or some function of L such as log L, partially with respect to each of the unknown quantities λ_k, μ_k, τ_k^2, $(k = 1, 2, \ldots, t)$, and equating the results to zero. We thus obtain the equations

$$\left.\begin{aligned} &\frac{\partial(\log L)}{\partial\lambda_k} = 0; \\ &\frac{\partial(\log L)}{\partial\mu_k} = 0; \\ &\frac{\partial(\log)\,L}{\partial(\tau_k^2)} = 0. \end{aligned}\right\}$$

Now

$$\log L = -\frac{(n-1)}{2}\left\{\log\Delta + \frac{1}{\Delta}\sum_{i,j}(\Delta_{ij}a_{ij})\right\} + \text{a function independent of } \lambda_k, \mu_k, \tau_k.$$

Hence

$$-\frac{2}{(n-1)}\frac{\partial(\log L)}{\partial\lambda_k} = \left\{\frac{1}{\Delta} - \frac{1}{\Delta^2}\sum_{i,j}(\Delta_{ij}a_{ij})\right\} \cdot 2\sum_{r=1}^{t}(\lambda_r\Delta_{kr})$$
$$+\frac{2}{\Delta}\sum_{\substack{i \neq k \\ j \neq k}}\left\{\left[\sum_{r \neq j}\lambda_r\Delta_{ij,kr}\right]a_{ij}\right\} + \frac{2}{\Delta}\sum_{i \neq k}\left\{\left[\sum_{r \neq k}\lambda_r\Delta_{ik,kr}\right]a_{ik}\right\},$$

where $\Delta_{ij,pq}$ is the cofactor of c_{pq} in the expression as a determinant of Δ_{ij}, $\Delta_{ij,pq}$ being in fact plus or minus the determinant formed from Δ by crossing out the ith and pth rows and the jth and qth columns.

Hence

$$-\frac{\Delta}{(n-1)}\frac{\partial(\log L)}{\partial\lambda_k} = \left\{1 - \frac{1}{\Delta}\sum_{i,j}(\Delta_{ij}a_{ij})\right\}\sum_{r}(\lambda_r\Delta_{kr}) + \sum_{\substack{i,j \\ i \neq k}}\left\{\left[\sum_{r \neq j}\lambda_r\Delta_{ij,kr}\right]a_{ij}\right\}.$$

Similarly, for

$$\frac{\partial(\log L)}{\partial\mu_k},$$

and

$$-\frac{2}{(n-1)}\frac{\partial(\log L)}{\partial(\tau_k^2)}=\frac{\Delta_{kk}}{\Delta}-\frac{\Delta_{kk}}{\Delta}\sum_{i,j}(\Delta_{ij}a_{ij})+\frac{1}{\Delta}\sum_{\substack{i\neq k\\ j\neq k}}(\Delta_{ij,\,kk}a_{ij}).$$

The equations required are therefore

$$\left\{1-\frac{1}{\Delta}\sum_{i,j}(\Delta_{ij}a_{ij})\right\}\sum_{r}(\lambda_r\Delta_{kr})+\sum_{\substack{i,j\\ i\neq k}}\left\{\left[\sum_{r\neq j}\lambda_r\Delta_{ij,\,kr}\right]a_{ij}\right\}=0, \quad . \quad (1)$$

$$\left\{1-\frac{1}{\Delta}\sum_{i,j}(\Delta_{ij}a_{ij})\right\}\sum_{r}(\mu_r\Delta_{kr})+\sum_{\substack{i,j\\ i\neq k}}\left\{\left[\sum_{r\neq j}\mu_r\Delta_{ij,\,kr}\right]a_{ij}\right\}=0, \quad . \quad (2)$$

$$\left\{1-\frac{1}{\Delta}\sum_{i,j}(\Delta_{ij}a_{ij})\right\}\Delta_{kk}+\sum_{\substack{i\neq k\\ j\neq k}}\{\Delta_{ij,\,kk}a_{ij}\}=0, \quad . \quad (3)$$

where $k=1, 2, \ldots, t$.

These equations appear somewhat complicated, but we can however derive a very much simpler set from them. Firstly, multiply (1) by λ_k, (2) by μ_k, (3) by τ_2^k and add. We thus obtain

$$\sum_{j}(\Delta_{kj}a_{kj})=\Delta, \qquad (k=1, 2, \ldots, t). \quad . \quad (4)$$

Now multiply (1) by c_{kh}, where h may be any of the numbers $1, 2, \ldots, t$, and sum over all values of k. The resulting equations then reduce to

$$\frac{1}{\Delta}\sum_{r,i}(\lambda_r\Delta_{ri}a_{ih})=\lambda_h, \qquad (h=1, 2, \ldots, t). \quad . \quad (5)$$

Similarly, we obtain

$$\frac{1}{\Delta}\sum_{r,i}(\mu_r\Delta_{ri}a_{ih})=\mu_h, \qquad (h=1, 2, \ldots, t). \quad . \quad (6)$$

The equations (4), (5), and (6) are strictly equations for the estimates of λ_i, μ_i, τ_i, and c_{ij}, but in what follows we shall use these symbols to denote the estimated, and not the true, values.

If we let A denote the matrix

$$[a_{ij}]=\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1t} \\ a_{12} & a_{22} & \ldots & a_{2t} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{1t} & a_{2t} & \ldots & a_{tt} \end{bmatrix},$$

and

$$L=[\lambda_1 \quad \lambda_2 \ldots \lambda_t],$$

$$M=[\mu_1 \quad \mu_2 \ldots \mu_t],$$

then equations (5) and (6) can be expressed in the form

$$\left.\begin{aligned} LC^{-1}A&=L \\ MC^{-1}A&=M \end{aligned}\right\}, \qquad (7)$$

or alternatively

$$\left.\begin{aligned} LA^{-1}C&=L \\ MA^{-1}C&=M \end{aligned}\right\}. \qquad (8)$$

Now if H is the diagonal matrix

$$\begin{bmatrix} \tau_1^2 & & & & \\ & \tau_2^2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & \tau_t^2 \end{bmatrix},$$

then we have that

$$L'L+M'M+H=C. \qquad (9)$$

But from (7)

$$(L'L+M'M)C^{-1}A=L'L+M'M.$$

Hence

$$(C-H)C^{-1}A=C-H,$$

or

$$HC^{-1}A=H-C+A. \qquad (10)$$

Now equate the diagonal elements of the matrices on both sides of (10), and at the same time use equations (4). This gives the result

$$c_{ii}=a_{ii}, \qquad (i=1, 2, \ldots, t). \qquad (11)$$

Thus the matrices C and A have the same diagonal elements; a result which might have been expected.

Equations (11) together with either (7) or (8) are equivalent to the original equations. Before, however, discussing their practical solution there are a few points which must be noted.

(i) The quantities $\{a_{ij}\}$ provide a set of estimates of the population variances and covariances, but not the best set under the assumed hypothesis concerning the factors required. The best set is provided by the quantities $\{c_{ij}\}$ estimated from the equations above.

(ii) The equations do not give a unique set of solutions for L and M. For suppose that one set of solutions of (7) is given by

$$L=L_1, \qquad M=M_1.$$

Then if $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$ is any orthogonal matrix, it is clear that putting

$$L=L_2=pL_1+qM_1,$$
$$M=M_2=rL_1+sM_1,$$

also satisfies (7).

It will be noticed, however, that

$$L_2'L_2 + M_2'M_2 = L_1'L_1 + M_1'M_1,$$

and thus that $L'L + M'M$ is invariant under all orthogonal transformations. Since

$$C = L'L + M'M + H,$$

this means that C is uniquely determined and that so also are the quantities $\tau_1^2, \tau_2^2, \ldots, \tau_t^2$, which are the elements of H.

The geometrical meaning of all this is obvious: the arbitrariness is due to the fact that we are at liberty to rotate the factor axes within the common factor space, which in this particular case is one of two dimensions.

(iii) Let V be the diagonal matrix

$$\begin{bmatrix} a_{11} & & & & \\ & a_{22} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & a_{tt} \end{bmatrix}.$$

Then we define matrices P and R by the relations

$$C = V^{\frac{1}{2}}PV^{\frac{1}{2}},$$
$$A = V^{\frac{1}{2}}RV^{\frac{1}{2}}.$$

R is thus the sample correlation matrix

$$\begin{bmatrix} 1 & r_{12} & \cdots & r_{1t} \\ r_{12} & 1 & \cdots & r_{2t} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ r_{1t} & r_{2t} & \cdots & 1 \end{bmatrix},$$

while P is the estimate of the population correlation matrix

$$\begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1t} \\ \rho_{12} & 1 & \cdots & \rho_{2t} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \rho_{1t} & \rho_{2t} & \cdots & 1 \end{bmatrix}.$$

If we also define row-matrices l and m as follows:—

$$\left.\begin{aligned} l &= LV^{-\frac{1}{2}} = [\lambda_1/\sqrt{a_{11}} \quad \lambda_2/\sqrt{a_{22}} \; . \; . \; . \; \lambda_t/\sqrt{a_{tt}}] \\ m &= MV^{-\frac{1}{2}} = [\mu_1/\sqrt{a_{11}} \quad \mu_2/\sqrt{a_{22}} \; . \; . \; . \; \mu_t/\sqrt{a_{tt}}] \end{aligned}\right\}, \qquad . \quad . \quad . \qquad (12)$$

then we may easily transform equations (7) to the form

$$\left.\begin{array}{l} lP^{-1}R = l \\ mP^{-1}R = m \end{array}\right\}, \qquad (13)$$

and

$$P = l'l + m'm + h,$$

where

$$h = V^{-\frac{1}{2}} H V^{-\frac{1}{2}} = \begin{bmatrix} \tau_1^2/a_{11} & & & & \\ & \tau_2^2/a_{22} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \tau_t^2/a_{tt} \end{bmatrix}.$$

Here l, m, and h represent the loadings and specific variances of the standardised scores, and it is clear by virtue of the relations (12) that we shall obtain similar results whether we work with variances and covariances or correlations, since the loadings in the one case are proportional to those in the other. The equations of estimation thus have the advantage of giving results which are independent of the scale of measurement of the test scores.

(iv) Consider the equations

$$\left.\begin{array}{r} LA^{-1}(A - H) = \theta L \\ MA^{-1}(A - L'L - H) = \phi M \end{array}\right\}, \qquad (14)$$

where

$$\theta = LA^{-1}L', \qquad \phi = MA^{-1}M'.$$

If these equations are satisfied by L and M, then it is easy to show that

$$LA^{-1}M' = MA^{-1}L' = 0.$$

Hence

$$\begin{aligned} LA^{-1}(A - H) &= LA^{-1}(L'L + M'M) \\ &= LA^{-1}(C - H), \end{aligned}$$

since

$$C = L'L + M'M + H;$$

and therefore

$$L = LA^{-1}C.$$

Similarly

$$M = MA^{-1}C.$$

Thus any set of solutions of equations (14) also satisfies equations (8), and if we choose θ and ϕ to be the largest latent roots of the matrices $A^{-1}(A - H)$ and $A^{-1}(A - L'L - H)$ respectively, then equations (14) are satisfied uniquely. We are thus led to a practical method of obtaining a particular set of solutions of the equations of estimation satisfying the condition

$$LA^{-1}M' = MA^{-1}L' = 0,$$

which appears to have a certain analogy with the condition of orthogonality satisfied by Hotelling's "principal components."

3. The following iterative method of procedure is put forward in a somewhat tentative spirit. We have assumed that it converges, though no attempt has been made to prove this. It has however been found to do so in the simple examples on which it has as yet been tried.

(i) Calculate the reciprocal matrix A^{-1} of the sample covariance matrix A.

(ii) Guess a first approximation to the diagonal matrix H, which we will denote by H_1, and then find an approximate solution L of the equation

$$LX_1 = \theta L, \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where

$$X_1 = A^{-1}(A - H_1) \quad \text{and} \quad LA^{-1}L' = \theta,$$

θ being the largest latent root of X_1. For doing this certain methods of Aitken (1937) may be found useful.

If $L = L_1$ is a first approximation, then if we calculate

$$\begin{aligned} L_1X_1 &= L_1A^{-1}(A - H_1) \\ &= L_1 - L_1A^{-1}H_1, \end{aligned}$$

and also

$$\theta_1 = L_1A^{-1}L',$$

a second approximation L_2 is given by $L_2 = 1/\theta_1 L_1X_1$.

(iii) Assuming $L = L_2$ to be close enough to the true solution of equation (15), we now find an approximate solution M of

$$MY_1 = \phi M, \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where

$$Y_1 = A^{-1}(A - H_1 - L_2'L_2) \quad \text{and} \quad MA^{-1}M' = \phi,$$

ϕ being the largest latent root of Y_1.

If $M = M_1$ is a first approximation, then we calculate

$$M_1Y_1 = M_1 - M_1A^{-1}H_1 - (M_1A^{-1}L_2')L_2,$$

and also

$$\phi_1 = M_1A^{-1}M_1'.$$

A second approximation M_2 is now given by $M_2 = 1/\phi_1 M_1Y_1$.

(iv) If

$$L_2 = [\lambda_{12} \quad \lambda_{22} \; . \; . \; . \; \lambda_{t2}],$$

and

$$M_2 = [\mu_{12} \quad \mu_{22} \; . \; . \; . \; \mu_{t2}],$$

then

$$H_2 = \begin{bmatrix} a_{11} - \lambda_{12}^2 - \mu_{12}^2 & & & & \\ & a_{22} - \lambda_{22}^2 - \mu_{22}^2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & a_{tt} - \lambda_{t2}^2 - \mu_{t2}^2 \end{bmatrix}$$

is a second approximation to H.

This process is continued until the required degree of approximation is reached, which may be decided by comparing successive stages in the calculation. The process can obviously be generalised to include cases where there are more than two general factors. It must be pointed out, however, that we are free to rotate the factors as we please.

We shall give a (somewhat artificial) numerical example of the practical method of solution of the equations of estimation, though the calculations in this example will be very simple, since we shall assume the presence of only one general factor. Suppose that the matrix A, calculated from a sample, is

$$\begin{bmatrix} 1{\cdot}0 & {\cdot}4 & {\cdot}4 & {\cdot}2 \\ {\cdot}4 & 1{\cdot}0 & {\cdot}7 & {\cdot}3 \\ {\cdot}4 & {\cdot}7 & 1{\cdot}0 & {\cdot}3 \\ {\cdot}2 & {\cdot}3 & {\cdot}3 & 1{\cdot}0 \end{bmatrix}.$$

The matrix A^{-1} is then

$$\begin{bmatrix} 1{\cdot}2378 & -{\cdot}2769 & -{\cdot}2769 & -{\cdot}0814 \\ -{\cdot}2769 & 2{\cdot}0577 & -1{\cdot}2760 & -{\cdot}1791 \\ -{\cdot}2769 & -1{\cdot}2760 & 2{\cdot}0577 & -{\cdot}1791 \\ -{\cdot}0814 & -{\cdot}1791 & -{\cdot}1791 & 1{\cdot}1237 \end{bmatrix}.$$

As a first guess we will take the loadings given by Spearman's formula $(A^2 - A')^{\frac{1}{2}}/(T - 2A)^{\frac{1}{2}}$, namely

$$L_1 = [{\cdot}496 \quad {\cdot}823 \quad {\cdot}823 \quad {\cdot}374];$$

also

$$H_1 = \begin{bmatrix} {\cdot}754 & & & \\ & {\cdot}322 & & \\ & & {\cdot}322 & \\ & & & {\cdot}860 \end{bmatrix}.$$

The steps in the calculation are then as follows:—

$$\begin{array}{rllll} L_1 = & {\cdot}496 & {\cdot}823 & {\cdot}823 & {\cdot}374 \\ L_1A^{-1} = & {\cdot}1277 & {\cdot}4390 & {\cdot}4390 & {\cdot}0851 \\ L_1A^{-1}H_1 = & {\cdot}0963 & {\cdot}1414 & {\cdot}1414 & {\cdot}0732 \\ L_1X_1 = & {\cdot}3997 & {\cdot}6816 & {\cdot}6816 & {\cdot}3008 \end{array} \quad H_1 = \begin{bmatrix} {\cdot}754 & & & \\ & {\cdot}322 & & \\ & & {\cdot}322 & \\ & & & {\cdot}860 \end{bmatrix}$$

$$\theta_1 = L_1A^{-1}L_1' = {\cdot}81776$$

$$\begin{array}{rllll} L_2 = & {\cdot}4888 & {\cdot}8335 & {\cdot}8335 & {\cdot}3678 \\ L_2A^{-1} = & {\cdot}1135 & {\cdot}4503 & {\cdot}4503 & {\cdot}0749 \\ L_2A^{-1}H_2 = & {\cdot}0864 & {\cdot}1375 & {\cdot}1375 & {\cdot}0648 \\ L_2X_2 = & {\cdot}4024 & {\cdot}6960 & {\cdot}6960 & {\cdot}3030 \end{array} \quad H_2 = \begin{bmatrix} {\cdot}7611 & & & \\ & {\cdot}3053 & & \\ & & {\cdot}3053 & \\ & & & {\cdot}8647 \end{bmatrix}$$

$$\theta_2 = L_2A^{-1}L_2' = {\cdot}83368$$

$$\begin{array}{rllll} L_3 = & {\cdot}4827 & {\cdot}8349 & {\cdot}8349 & {\cdot}3635 \\ L_3A^{-1} = & {\cdot}1055 & {\cdot}4539 & {\cdot}4539 & {\cdot}0701 \\ L_3A^{-1}H_3 = & {\cdot}0809 & {\cdot}1375 & {\cdot}1375 & {\cdot}0608 \\ L_3X_3 = & {\cdot}4018 & {\cdot}6974 & {\cdot}6974 & {\cdot}3027 \end{array} \quad H_3 = \begin{bmatrix} {\cdot}7670 & & & \\ & {\cdot}3029 & & \\ & & {\cdot}3029 & \\ & & & {\cdot}8679 \end{bmatrix}$$

$$\theta_3 = L_3A^{-1}L_3' = {\cdot}83433$$

$$\begin{array}{rllll} L_4 = & {\cdot}4816 & {\cdot}8359 & {\cdot}8359 & {\cdot}3628 \\ L_4A^{-1} = & {\cdot}1037 & {\cdot}4551 & {\cdot}4551 & {\cdot}0691 \\ L_4A^{-1}H_4 = & {\cdot}0797 & {\cdot}1371 & {\cdot}1371 & {\cdot}0600 \\ L_4X_4 = & {\cdot}4019 & {\cdot}6988 & {\cdot}6988 & {\cdot}3028 \end{array} \quad H_4 = \begin{bmatrix} {\cdot}7681 & & & \\ & {\cdot}3013 & & \\ & & {\cdot}3013 & \\ & & & {\cdot}8684 \end{bmatrix}$$

$$\theta_4 = L_4A^{-1}L_4' = {\cdot}83585$$

$$\begin{array}{rllll} L_5 = & {\cdot}4808 & {\cdot}8360 & {\cdot}8360 & {\cdot}3623 \\ L_5A^{-1} = & {\cdot}1027 & {\cdot}4555 & {\cdot}4555 & {\cdot}0685 \\ L_5A^{-1}H_5 = & {\cdot}0790 & {\cdot}1372 & {\cdot}1372 & {\cdot}0595 \\ L_5X_5 = & {\cdot}4018 & {\cdot}6988 & {\cdot}6988 & {\cdot}3028 \end{array} \quad H_5 = \begin{bmatrix} {\cdot}7688 & & & \\ & {\cdot}3011 & & \\ & & {\cdot}3011 & \\ & & & {\cdot}8687 \end{bmatrix}$$

$$\theta_5 = L_5A^{-1}L_5' = {\cdot}83579$$

$$\begin{array}{rllll} L_6 = & {\cdot}4807 & {\cdot}8361 & {\cdot}8361 & {\cdot}3623 \end{array} \quad H_6 = \begin{bmatrix} {\cdot}7689 & & & \\ & {\cdot}3009 & & \\ & & {\cdot}3009 & \\ & & & {\cdot}8687 \end{bmatrix}$$

It is clear that, correct to three figures, the loadings of the four tests in the general factor are given by

$$L = [{\cdot}481 \quad {\cdot}836 \quad {\cdot}836 \quad {\cdot}362],$$

and the specific variances by

$$H = \begin{bmatrix} {\cdot}769 & & & \\ & {\cdot}301 & & \\ & & {\cdot}301 & \\ & & & {\cdot}869 \end{bmatrix}.$$

The matrix C is also found to be

$$\begin{bmatrix} 1{\cdot}000 & {\cdot}402 & {\cdot}402 & {\cdot}174 \\ {\cdot}402 & 1{\cdot}000 & {\cdot}699 & {\cdot}303 \\ {\cdot}402 & {\cdot}699 & 1{\cdot}000 & {\cdot}303 \\ {\cdot}174 & {\cdot}303 & {\cdot}303 & 1{\cdot}000 \end{bmatrix}.$$

We give below a comparison of these results with those obtained by applying the methods of C. Spearman $\left(\text{viz. formula } \dfrac{A^2 - A'}{T - 2A}\right)$ and of L. L. Thurstone (repeated application of the centroid method *), and also the modification of H. Hotelling's method by Thomson (1934).

Method.	Squares of Loadings Obtained.			
Spearman	·246	·678	·678	·140
Thurstone	·244	·684	·684	·139
Thomson	·236	·694	·694	·135
Maximum likelihood	·231	·699	·699	·131

It will be noticed that our results are closest to those of Thomson's method.

4. If we are working with correlation coefficients our equations for estimating the loadings L of the standardised scores in a single general factor are

$$LR^{-1}(R-H) = (LR^{-1}L')L, \qquad (17)$$

where

$$L = [\lambda_1 \quad \lambda_2 \; . \; . \; . \; \lambda_t],$$

$$H = \begin{bmatrix} 1-\lambda_1^2 & & & & \\ & 1-\lambda_2^2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \quad 1-\lambda_t^2 \end{bmatrix},$$

and R is the sample correlation matrix.

Equations (17) are equivalent to

$$L = LR^{-1}P,$$

or

$$LP^{-1} = LR^{-1}, \qquad (18)$$

---

* By repeated application of the centroid method is meant that starting with guessed communalities x factors (here only one factor) are taken out, and their loadings squared and summed to give new guessed communalities, with which the whole process is repeated, again and again (Thomson, 1939).

where

$$P=L'L+H=\begin{bmatrix} 1 & \lambda_1\lambda_2 & \dots & \lambda_1\lambda_t \\ \lambda_1\lambda_2 & 1 & \dots & \lambda_2\lambda_t \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \lambda_1\lambda_t & \lambda_2\lambda_t & \dots & 1 \end{bmatrix}.$$

It is easy to see that the equations of estimation in Thomson's modified Hotelling method are

$$L(R-H)=kL, \qquad (19)$$

where

$$k=LL',$$

and these equations are equivalent to

$$LP=LR. \qquad (20)$$

There is thus a certain formal similarity between the two methods.

5. In the cases to which we have so far applied the maximum likelihood method the matrix of factor loadings K took the form

$$\begin{bmatrix} \lambda_1 & \mu_1 & \dots & \nu_1 & \tau_1 & 0 & \dots & 0 \\ \lambda_2 & \mu_2 & \dots & \nu_2 & 0 & \tau_2 & \dots & 0 \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \lambda_t & \mu_t & \dots & \nu_t & 0 & 0 & \dots & \tau_t \end{bmatrix},$$

K being such that

$$x=Kf,$$

where x is the column matrix of test scores

$$\{x_1 \quad x_2 \dots x_t\},$$

and f is the column matrix of factors

$$\{f \quad g \dots h \quad s_1 \quad s_2 \dots s_t\}.$$

If we define the matrix Q by the equation

$$Q=K'C^{-1}-K'C^{-1}AC^{-1}, \qquad (21)$$

where

$$KK'=C,$$

then Q is of the same shape as K', and it may easily be verified that every element of Q which does not correspond in position to an assumed zero element of K' may be equated to zero. Using the methods of section 2 it may further be shown that this statement remains true whatever hypothesis is made concerning the form of K. This, therefore, provides us with completely general equations of estimation. Practical methods of solution suitable for types of factor analysis other than those already considered have not yet, however, been fully worked out.

6. Closely connected with the problem of estimation is that of testing the hypothesis which is made concerning the number of factors required. Let us suppose, for example, that we have assumed the presence of two general factors in addition to specifics, and that it is desired to test whether these may be considered sufficient. One method which has been used for doing this is to calculate the residuals $(r_{ij}-\check{\rho}_{ij})$ left after "taking out" the two factors, where

$$\check{\rho}_{ij}=\lambda_i\lambda_j+\mu_i\mu_j,$$

and where

$$L=[\lambda_1 \quad \lambda_2 \;.\;.\;.\; \lambda_t],$$
$$M=[\mu_1 \quad \mu_2 \;.\;.\;.\; \mu_t]$$

are the estimated loadings of the standardised scores, and comparing these residuals with the standard deviations of the correlation coefficients as given by the expression $(1-r_{ij}^2)/\sqrt{n}$. If the residuals are found to be "significant," then the hypothesis is considered to be disproved and the presence of another factor is assumed.

One objection to this method is that the assumption of approximate normality of the distribution of r_{ij} is only justified when ρ_{ij} is moderate or small and the sample number n is large, and if this assumption is not justified the test of significance is rendered untrustworthy. The above conditions are however realised in most practical psychological work. There is nevertheless a more serious objection: the estimates $\check{\rho}_{ij}$ are subject to variation from one sample to another, and this fact must be taken into account when finding the standard deviation of

$$d_{ij}=r_{ij}-\check{\rho}_{ij}.$$

Let us assume for the sake of argument that r_{ij} and $\check{\rho}_{ij}$ are both normally distributed and that their respective variances for samples of a given size are σ_r^2 and σ_ρ^2. Also let $\rho_{r\rho}$ be the correlation between r_{ij} and $\check{\rho}_{ij}$. The variance σ_d^2 of the residual d_{ij} is then given by the formula

$$\sigma_d^2=\sigma_r^2+\sigma_\rho^2-2\rho_{r\rho}\sigma_r\sigma_\rho.$$

Now it is clear that $\rho_{r\rho}$ will generally be very near unity and that σ_ρ will be very near σ_r. We may thus expect σ_d^2 to be very small, much smaller in fact than σ_r^2. Hence it may well happen that residuals which are not adjudged significant by the process described above would certainly be so if they could be compared with their true standard deviations. The practical effect of this is that more factors may be required to explain the test scores obtained than are at present considered necessary. What is clearly required is some criterion which is appropriate for testing the assumed hypothesis and which is such that

(i) it is a function of the observed data only, and not the population parameters;

(ii) its sampling distribution, or at least its sampling variance, can be calculated.

It is somewhat difficult to find such a criterion as this which can be used when dealing with small samples, but we shall however derive one which is suitable for large samples.

7. In future we shall use an inverted circumflex accent to denote the estimated values of population parameters as distinct from their true values.

Let us again consider the likelihood function L as defined in section 2. We have (changing our notation) that

$$\log_e \mathrm{L} = \frac{(n-t-2)}{2} \log_e |A| - \frac{(n-1)}{2}\left\{\log_e |C| + \sum_{i,j} a_{ij} c^{ij}\right\} + \text{a constant},$$

where c^{ij} denotes an element of the inverse matrix C^{-1}. Now let

$$[\log \mathrm{L}]_0, \qquad \left[\frac{\partial (\log \mathrm{L})}{\partial c_{pq}}\right]_0, \qquad \left[\frac{\partial^2 (\log \mathrm{L})}{\partial c_{pq} \partial c_{rs}}\right]_0$$

denote the values obtained by replacing the set of quantities $\{c_{ij}\}$ by the set $\{a_{ij}\}$ in the expressions for

$$\log \mathrm{L}, \qquad \frac{\partial (\log \mathrm{L})}{\partial c_{pq}}, \qquad \frac{\partial^2 (\log \mathrm{L})}{\partial c_{pq} \partial c_{rs}}$$

respectively. Then by Taylor's theorem we have

$$\log \mathrm{L} = [\log \mathrm{L}]_0 + \sum_{p \leqslant q} \left\{(c_{pq} - a_{pq}) \left[\frac{\partial (\log \mathrm{L})}{\partial c_{pq}}\right]_0\right\}$$
$$+ \tfrac{1}{2} \sum_{\substack{p \leqslant q \\ r \leqslant s}} \left\{(c_{pq} - a_{pq})(c_{rs} - a_{rs}) \left[\frac{\partial^2 (\log \mathrm{L})}{\partial c_{pq} \partial c_{rs}}\right]_0\right\} + \text{residual terms},$$

and since the quantities $(c_{pq} - a_{pq})$ are of order $1/\sqrt{n}$, the residual terms on the right-hand side of this equation are also of order $1/\sqrt{n}$.

Now the quantities $\left[\frac{\partial (\log \mathrm{L})}{\partial c_{pq}}\right]_0$ are all zero; which means that log L, considered only as a function of the set $\{c_{ij}\}$, and supposing no restrictions to be imposed on these quantities, is a maximum when $c_{ij}=a_{ij}$. Hence, neglecting terms of order $1/\sqrt{n}$, we have

$$\log L-[\log L]_0=-\tfrac{1}{2}\sum_{\substack{p\leq q\\ r\leq s}}(a_{pq,\,rs}\epsilon_{pq}\epsilon_{rs}), \qquad (22)$$

where $\epsilon_{pq}=(a_{pq}-c_{pq})$ and $a_{pq,\,rs}$ denotes the average value of

$$-\left[\frac{\partial^2(\log L)}{\partial c_{pq}\partial c_{rs}}\right]_0.$$

Also it may easily be verified that $[\log L]_0$ differs from its mean value only by a quantity of order $1/\sqrt{n}$. Thus, with the same order of approximation as before, we have

$$L\propto\exp\left[-\tfrac{1}{2}\sum_{\substack{p\leq q\\ r\leq s}}(a_{pq,\,rs}\epsilon_{pq}\epsilon_{rs})\right].$$

This means that, approximately when n is large, the $\frac{1}{2}t(t+1)$ quantities $\{\epsilon_{ij}\}$ are distributed about zero means in a normal multivariate distribution, while the expression

$$\lambda=\sum_{\substack{p\leq q\\ r\leq s}}(a_{pq,\,rs}\epsilon_{pq}\epsilon_{rs}),$$

which is a positive definite quadratic function of these quantities, is distributed in Karl Pearson's χ^2-distribution with $\frac{1}{2}t(t+1)$ degrees of freedom.

Now from equation (22) we have

$$\begin{aligned}\lambda&=-2\log L+2[\log L]_0\\ &=(n-1)\Big\{\Big(\log|C|+\sum_{i,j}a_{ij}c^{ij}\Big)-(\log|A|+t)\Big\}.\end{aligned}$$

Hence, writing for convenience n in place of $(n-1)$, since we are neglecting quantities of order $1/\sqrt{n}$, we have approximately

$$\lambda=n\left\{\log_e\frac{|C|}{|A|}-\sum_{i,j}(c_{ij}-a_{ij})c^{ij}\right\}. \qquad (23)$$

It is evident that λ gives a measure of the discrepancy between observation and hypothesis, and hence if the population parameters $\{c_{ij}\}$ were known we could use λ as a criterion for testing our hypothesis. In practice, however, the quantities $\{c_{ij}\}$ are not known exactly and must be replaced by estimated values $\{\bar{c}_{ij}\}$.

The new value of λ is then given by

$$\lambda = n\left\{\log_e \frac{|\check{C}|}{|A|} - \sum_{i,j}(\check{c}_{ij} - a_{ij})\check{c}^{ij}\right\}$$

$$= n \log_e \frac{|\check{C}|}{|A|}, \quad \text{using equation (4).}$$

The distribution of λ will of course be altered owing to the fact that the estimates $\{\check{c}_{ij}\}$ are subject to sampling error. Using an argument employed by Fisher (1924), it may be shown that while λ will still be distributed as χ^2, the degrees of freedom will be reduced by an amount p, where p is the number of independent equations needed to determine the quantities $\{\check{c}_{ij}\}$, this being equal to the number of independent factor loadings. Another way of defining p is to state that p is equal to the total number of factor loadings minus the number of possible independent rotations of the factor axes. We thus find that the expression

$$\lambda = n \log_e \frac{|\check{C}|}{|A|}$$

is distributed approximately as χ^2 with k degrees of freedom, where $k = \{\frac{1}{2}t(t+1) - p\}$ and must always be positive.

If the value of λ calculated from a sample is found to be significantly large, then we may consider our original hypothesis concerning the required number of factors to be disproved. This method of dealing with the problem is applicable whatever the form of this hypothesis, and it is also immaterial whether we work with variances and covariances or with correlation coefficients. In the latter case we should put

$$\lambda = n \log_e \frac{|\check{P}|}{|R|}.$$

There is in the case we have considered a very simple method of calculating λ, using the diagonal matrix $\check{H}$, whose elements are the estimated specific variances $\check{\tau}_1^2, \check{\tau}_2^2, \ldots, \check{\tau}_t^2$. For from equation (10) (changing the notation) we have

$$\begin{aligned}
\check{C}^{-1}A &= \check{H}^{-1}(\check{H} - \check{C} + A) \\
&= I + \check{H}^{-1}(A - \check{C}) \\
&= \begin{bmatrix}
1 & \dfrac{a_{12} - \check{c}_{12}}{\check{\tau}_1^2} & \dfrac{a_{13} - \check{c}_{13}}{\check{\tau}_1^2} & \cdots & \dfrac{a_{1t} - \check{c}_{1t}}{\check{\tau}_1^2} \\
\dfrac{a_{12} - \check{c}_{12}}{\check{\tau}_2^2} & 1 & \dfrac{a_{23} - \check{c}_{23}}{\check{\tau}_2^2} & \cdots & \dfrac{a_{2t} - \check{c}_{2t}}{\check{\tau}_2^2} \\
\cdot & & & & \cdot \\
\cdot & & & & \cdot \\
\cdot & & & & \cdot \\
\dfrac{a_{1t} - \check{c}_{1t}}{\check{\tau}_t^2} & \cdot & \cdot & \cdots & 1
\end{bmatrix}.
\end{aligned}$$

Hence, equating the determinants of the matrices on either side of this equation, we have

$$\frac{|A|}{|\check{C}|} \doteqdot 1 - \sum_{i<j}\left\{\frac{(\check{c}_{ij}-a_{ij})^2}{\check{v}_i^2\check{v}_j^2}\right\}.$$

Thus

$$\begin{aligned}\lambda &\doteqdot n\log_e\frac{|\check{C}|}{|A|}\\ &\doteqdot n\left\{1-\frac{|A|}{|\check{C}|}\right\} \quad \left(\text{since } \frac{|A|}{|\check{C}|} \text{ is near unity}\right)\\ &\doteqdot n\sum_{i<j}\left\{\frac{(\check{c}_{ij}-a_{ij})^2}{\check{v}_i^2\check{v}_j^2}\right\}.\end{aligned}$$

8. As an example of the method described in the previous section, let us consider again the matrix

$$A = \begin{bmatrix} 1{\cdot}000 & {\cdot}400 & {\cdot}400 & {\cdot}200 \\ {\cdot}400 & 1{\cdot}000 & {\cdot}700 & {\cdot}300 \\ {\cdot}400 & {\cdot}700 & 1{\cdot}000 & {\cdot}300 \\ {\cdot}200 & {\cdot}300 & {\cdot}300 & 1{\cdot}000 \end{bmatrix}.$$

Let us suppose that these correlation coefficients have been calculated from a sample of 10,000 individuals. We have previously found (in section 3), assuming the presence of one general factor in addition to specifics, that

$$\check{C} = \begin{bmatrix} 1{\cdot}000 & {\cdot}402 & {\cdot}402 & {\cdot}174 \\ {\cdot}402 & 1{\cdot}000 & {\cdot}699 & {\cdot}303 \\ {\cdot}402 & {\cdot}699 & 1{\cdot}000 & {\cdot}303 \\ {\cdot}174 & {\cdot}303 & {\cdot}303 & 1{\cdot}000 \end{bmatrix},$$

and

$$\check{H} = \begin{bmatrix} {\cdot}769 & & & \\ & {\cdot}301 & & \\ & & {\cdot}301 & \\ & & & {\cdot}869 \end{bmatrix}.$$

Hence, putting $n = 10{,}000$, we have

$$\begin{aligned}\lambda &= 10{,}000\left\{\frac{({\cdot}002)^2}{{\cdot}769\times{\cdot}301}\times 2+\frac{({\cdot}026)^2}{{\cdot}769\times{\cdot}869}+\frac{({\cdot}001)^2}{({\cdot}301)^2}+\frac{({\cdot}003)^2}{{\cdot}301\times{\cdot}869}\times 2\right\}\\ &= 10{,}000\times{\cdot}00113\\ &= 11{\cdot}3.\end{aligned}$$

The number of tests t is 4, and the number of independent factor loadings p is 8. Hence the number of degrees of freedom k is equal to

$$\tfrac{1}{2}\times 4\times 5 - 8 = 2.$$

Now for χ^2 with 2 degrees of freedom the probability P of exceeding by chance the value 11·3 is less than ·01, and the value obtained for λ is thus significant at the 1 per cent. point. This may be taken as disproving our hypothesis that one general factor, together with specifics, is sufficient to "explain" the observed correlation coefficients. It will be noted that the chief contribution to the value of λ is made by the discrepancy of ·026 between the observed and the calculated values of the correlation between tests 1 and 4. If the sample number n were 1000 instead of 10,000 then the value of λ would only be 1·13, which value is certainly not significant since it is exceeded by chance in about 57 per cent. of all cases.

It may be remarked that the above test of significance is not applicable when other, inefficient, methods of estimation are used. Let us now find the value of λ using the estimates of $\{c_{ij}\}$ calculated from the loadings given by Spearman's formula $\frac{(A^2 - A')^{\frac{1}{2}}}{(T-2A)^{\frac{1}{2}}}$, which have already been found in section 3. We obtain a new matrix of estimates $\hat{C}$ given by

$$\hat{C} = \begin{bmatrix} 1·000 & ·408 & ·408 & ·186 \\ ·408 & 1·000 & ·678 & ·308 \\ ·408 & ·678 & 1·000 & ·308 \\ ·186 & ·308 & ·308 & 1·000 \end{bmatrix}.$$

The value of λ must in this case be calculated directly, using equation (23). Thus, letting $\hat{c}^{ij}$ denote an element of the inverse matrix $\hat{C}^{-1}$, we have

$$\lambda = n\left\{\log_e \frac{|\hat{C}|}{|A|} - \sum_{i,j} (\hat{c}_{ij} - a_{ij})\hat{c}^{ij}\right\}.$$

Now

$$|\hat{C}| = ·383433,$$
$$|A| = ·368400;$$

so that

$$\frac{|\hat{C}|}{|A|} = 1·040805,$$

and

$$\log_e \frac{|\hat{C}|}{|A|} = ·039994.$$

Also

$$\sum_{i,j} (\hat{c}_{ij} - a_{ij})\hat{c}^{ij} = ·036254.$$

Hence

$$\lambda = 10{,}000(·039994 - ·036254)$$
$$= 37·4.$$

This value of λ is over three times the previous value of 11·3, which may be taken as an indication that Spearman's method of estimation cannot be considered as "efficient," using the word in the same sense as Fisher. It is also clear that to use the χ^2 test in this case would be highly misleading, and would greatly exaggerate the significance of the discrepancy.

SUMMARY.

9. The method of maximum likelihood is used to estimate the loadings of a battery of tests in the factors which are assumed to be present. In a simple numerical example the results obtained are compared with those found by using various other methods. A method, suitable for large samples, is then given for testing the truth of any hypothesis made concerning the number of factors required.

In conclusion I should like to thank Professor Godfrey H. Thomson for his help and encouragement in the writing of this paper.

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VII—Further Investigations into the Periodic Lamé Functions. By E. L. Ince, D.Sc.

(MS. received January 10, 1940. Read March 4, 1940.)

1. BASIS OF THE PAPER.

HITHERTO the Lamé functions of u, of real periods $2K$ and $4K$, and the Lamé polynomials in particular, have been expressed as series of integral powers of sn u (multiplied, in appropriate cases, by cn u, dn u, or cn u dn u). The necessity to use these, or the corresponding Weierstrassian forms, is a serious handicap to research, because the terms of the expression are not mutually orthogonal. In particular, results that can be deduced from the orthogonal properties of the Fourier series in which periodic Mathieu functions are developed, find no analogues in the Lamé functions as expressed in the customary forms. For instance, the integral equations that generate the Mathieu functions have proved to be valuable sources of information, but those for the Lamé functions cannot be much more than scientific curiosities until the development of Lamé functions as series of orthogonal functions enables them to be put to use.

This paper springs from a question raised by Hermite in a letter to Dini dated 13th September 1880 (*Ann. Mat. pura appl.* (2), vol. x, p. 144; *Œuvres de Charles Hermite*, vol. iv, p. 47) as to whether the Fourier series does not find a generalisation, in the theory of elliptic functions, in the formula

$$\sum (A_p \cos p \operatorname{am} u + B_p \sin p \operatorname{am} u),$$

a point which does not seem to have received the attention that it deserves. It would indeed appear that the disuse of Jacobi's original notation has meant the neglect of a valuable tool; the extraordinary complexity of the expression for sn pu in terms of functions of argument u seems to indicate that the true analogue of $\sin px$ is $\sin p \operatorname{am} u$ rather than sn pu.

It will be shown that all Lamé functions of u, of real periods $2K$ or $4K$, can be expressed as series of the above type, and that these expressions terminate in precisely those cases in which the series of powers of sn u terminate, when the order n is a positive integer. When n is non-integral, the series converge provided that $k^2 < 1$. In §§ 5 and 6 are examples to show that manipulations which are quite impossible when the functions are expressed as series of powers of $\operatorname{sn} u$, become practicable, in fact simple, when what might be called the Fourier-Jacobi series are used. The remaining sections deal with functions of periods greater than $4K$ in general, but in particular with those of period $8K$. Such functions can exist for all values of n, and are expressible in a finite form when $2n$ is an odd integer. These finite forms have been considered in some detail by Halphen and others, but the results given here are simpler and more significant than any hitherto discovered.

2. THE DIFFERENTIAL EQUATION TRANSFORMED.

It is convenient to write

$$v = \operatorname{am}(u, k),$$

so that

$$\frac{dv}{du} = \sqrt{(1 - k^2 \sin^2 v)} = \Delta v = \operatorname{dn} u.$$

The Lamé equation, whose standard Jacobian form is

$$\frac{d^2y}{du^2} + \{h - n(n+1)k^2 \operatorname{sn}^2 u\}y = 0, \quad . \quad . \quad . \quad (2.1)$$

then becomes

$$(1 - k^2 \sin^2 v)\frac{d^2y}{dv^2} - k^2 \sin v \cos v \frac{dy}{dv} + \{h - n(n+1)k^2 \sin^2 v\}y. \quad . \quad (2.2)$$

In dealing with cases in which the factor $\operatorname{dn} u$ is explicit, it is convenient to make the substitution

$$y = w \operatorname{dn} u = w\sqrt{(1 - k^2 \sin^2 v)},$$

by which (2.2) is transformed into

$$(1 - k^2 \sin^2 v)\frac{d^2w}{dv^2} - 3k^2 \sin v \cos v \frac{dw}{dv} + \{h - k^2 - (n-1)(n+2)k^2 \sin^2 v\}w = 0. \quad (2.3)$$

In what follows it will be convenient to use the term "natural period" and the symbol P for the least real period common to the coefficients of the differential equation, that is $2K$ when the equation is in the form (2.1) and π when in the form (2.2) or (2.3). If any solution is periodic, then its period must necessarily be equal to, or an integral multiple of, the natural period.

3. FUNCTIONS OF PERIOD P.

An even function of period P may be developed as

$$y = A_0 + \sum A_{2r} \cos 2rv. \quad . \quad . \quad . \quad . \quad (3.1)$$

When η is written for $2h-n(n+1)k^2$, the recurrence-relations between the coefficients are

$$\tfrac{1}{2}(n-1)(n+2)k^2A_2+\eta A_0=0,$$

$$\tfrac{1}{2}(n-3)(n+4)k^2A_4+\{\eta-4(2-k^2)\}A_2+n(n+1)k^2A_0=0,$$

$$\begin{aligned}\tfrac{1}{2}(n-2r-1)(n+2r+2)k^2A_{2r+2}+\{\eta-4r^2(2-k^2)\}A_{2r}&\\ +\tfrac{1}{2}(n+2r-1)(n-2r+2)k^2A_{2r-2}=0&\quad (r>1).\end{aligned}$$

The continued fraction for A_{2r}/A_{2r-2}, derived from these relations, is convergent for $k^2<1$, and shows that, as $r\to\infty$,

$$\frac{A_{2r}}{A_{2r-2}}\to-\frac{(1-k')^2}{k^2},$$

where $k'=\sqrt{(1-k^2)}$. Consequently

$$\left|\frac{A_{2r}}{A_{2r-2}}\right|<k^2$$

for sufficiently large r, when $k^2<1$. Thus, term for term, the convergence is more rapid than that of the development in powers of sn u, where the limiting ratio is equal to k^2 (*Proc. Roy. Soc. Edin.*, vol. lx, p. 49).

The characteristic equation for η, obtained in the usual manner, is

$$\eta=-\frac{\tfrac{1}{8}(n^2-1)n(n+2)k^4}{2-k^2-\tfrac{1}{4}\eta}\;-\;\frac{(n^2-9)(n-2)(n+4)k^4/16^2}{2-k^2-\tfrac{1}{16}\eta}\;-\;\frac{(n^2-25)(n-4)(n+6)k^4/48^2}{2-k^2-\tfrac{1}{36}\eta}\;-\;\ldots \qquad (3.2)$$

Its roots are the characteristic numbers $a_n^{2m}(k^2)$.

If, in place of (3.1), the even function of period P is assumed to have the form

$$y=\sqrt{(1-k^2\sin^2 v)}\{C_0+\textstyle\sum C_{2r}\cos 2rv\}, \quad . \quad . \quad . \quad (3.3)$$

the recurrence-relations for the coefficients C naturally differ from those given above for the coefficients A, but they lead to the same characteristic equation (3.2) for h, confirming the fact that (3.1) and (3.3) are identical if C_0 is a definite multple of A_0. The continued fraction terminates when n is an integer; if $n=2\nu$ or $2\nu+1$ (assuming $n>0$), the equation is algebraic and of degree $\nu+1$ in η, and its roots are the characteristic numbers corresponding to Lamé functions of a finite form, of type (3.1) when n is even and type (3.3) when n is odd.

To sum up, in terms of the original variable u, the even Lamé functions of period $2K$ may be expressed as

$$\begin{aligned}\mathrm{Ec}_n^{2m}(u)&=A_0+\textstyle\sum A_{2r}\cos 2r\,(\mathrm{am}\,u)\\ &=\mathrm{dn}\,u\{C_0+\textstyle\sum C_{2r}\cos 2r\,(\mathrm{am}\,u)\},\end{aligned}$$

where m is a positive integer, or zero. The degree n is unrestricted; when n is a positive integer there exist distinct values of η, in number equal to $\frac{1}{2}n+1$ or to $\frac{1}{2}(n+1)$, whichever is an integer, to which correspond terminating solutions. But for every integral value of n there also exists an infinity of values of η for which the solution does not terminate. For example, when $n=0$ or 1, these values of η are the roots of the equation obtained by equating to zero the entire denominator of $\frac{1}{8}(n^2-1)n(n+2)k^4$ in (3.2), and similarly for higher values of n. When the series do not terminate they are convergent if $(1-k')^2 < k^2$, and in particular if $k^2 < 1$.

Likewise, the odd Lamé functions of period $2K$ are expressible as

$$\begin{aligned}\mathrm{Es}_n^{2m}(u) &= \sum B_{2r} \sin 2r\,(\mathrm{am}\, u)\\ &= \mathrm{dn}\, u \sum D_{2r} \sin 2r\,(\mathrm{am}\, u). \qquad (3.4)\end{aligned}$$

The corresponding values of η are the roots of the equation

$$\eta = 4(2-k^2) - \frac{(n^2-9)(n-2)(n+4)k^4/8^2}{2-k^2-\frac{1}{16}\eta} - \frac{(n^2-25)(n-4)(n+6)k^4/48^2}{2-k^2-\frac{1}{36}\eta} - \dots \qquad (3.5)$$

4. FUNCTIONS OF PERIOD $2P$.

In the same way, even functions $\mathrm{Ec}_n^{2m+1}(u)$ and odd functions $\mathrm{Es}_n^{2m+1}(u)$, both of period $2P$, may be shown to exist, each in two forms. In particular, the recurrence-relations for the even solution

$$y = \sum A_{2r+1} \cos (2r+1)v \qquad (4.1)$$

of (2.2) are

$$\begin{aligned}&\tfrac{1}{2}(n-2)(n+3)k^2 A_3 + \{\eta - (2-k^2) + \tfrac{1}{2}n(n+1)k^2\}A_1 = 0,\\ &\tfrac{1}{2}(n-2r-2)(n+2r+3)k^2 A_{2r+3} + \{\eta - (2r+1)^2(2-k^2)\}A_{2r+1}\\ &\qquad + \tfrac{1}{2}(n-2r+1)(n+2r)k^2 A_{2r-1} = 0 \qquad (r \geqslant 1).\end{aligned}$$

They lead to the characteristic equation:

$$\eta = -\tfrac{1}{2}n(n+1)k^2 + 2 - k^2 - \frac{(n^2-4)(n-1)(n+3)k^4/6^2}{2-k^2-\frac{1}{9}\eta} - \frac{(n^2-16)(n-3)(n+5)k^4/30^2}{2-k^2-\frac{1}{25}\eta} - \dots \qquad (4.2)$$

Those for the odd solution

$$y = \sum B_{2r+1} \sin (2r+1)v \qquad (4.3)$$

are the same, with B written for A, except the first, which is now

$$\tfrac{1}{2}(n-2)(n+3)k^2 B_3 + \{\eta - (2-k^2) - \tfrac{1}{2}n(n+1)k^2\}B_1 = 0.$$

Thus the characteristic equation is identical with (4.2), with the exception of the first term on the right, whose sign is changed. This leads to a very important conclusion, which will be reached towards the end of the following section.

Regarded as bases for computation, the characteristic equations given here and in the previous section are definitely superior in all non-terminating cases to those previously published, particularly for values of k^2 approaching unity. In fact they give the characteristic values of h in those cases, when $k^2=0{\cdot}9$, with less than half the labour involved in the earlier method. This is because the limiting value of the tail of the continued fractions in (3.2), etc., is $-\frac{1}{2}(1-k')^2$; *e.g.* $-0{\cdot}23377\ldots$ when $k^2=0{\cdot}9$, as compared with $-k^2$ before, and the more closely the numerical value of the tail approaches unity, the more difficult and sensitive to error does computation become.

5. THE COEXISTENCE QUESTION.

It is known that when n is an integer and h is such that one solution of the differential equation is a Lamé polynomial, the second solution cannot be periodic (Whittaker and Watson, 1927, § 23.47). It will now be proved that when n is not an integer, there can never be two independent solutions of period P or $2P$. The method is identical in principle with that by which the present author proved the non-coexistence of Mathieu functions of period π or 2π (*Proc. Camb. Phil. Soc.*, vol. xxi, 1922, p. 117). It is not applicable when the functions are expressed as series of powers of elliptic functions, but is effective only when the present series of orthogonal functions are employed.

If two independent periodic solutions actually did exist, they would be of the same period, P or $2P$ as the case may be. The non-existence proof is essentially the same in either case; considering that of period P, let it be assumed that for one and the same value of h, and therefore of η, the two solutions (4.1) and (4.3) exist simultaneously. The result of eliminating the central terms in the two general recurrence-relations may be written

$$\begin{vmatrix} A_{2r+3} & A_{2r+1} \\ B_{2r+3} & B_{2r+1} \end{vmatrix} = \frac{(2r-n-1)(2r+n)}{(2r-n+2)(2r+n+3)} \begin{vmatrix} A_{2r+1} & A_{2r-1} \\ B_{2r+1} & B_{2r-1} \end{vmatrix}. \qquad (5.1)$$

Repeated application of this relation gives

$$\begin{vmatrix} A_{2r+3} & A_{2r+1} \\ B_{2r+3} & B_{2r+1} \end{vmatrix} = \frac{(2r-n-1)(2r-n-3)\ \ldots\ (1-n)(2r+n)\ \ldots\ (2+n)}{(2r-n+2)(2r-n)\ \ldots\ (4-n)(2r+n+3)\ \ldots\ (5+n)} \begin{vmatrix} A_3 & A_1 \\ B_3 & B_1 \end{vmatrix}.$$

Hence, as $r \to \infty$,

$$\frac{\Gamma(2-\frac{1}{2}n)\Gamma(2\frac{1}{2}+\frac{1}{2}n)}{\Gamma(\frac{1}{2}-\frac{1}{2}n)\Gamma(1+\frac{1}{2}n)} \begin{vmatrix} A_3 & A_1 \\ B_3 & B_1 \end{vmatrix} = \lim \begin{vmatrix} A_{2r+3} & A_{2r+1} \\ B_{2r+3} & B_{2r+1} \end{vmatrix}. \qquad (5.2)$$

But since, as in § 3,

$$\lim \frac{A_{2r+3}}{A_{2r+1}} = \lim \frac{B_{2r+3}}{B_{2r+1}} = -\frac{(1-k')^2}{k^2},$$

the above limit is zero. Further, since n is not an integer, each of the Γ-functions is finite and distinct from zero; consequently

$$A_3B_1 - A_1B_3 = 0.$$

Now the elimination of η from the two leading recurrence-relations shows that

$$(n-2)(n+3)(A_3B_1 - A_1B_3) = -2n(n+1)A_1B_1,$$

whence $A_1B_1 = 0$. But the vanishing of A_1 or of B_1 implies the identical vanishing of the whole series of which it is the initial coefficient. Therefore solutions of the forms (4.1) and (4.3) cannot exist simultaneously, which establishes the theorem.

But in cases where n is a positive integer (or zero) and the solution does not terminate (*i.e.* when the index exceeds the suffix n), circumstances are entirely different. When $n = 2N$ or $2N-1$, (4.2) is replaced by the equation obtained by equating to zero the complete Nth denominator of the continued fraction. In particular, when $n = 2N$ the characteristic equation for the non-terminating even solutions of period $2P$ is

$$\eta = (2N+1)^2(2-k^2) - \cfrac{(2N+1)(4N+3)k^4/(2N+3)^2}{2-k^2-\eta/(2N+3)^2 - \cfrac{6(2N+2)(4N+5)k^4/(2N+3)^2(2N+5)^2}{2-k^2-\eta/(2N+5)^2 - \dots}} \tag{5.3}$$

Now the general characteristic equation for odd solutions of period $2P$ differs from (4.2) only in the leading term of the right-hand member, and in forming the equation corresponding to (5.3) this term is immaterial. Thus for the non-terminating odd solutions of period $2P$ we obtain a characteristic equation identical with (5.3). The characteristic numbers for the even and odd solutions of this type are therefore identical when $n = 2N$, *i.e.*

$$b_{2N}^{2m-1} = a_{2N}^{2m-1} \qquad \text{if } m > N,$$

and the even and odd solutions coexist.

Actually (5.3) is equivalent to the discriminant equation for the set of recurrence-relations in which $r \geqslant N$, both for the coefficients A_{2r+1} and for B_{2r+1}. But the roots of this equation for small k^2 are $\eta = 2(2r+1)^2 + O(k^2)$, where $r \geqslant N$. Now consider the first N recurrence-relations for the even functions; they involve the N coefficients $A_1, A_3, \dots, A_{2N-1}$, and no others. The discriminant equation for this set has the roots

$\eta=2(2r+1)^2+O(k^2)$, where $r<N$. They are therefore inconsistent with the characteristic values that are roots of (5.3), and consequently $A_{2r+1}=0$ for $r<N$. But the recurrence-relations for the remaining coefficients of the odd solutions are identical with those for the even solutions; taking the initial coefficient to be A_{2N+1} in both solutions, we conclude that, when $m>N$, the solutions

$$\mathrm{Ec}_{2N}^{2m-1}(u)=\sum_{r=N}^{\infty}A_{2r+1}\cos\{(2r+1)\operatorname{am}u\},$$

$$\mathrm{Es}_{2N}^{2m-1}(u)=\sum_{r=N}^{\infty}A_{2r+1}\sin\{(2r+1)\operatorname{am}u\},$$

exist for the same characteristic value of h.

Similarly, it may be shown that

$$b_{2N-1}^{2m-1}=a_{2N-1}^{2m-1}\qquad\text{if } m>N;$$

in this case the forms of the coexistent solutions most nearly analogous to the above are

$$\mathrm{Ec}_{2N-1}^{2m-1}(u)=\operatorname{dn}u\sum_{r=N}^{\infty}C_{2r+1}\cos\{(2r+1)\operatorname{am}u\},$$

$$\mathrm{Es}_{2N-1}^{2m-1}(u)=\operatorname{dn}u\sum_{r=N}^{\infty}C_{2r+1}\sin\{(2r+1)\operatorname{am}u\}.$$

Similar results are true with regard to the even and odd functions of period P. In fact

$$b_n^m=a_n^m\qquad\text{if } m>n,$$

for all positive integral (and zero) values of n. Thus for fixed m (and k) the graphs of a_n^m and b_n^m intersect *only* in the m points for which $n=0, 1, \ldots, m-1$, and in the symmetrically situated points for which $n=-1, -2, \ldots, -m$. For $-1<n<0$, $a_n^m>b_n^m$, whereas for large n, $a_n^m >$ or $< b_n^m$ according as m is even or odd.

6. DEVELOPMENT OF THE NUCLEUS OF AN INTEGRAL EQUATION.

In this present section, n will be restricted to be a positive integer, not only because interest mainly lies in that case, but also in order to avoid complicating the argument by questions of convergence. There should, however, be little difficulty in removing that restriction, if that ever became desirable.

It is known that $P_{2n}(k\operatorname{sn}u\operatorname{sn}t)$ is the nucleus of a homogeneous integral equation whose characteristic functions are the set of even Lamé polynomials of period $2K$:

$$\mathrm{Ec}_{2n}^{0}(u),\qquad \mathrm{Ec}_{2n}^{2}(u),\qquad \ldots,\qquad \mathrm{Ec}_{2n}^{2n}(u).\qquad (6.1)$$

The definition of these symbols as given in a recent paper already quoted determined the functions except for a constant multiplier. It is now absolutely necessary to complete the definition; this is achieved by adding the supplementary condition that

$$\int_0^{4K} \{\mathrm{Ec}_{2n}^{2m}(u)\}^2 du = \pi. \qquad . \quad . \quad . \quad . \quad (6.2)$$

The constants $a_0, \ldots, a_{2n}$ in the relation

$$P_{2n}(k \operatorname{sn} u \operatorname{sn} t) = \sum_{m=0}^{n} a_{2m} \mathrm{Ec}_{2n}^{2m}(u) \mathrm{Ec}_{2n}^{2m}(t)$$

can now be determined. Putting $t=0$, we have immediately

$$(-)^n \frac{(2n)!}{2^{2n}(n!)^2} = \sum_{m=0}^{n} a_{2m} \mathrm{Ec}_{2n}^{2m}(0) \mathrm{Ec}_{2n}^{2m}(u).$$

To determine a_{2m} this equation is multiplied through by $\mathrm{Ec}_{2n}^{2m}(u)$ and both members are integrated over the range $(0, 4K)$. In consequence of the orthogonal properties of the set of functions (6.1), coupled with the definition (6.2), the result obtained is

$$(-)^n \frac{(2n)!}{2^{2n}(n!)^2} \int_0^{4K} \mathrm{Ec}_{2n}^{2m}(u) du = \pi a_{2m} \mathrm{Ec}_{2n}^{2m}(0).$$

It is the difficulty of evaluating this integral that has hitherto blocked any further progress, but now if the Lamé function is expressed in terms of am u, the evaluation is simple. The coefficients of the developments in § 3 depend upon the particular characteristic number a_n^{2m} involved, that is to say, upon m. A fuller notation, explicitly indicating this dependence, will be necessary; thus

$$\begin{aligned} \mathrm{Ec}_{2n}^{2m}(u) &= A_0^{2m} + \sum_{r=1}^{m} A_{2r}^{2m} \cos 2rv \\ &= \Delta v \left\{ C_0^{2m} + \sum_{r=1}^{\infty} C_{2r}^{2m} \cos 2rv \right\}, \end{aligned}$$

where $v = \operatorname{am} u$.

Now $du = dv/\Delta v$, which necessitates the use of the second form, giving

$$\int_0^{4K} \mathrm{Ec}_{2n}^{2m}(u) du = 2\pi C_0^{2m}.$$

Thus a_{2m} is obtained, and the nuclear development is found to be

$$P_{2n}(k \operatorname{sn} u \operatorname{sn} t) = (-)^n \frac{(2n)!}{2^{2n-1}(n!)^2} \sum_{m=0}^{n} \frac{C_0^{2m}}{\mathrm{Ec}_{2n}^{2m}(0)} \mathrm{Ec}_{2n}^{2m}(u) \mathrm{Ec}_{2n}^{2m}(t). \quad . \quad (6.3)$$

In the same way it may be shown that

$$\operatorname{dn} u \operatorname{dn} t\, P_{2n}'(k \operatorname{sn} u \operatorname{sn} t)$$
$$= (-)^{n+1}\frac{(2n+2)!\,k}{2^{2n}(n-1)!\,(n+1)!}\sum_{m=1}^{n}\frac{B_1^{2m-1}}{\mathrm{Es}_{2n}^{2m-1\prime}(0)}\,\mathrm{Es}_{2n}^{2m-1}(u)\,\mathrm{Es}_{2n}^{2m-1}(t), \quad (6.4)$$

where

$$\mathrm{Es}_{2n}^{2m-1}(u) = \sum_{r=1}^{\infty} B_{2r-1}^{2m-1}\sin(2r-1)v$$
$$= \Delta v \sum_{r=1}^{m} D_{2r-1}^{2m-1}\sin(2r-1)v.$$

All other nuclei can be developed in the same manner.

If (6.3) is differentiated with respect to u, and P_{2n}' is eliminated between the result and (6.4), there arises the relation

$$\sum_{m=0}^{n}\frac{C_0^{2m}}{\mathrm{Ec}_{2n}^{2m}(0)}\frac{d}{du}\{\mathrm{Ec}_{2n}^{2m}(u)\}\,\mathrm{Ec}_{2n}^{2m}(t)$$
$$= -n(2n+1)k^2 \operatorname{cn} u \operatorname{sd} t \sum_{m=1}^{n}\frac{B_1^{2m-1}}{\mathrm{Es}_{2n}^{2m-1\prime}(0)}\,\mathrm{Es}_{2n}^{2m-1}(u)\,\mathrm{Es}_{2n}^{2m-1}(t).$$

Let each member be multiplied by $\mathrm{Ec}_{2n}^{2m}(t)/\pi$ and then integrated with respect to t over the range $(0, 4K)$; if the original m on the right is replaced by μ the equation becomes

$$\frac{C_0^{2m}}{\mathrm{Ec}_{2n}^{2m}(0)}\frac{d}{du}\{\mathrm{Ec}_{2n}^{2m}(u)\}$$
$$= -n(2n+1)k^2 \operatorname{cn} u \sum_{\mu=1}^{n}\frac{B_1^{2\mu-1}}{\pi\,\mathrm{Es}_{2n}^{2\mu-1\prime}(0)}\,\mathrm{Es}_{2n}^{2\mu-1}(u)\int_0^{4K}\mathrm{Es}_{2n}^{2\mu-1}(t)\,\mathrm{Ec}_{2n}^{2m}(t)\operatorname{sd} t\,dt.$$

The integral on the right has the value

$$\pi\left\{C_0^{2m}D_1^{2\mu-1} + \tfrac{1}{2}\sum_{r=1}^{m-1}C_{2r}^{2m}(D_{2r+1}^{2\mu-1} - D_{2r-1}^{2\mu-1}) - \tfrac{1}{2}C_{2m}^{2m}D_{2m-1}^{2\mu-1}\right\},$$

and thus the derivative of $\mathrm{Ec}_{2n}^{2m}(u)$ has been expressed linearly in terms of the n function $\operatorname{scn} u\,\mathrm{Es}_{2n}^{2\mu-1}(u)$, where $\mu = 1, 2, \ldots, n$. This is the analogue of a relation between elliptic-cylinder functions recorded in an earlier paper (*Proc. Roy. Soc. Edin.*, vol. lix, 1939, p. 181).

The examples of this and the preceding section will suffice to establish the fact that the expression of Lamé functions of period P or $2P$ as Fourier-Jacobi series renders manipulations possible that are impracticable with any other forms of the functions that have yet been discovered. The remainder of the paper will be devoted to functions of periods greater than $2P$.

7. PERIODIC SOLUTIONS IN GENERAL.

We assume a solution of the form

$$y = \sum A_r \cos (2r - \theta)v,$$

where θ is a rational number other than an integer. If $\theta = p/s$, where p is prime to s, the period will be $2sP$. The recurrence-relation has the general form

$$\tfrac{1}{2}(n - \theta + 2r + 2)(n + \theta - 2r - 1)k^2 A_{r+1} + \{\eta - (2r - \theta)^2(2 - k^2)\}A_r \\ + \tfrac{1}{2}(n - \theta + 2r - 1)(n + \theta - 2r + 2)k^2 A_{r-1} = 0, \qquad (7.1)$$

in which r assumes all integral values, positive, zero, and negative.

We set aside for the present the case in which θ is an odd multiple of $\frac{1}{2}$, and note that, in the entire set of relations, the coefficient of only one extreme term can vanish. But since the form of the assumed solution is unchanged when the sign of θ is reversed, or when θ is increased or decreased by an even integer, there are in reality only two distinct cases to consider, namely $\theta = n$ and $\theta = -n - 1$.

In either of these cases, the coefficient of A_0 in the relation between A_0, A_{-1}, and A_{-2} will be the sole vanishing coefficient; if A_{-1} is made zero, then A_{-2} will vanish, and so also will every coefficient with a negative suffix. Thus

$$y = \sum_{r=0}^{\infty} A_r \cos (2r - \theta)v$$

will be a solution if $\theta = n$ or $\theta = -n - 1$, and if the relations (7.1) are consistent for all positive values of r. The latter condition leads to the characteristic equation for the two cases in question, namely

$$\eta = \theta^2(2 - k^2) + \frac{\theta(1 - 2\theta)k^4/(2 - \theta)^2}{2 - k^2 - \eta/(2 - \theta)^2} - \frac{2 . 3(1 - \theta)(3 - 2\theta)k^4/(2 - \theta)^2(4 - \theta)^2}{2 - k^2 - \eta/(4 - \theta)^2} - \dots \qquad (7.2)$$

As in § 3, the continued fraction and the series may be proved convergent when $k^2 < 1$.

Thus when $n = p/s$, where $s > 2$, there are two possible forms of even solution, namely

$$y = \sum_{r=0}^{\infty} A_r \cos (2r - n)v, \qquad (7.3)$$

$$y = \sum_{r=0}^{\infty} B_r \cos (2r + n + 1)v, \qquad (7.4)$$

in which B_r is the same function of $-n - 1$ as A_r is of n. Both are of period $2sP$; the fundamental difference between them lies in their respective number of zeros in the interval $(0, sP)$. When s is odd, that number is of the same parity as p for (7.3) but of the opposite parity for (7.4). When s is even, the number of zeros is odd in both cases, but is congruent to 1 and to 3 (mod. 4) in (7.3) and (7.4) respectively. This is easily verified by considering the sign of y when $v = s\pi$, and when that is not conclusive, the sign of y' when $v = \frac{1}{2}s\pi$.

There is no difficulty in adapting the notation $\mathrm{Ec}_n^m(u)$ to cases in which the functions are of period $2sP$ (including $s = 2$). The suffix n is invariably the order; when the functions were of period P or $2P$ the index m indicated the number of zeros in the interval $(0, P)$ in which the pattern of zeros was complete. But in the present case the pattern is spread over an interval of length sP, and is presumably not symmetrical within that interval. If m is the number of zeros in any continuous interval of length sP (only one end-point being reckoned), we shall denote the even function by

$$\mathrm{Ec}_{p/s}^{m/s}(u),$$

where $n = p/s$. The notation is consistent with its use for the functions of period P or $2P$, and is definite, for the parity of m, or its congruence modulo 4, suffices to distinguish between (7.3) and (7.4).

Since (2.2) is unaltered when v is replaced by $\pi + v$, another solution is given by

$$\begin{aligned} y &= \sum_{r=0}^{\infty} A_r \cos(2r - \theta)(\pi + v) \\ &= \cos\theta\pi \sum_{r=0}^{\infty} A_r \cos(2r - \theta)v + \sin\theta\pi \sum_{r=0}^{\infty} A_r \sin(2r - \theta)v, \end{aligned}$$

so that, if θ is not an integer, a second and obviously independent solution is

$$y = \sum_{r=0}^{\infty} A_r \sin(2r - \theta)v.$$

Thus simultaneously with (7.3) and (7.4) respectively we have the two odd solutions

$$y = \sum_{r=0}^{\infty} A_r \sin(2r - n)v, \qquad . \quad . \quad . \quad . \quad (7.5)$$

$$y = \sum_{r=0}^{\infty} B_r \sin(2r + n + 1)v, \qquad . \quad . \quad . \quad . \quad (7.6)$$

which may be denoted by $\mathrm{Es}_{p/s}^{m/s}(u)$.

The investigation of this section is valid also in cases where n, and therefore θ, is irrational; a pair of independent solutions can always be found to correspond to each characteristic number determined from (7.2), but as the solutions are not periodic, the notation would be something like $\mathrm{Ec}_n^{mn}(u)$, where m is a rational number.

8. SOLUTIONS OF PERIOD $4P$.

There will be no loss of generality in assuming $\theta = \frac{1}{2}$; the general recurrence-relation for

$$y = \sum A_r \cos (2r - \tfrac{1}{2})v \qquad . \quad . \quad . \quad . \quad (8.1)$$

is

$$\tfrac{1}{2}(2n + 4r + 3)(2n - 4r - 1)k^2 A_{r+1} + \{4\eta - (4r - 1)^2(2 - k^2)\}A_r \\ + \tfrac{1}{2}(2n + 4r - 3)(2n - 4r + 5)k^2 A_{r-1} = 0. \quad . \quad (8.2)$$

On the left, the coefficients of A_{p+1} and A_{-p} vanish when $2n = 4p + 1$; on the right, those of A_p and A_{-p} vanish when $2n = 4p - 1$. Thus when $2n$ is any odd integer, other than -1, two of the extreme coefficients vanish, both on the left or both on the right.

When $2n = 4p + 1$, we take $A_{-p-1} = 0$ and deduce that $A_r = 0$ for $r < -p - 1$. The recurrence-relations for $r = -p, -p + 1, \ldots, p$ furnish $2p + 1$ homogeneous linear relations between the coefficients $A_{-p}, A_{-p+1}, \ldots, A_p$; elimination of these coefficients leads to an algebraic equation of degree $2p + 1$ or $n + \frac{1}{2}$ in η whose roots are characteristic numbers. The characteristic functions are not, however, obtained in a finite form, as they involve not only the coefficients from A_{-p} to A_p, but also the coefficients A_r for $r > p$, which are infinite in number. Similarly, when $2n = 4p - 1$ $(p \neq 0)$ we take $A_{p+1} = 0$ and proceed in the same way; the characteristic equation is of degree $2p$, or again $n + \frac{1}{2}$.

To obtain the characteristic functions in a finite form we have to proceed on other lines, a clue to which is found by applying the above process to the case $n = \frac{1}{2}$.

When $n = \frac{1}{2}$, (8.2) becomes

$$8r(r + 1)k^2 A_{r+1} - \{4\eta - (4r - 1)^2(2 - k^2)\}A_r + 2(2r - 1)(2r - 3)k^2 A_{r-1} = 0.$$

We take $A_{-1} = 0$ and then every A_r with a negative suffix vanishes. The equation with $r = 0$ gives the characteristic equation

$$4\eta = 2 - k^2, \quad \text{where} \quad \eta = 2h - \tfrac{3}{4}k^2$$

or

$$h = \tfrac{1}{4}(1 + k^2).$$

The set of recurrence-relations now reduces to

$$4r(r + 1)k^2 A_{r+1} + 4r(2r - 1)(2 - k^2)A_r + (2r - 1)(2r - 3)k^2 A_{r-1} = 0$$

for $r \geqslant 1$. Here there is a fundamental difference from the cases of functions of periods P and $2P$, for whereas in those cases the leading relation contained two terms, now it involves three. The consequence is that $A_1, A_2, \ldots$ are not determined directly in terms of an arbitrary A_0, but indirectly as a consequence of the necessary consistency of the relations.

In fact A_1/A_0 is obtained by eliminating the remaining ratios A_2/A_1, A_3/A_2, . . . from the entire set of relations. The result is best expressed by means of a continued fraction; when simplified it reads

$$\frac{A_1}{A_0}=\frac{k^2/4}{2-k^2}-\frac{k^4/4}{2-k^2}-\dots$$

As the continued fraction recurs, from the second component fraction onwards, it may be evaluated; we find

$$\frac{A_1}{A_0}=\frac{(1-k')^2}{2k^2}=\rho \qquad \text{(say).}$$

Similarly, discarding the first relation, then the first two, the first three, and so on, and expressing the condition that the remainder are consistent, we find in turn that

$$\frac{A_2}{A_1}=-\tfrac{1}{2}\rho, \qquad \frac{A_3}{A_2}=-\rho, \qquad \frac{A_4}{A_3}=-\tfrac{5}{4}\rho, \qquad \dots,$$

and in general

$$\frac{A_{r+1}}{A_r}=-\frac{2r-1}{r+1}\rho.$$

Thus if we take $A_0=1$ and denote the solution for the moment as $y=C(v)$, we have

$$C(v)=\cos\frac{v}{2}+\rho\cos\frac{3v}{2}-\tfrac{1}{2}\rho^2\cos\frac{7v}{2}+\frac{1\cdot 3}{2\cdot 3}\rho^3\cos\frac{11v}{2}-\frac{1\cdot 3\cdot 5}{2\cdot 3\cdot 4}\rho^4\cos\frac{15v}{2}+\dots$$

The second solution is $S(v)=C(\pi+v)$, that is

$$S(v)=-\sin\frac{v}{2}+\rho\sin\frac{3v}{2}-\tfrac{1}{2}\rho^2\sin\frac{7v}{2}+\frac{1\cdot 3}{2\cdot 3}\rho^3\sin\frac{11v}{2}-\frac{1\cdot 3\cdot 5}{2\cdot 3\cdot 4}\rho^4\sin\frac{15v}{4}+\dots$$

So if $z=e^{iv}$,

$$C+iS=z^{-\frac{1}{2}}\left(1+\rho z^2-\tfrac{1}{2}\rho^2z^4+\frac{1\cdot 3}{2\cdot 3}\rho^3z^6-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\rho^4z^8+\dots\right)$$
$$=(z^{-1}+2\rho z)^{\frac{1}{2}}=\{(2\rho+1)\cos v+i(2\rho-1)\sin v\}^{\frac{1}{2}}.$$

Thus we have, on squaring,

$$2CS=(2\rho-1)\sin v=-(2\rho+1)k'\sin v, \qquad C^2-S^2=(2\rho+1)\cos v,$$

whence we obtain

$$C^2+S^2=(2\rho+1)(1-k^2\sin^2 v)^{\frac{1}{2}}=(2\rho+1)\Delta v.$$

Discarding a constant factor, we thus have two independent solutions of the Lamé equation when $n=\frac{1}{2}$, $h=\frac{1}{4}(1+k^2)$:

$$y=\sqrt{(\Delta v+\cos v)}, \qquad y=\sqrt{(\Delta v-\cos v)},$$

or, reverting to the original variable u,

$$y=\sqrt{(\operatorname{dn} u+\operatorname{cn} u)}, \qquad y=\sqrt{(\operatorname{dn} u-\operatorname{cn} u)};$$

which may also be written (again dropping a constant multiplier),

$$y=\frac{\operatorname{cn}\frac{1}{2}u \operatorname{dn}\frac{1}{2}u}{\sqrt{(1-k^2\operatorname{sn}^4\frac{1}{2}u)}}, \qquad y=\frac{\operatorname{sn}\frac{1}{2}u}{\sqrt{(1-k^2\operatorname{sn}^4\frac{1}{2}u)}}.$$

In accordance with the notation proposed in § 7, these solutions will be denoted respectively by

$$\mathrm{Ec}_{\frac{1}{2}}^{\frac{1}{2}}(u), \qquad \mathrm{Es}_{\frac{1}{2}}^{\frac{1}{2}}(u);$$

the characteristic number will be $a_{\frac{1}{2}}^{\frac{1}{2}}=\frac{1}{4}(1+k^2)$.

The last pair of solutions represents the ultimate simplification of a complex result recorded by Jamet (*C.R. Acad. Sci. Paris*, vol. cxi, 1890, p. 639), and quoted by Whittaker and Watson (chap. xxiii, Ex. 18).

The extension to higher values of n, given in the following sections, has the advantage over all previous methods that it separates the even from the odd part of the solution. As there is a notable difference between the cases $2n=4p+1$ and $2n=4p-1$, these will be considered separately.

9. SOLUTIONS OF FINITE FORM WHEN $2n=4p+1$.

When the Lamé equation in its Jacobian form (2.1) is transformed by the substitution

$$y=w\sqrt{(\operatorname{dn} u+\operatorname{cn} u)},$$

and we write H for $h-\frac{1}{4}(1+k^2)$ and s, c, d for sn u, cn u, dn u respectively, it becomes

$$\frac{d^2w}{du^2}-\frac{1-cd}{s}\frac{dw}{du}+\{H-(n-\tfrac{1}{2})(n+\tfrac{3}{2})k^2s^2\}w=0. \qquad . \qquad . \qquad (9.1)$$

Taking $n=2p+\frac{1}{2}$, we assume

$$w=\sum A_r s^{2r}+cd\sum B_r s^{2r},$$

and obtain the double set of recurrence-relations

$$(2r+2)^2A_{r+1}+\{H-2r(2r+1)(1+k^2)\}A_r-4(p-r+1)(p+r)k^2A_{r-1}$$
$$=(2r+2)B_{r+1}-(2r+1)(1+k^2)B_r+2rk^2B_{r-1}, \quad (9.2a)$$
$$(2r+2)^2B_{r+1}+\{H-(2r+1)(2r+2)(1+k^2)\}B_r-4(p-r)(p+r+1)k^2B_{r-1}$$
$$=(2r+2)A_{r+1}. \quad (9.2b)$$

These relations hold for $r \geqslant 0$ provided that A_{-1} and B_{-1} are interpreted as zeros.

Assuming that $B_p=A_{p+1}=0$, it is easy to deduce that $B_r=A_{r+1}=0$

when $r > p$. Then to each of the $2p+1$ values of H for which the remaining $p+1$ equations (9.2*a*) together with the p remaining equations (9.2*b*) are consistent, there corresponds an even Lamé function of the form

$$\mathrm{Ec}_{2p+\frac{1}{2}}^{m+\frac{1}{2}}(u) = (\mathrm{dn}\, u + \mathrm{cn}\, u)^{\frac{1}{2}}\left\{\sum_{r=0}^{p} A_r \,\mathrm{sn}^{2r} u + \mathrm{cn}\, u\, \mathrm{dn}\, u \sum_{r=0}^{p-1} B_r \,\mathrm{sn}^{2r} u\right\},$$

where m is the number of zeros in the open interval $(0, 2K)$. The corresponding odd function will be $\mathrm{Ec}_{2p+\frac{1}{2}}^{m+\frac{1}{2}}(u+2K) = \mathrm{Es}_{2p+\frac{1}{2}}^{m+\frac{1}{2}}(u)$; that is, the above expression with $\mathrm{cn}\, u$ changed into $-\mathrm{cn}\, u$. The characteristic number associated with this pair of solutions is $h = a_{2p+\frac{1}{2}}^{m+\frac{1}{2}}$.

The number m may have any integral value from 0 to $2p$ inclusive. For the product of the two solutions is a polynomial in $\mathrm{sn}\, u$ of degree $4p+1$, and therefore has $4m+2$ zeros in the interval $0 \leqslant u < 4K$, where $m \leqslant 2p$. But by the Sturmian theory, the two solutions have exactly the same number of zeros in that interval, namely, $2m+1$. But there are $4p+1$ characteristic numbers, and the number of zeros of solutions corresponding to these are all unequal. Thus m cannot have the same value for two characteristic numbers, but assumes all integral values from 0 to $2p$, which was to be proved.

10. SOLUTIONS OF FINITE FORM WHEN $2n = 4p - 1$.

In this case we assume a solution

$$w = d \sum A_r s^{2r} + c \sum B_r s^{2r},$$

and find the recurrence-relations to be

$$(2r+2)^2 A_{r+1} + \{H - (2r+1)(2r+2)k^2 - 2r(2r+1)\}A_r - 4(p^2 - r^2)k^2 A_{r-1}$$
$$= (2r+2)B_{r+1} - (2r+1)B_r, \quad (10.1a)$$

$$(2r+2)^2 B_{r+1} + \{H - 2r(2r+1)k^2 - (2r+1)(2r+2)\}B_r - 4(p^2 - r^2)k^2 B_{r-1}$$
$$= (2r+2)A_{r+1} - (2r+1)k^2 A_r, \quad (10.1b)$$

for $r \geqslant 0$, with $A_{-1} = B_{-1} = 0$.

Taking $B_p = A_p = 0$, we obtain $B_r = A_r = 0$ when $r > p$. There remain p equations of each set; to each of the $2p$ values of H for which the whole system is consistent there corresponds an even Lamé function

$$\mathrm{Ec}_{2p-\frac{1}{2}}^{m+\frac{1}{2}}(u) = (\mathrm{dn}\, u + \mathrm{cn}\, u)^{\frac{1}{2}}\left\{\mathrm{dn}\, u \sum_{r=0}^{p-1} A_r \,\mathrm{sn}^{2r} u + \mathrm{cn}\, u \sum_{r=0}^{p-1} B_r \,\mathrm{sn}^{2r} u\right\},$$

where, as before, m denotes the number of zeros in the open interval $(0, 2K)$; m assumes all integral values from 0 to $2p-1$, inclusive. The odd function $\mathrm{Es}_{2p-\frac{1}{2}}^{m+\frac{1}{2}}(u)$ is derived as in § 9; the characteristic number is $h = a_{2p-\frac{1}{2}}^{m+\frac{1}{2}}$.

Thus we have demonstrated the existence of two distinct types of terminating solutions which may arise when $2n$ is an odd integer. The method is not very amenable to computation; in particular it is difficult to obtain H by elimination of the coefficients.

11. DETERMINATION OF h.

Separate characteristic equations for the cases $2n=4p+1$ and $2n=4p-1$ can be obtained in the usual way from the set of equations (8.2). Another form of equation which may be used in both these cases is found by considering the differential equation of the third order whose fundamental solutions are the squares and product of two independent solutions of the Lamé equation (2.1). This is the equation

$$\frac{d^3z}{du^3}+4\{h-n(n+1)k^2\,\mathrm{sn}^2\,u\}\frac{dz}{du}-4n(n+1)k^2\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u\;z=0. \quad . \quad (11.1)$$

It has always one solution (unique apart from a constant factor) which is an odd function of u. In view of the form of the general solution of the Lamé equation for arbitrary h (Whittaker and Watson, § 23.71), this odd solution will be a function of period $4K$ when, and only when, the Lamé equation has two distinct solutions of period $8K$, and that is precisely the case we have in mind.

We therefore substitute in (11.1) a series of the form

$$z=\sum A_r\,\mathrm{sn}^{2r+1}\,u$$

and obtain the recurrence-relations

$$(2r+1)(2r+2)(2r+3)A_{r+1}+(2r+1)\{4h-(2r+1)^2(1+k^2)\}A_r \\ +2r(2r-2n-1)(2r+2n+1)k^2A_{r-1}=0,$$

for $r>0$ (taking $A_{-1}=0$). Thence we deduce the characteristic equation

$$h=\frac{1+k^2}{4}+\frac{(2n-1)(2n+3)k^2/9}{1+k^2-\frac{4}{9}h}+\frac{16(2n-3)(2n+5)k^2/225}{1+k^2-\frac{4}{25}h}+\frac{36(2n-5)(2n+7)k^2/1225}{1+k^2-\frac{4}{49}h}+\ldots \quad (11.2)$$

in which the continued fraction terminates whenever $2n$ is an odd integer, other than -1.

The case $2n=1$ was fully considered in § 8. When $2n=3$,

$$a_{\frac{3}{2}}^{\frac{1}{2}}=\tfrac{5}{4}(1+k^2)-(1-k^2+k^4)^{\frac{1}{2}},$$

$$\mathrm{Ec}_{\frac{3}{2}}^{\frac{1}{2}}(u)=(\mathrm{dn}\,u+\mathrm{cn}\,u)^{\frac{1}{2}}\,[\mathrm{dn}\,u-\{1-k^2-(1-k^2+k^4)^{\frac{1}{2}}\}\mathrm{cn}\,u];$$

$a_{\frac{3}{2}}^{\frac{3}{2}}$ and the corresponding function are obtained by changing the sign of the radical.

12. VALUES OF $a_n^{m+\frac{1}{2}}$.

The following short table of values of h corresponding to the finite forms of §§ 9, 10 was obtained by the use of (11.2). This method is not easy to use for the higher values of m when k^2 approaches unity. It is likely that a method based on (8.2) would be more manageable in those cases, and also in computing the characteristic numbers of non-terminating solutions of period $8K$ which are not included in this table.

n	$a_n^{\frac{1}{2}}$	$a_n^{1\frac{1}{2}}$	$a_n^{2\frac{1}{2}}$	$a_n^{3\frac{1}{2}}$	$a_n^{4\frac{1}{2}}$
			$k^2=0{\cdot}1$		
$\frac{1}{2}$	0·2750				
$1\frac{1}{2}$	0·4211	2·3289			
$2\frac{1}{2}$	0·6473	2·5983	6·3794		
$3\frac{1}{2}$	0·9302	2·9944	6·7448	12·4305	
$4\frac{1}{2}$	1·2443	3·5278	7·2239	12·8971	20·4818
			$k^2=0{\cdot}5$		
$\frac{1}{2}$	0·3750				
$1\frac{1}{2}$	1·0090	2·7410			
$2\frac{1}{2}$	1·7292	4·3750	7·0208		
$3\frac{1}{2}$	2·4438	6·4600	9·2900	13·3062	
$4\frac{1}{2}$	3·1539	8·6371	12·3750	16·1129	21·5961
			$k^2=0{\cdot}9$		
$\frac{1}{2}$	0·4750				
$1\frac{1}{2}$	1·4211	3·3289			
$2\frac{1}{2}$	2·3706	6·1517	8·1027		
$3\frac{1}{2}$	3·3195	9·0052	12·7555	14·8198	
$4\frac{1}{2}$	4·2682	11·8529	17·5261	21·2222	23·5054

Here, as in the tables previously published, the case $k^2=\frac{1}{2}$ has features that have not yet been explained. It may, for instance, be conjectured that when $2n$ is an odd integer,

$$a_n^{m+\frac{1}{2}}+a_n^{n-m}=n(n+1),$$

and, in particular, when $2n=4p+1$,

$$a_{2p+\frac{1}{2}}^{p+\frac{1}{2}}=\tfrac{1}{8}(4p+1)(4p+3).$$

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VIII.—Reciprocity. Part II: Scalar Wave Functions. By **Professor Max Born,** F.R.S., University of Edinburgh, and **Klaus Fuchs,** D.Sc., Carnegie Research Fellow. (With Two Figures.)

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1. INTRODUCTION.

THIS paper contains an attempt to formulate more rigorously the postulate of reciprocity (Born, 1939) and to solve the problem of finding reciprocal wave functions. We have not yet completely succeeded in this task, though considerable progress has been made; but it turns out that the original discussion has to be modified in several important respects.

In the first place, it had not been observed that the 4 vectors **r**, t and **p**, E have different characters, the former being space-like ($r^2 > t^2$), the latter being time-like ($E^2 > p^2$). Instead of the equations (A) and (B) of Part I (which are also those of Landé, 1939), we have therefore

$$E^2 - p^2 = \epsilon^2, \quad . \quad . \quad . \quad . \quad . \quad (A)$$

$$r^2 - t^2 = a^2, \quad . \quad . \quad . \quad . \quad . \quad (B)$$

which represent hyper-surfaces whose sections by hyperplanes through the axis of symmetry are hyperboloids. We call these hyper-surfaces "pseudo-spheres," although this expression is used in another sense in differential geometry. The constant ϵ is a characteristic energy, the constant a a characteristic length; the precise physical interpretation of these constants cannot be given at the present stage.

The equations (A) and (B) have two roots, viz.

$$E = \pm\sqrt{p^2 + \epsilon^2}, \qquad t = \pm\sqrt{r^2 - a^2}.$$

The occurrence of negative energy values is known from Dirac's theory of the electron. A similar feature appears here with respect to the time; we have to take account of retarded and advanced times of propagation.

The last point is concerned with the formulation of the fundamental postulate (Part I, (3)). In the previous work two wave functions, $\psi(\mathbf{r}, t)$ and $\phi(\mathbf{p}, E)$, were introduced which satisfy the wave equations obtained from (A) and (B):

$$(E^2 - p^2 - \epsilon^2)\psi = 0, \qquad (t^2 - r^2 + a^2)\phi = 0, \quad . \quad . \quad . \quad (1.1)$$

where

$$E = -\frac{h}{i}\frac{\partial}{\partial t}, \quad p_x = \frac{h}{i}\frac{\partial}{\partial x}, \ldots$$

and

$$t = \frac{h}{i}\frac{\partial}{\partial E}, \quad x = -\frac{h}{i}\frac{\partial}{\partial p_x}, \ldots$$

In Part I we have represented each of the functions ψ, ϕ by a Fourier integral, for instance

$$\psi(\mathbf{r}, t) = \frac{1}{h^{3/2}}\int \Phi(\mathbf{p}) e^{\frac{i}{h}(\mathbf{r}\cdot\mathbf{p} - t\sqrt{\epsilon^2+p^2})} d\bar{p}, \quad . \quad . \quad . \quad (1.2)$$

where $\Phi(\mathbf{p})$ is a function of $\mathbf{p}$ alone, *a priori* quite independent of the wave function $\phi(\mathbf{p}, E)$. Our postulate consisted in identifying these two functions on the pseudo-sphere (A). This led to an integral equation for each of these functions.

The integral in (1.2) is not relativistically invariant. In order to make it so, we now form our Fourier integral over the relativistically invariant hyper-surfaces

$$r^2 - t^2 = R^2, \quad E^2 - p^2 = P^2. \quad . \quad . \quad . \quad . \quad (1.3)$$

This is equivalent to replacing the volume elements in $\mathbf{r}$-space and $\mathbf{p}$-space by

$$d\tilde{r} = \frac{R\,d\bar{r}}{\sqrt{(r^2 - R^2)}}, \quad d\tilde{p} = \frac{P\,d\bar{p}}{\sqrt{(p^2 + P^2)}}, \quad . \quad . \quad . \quad (1.4)$$

respectively.

Further, in the formulation of the postulate, we prefer here a more fundamental procedure. For this purpose we introduce the conception of a "*relativistically invariant Fourier integral,*" for which the Fourier coefficient is also a function of $\mathbf{p}$ *and* E.

It is very convenient here to introduce a notation similar to that of quantum mechanics. If ψ_1 and ψ_2 are two ψ-functions, we write

$$(\psi_1 \cdot \psi_2)_R = \int_{r^2 - t^2 = R^2} \psi_1^*(\mathbf{r}, t)\psi_2(\mathbf{r}, t) d\tilde{r}, \quad . \quad . \quad . \quad (1.5)$$

where integration is over the pseudo-sphere (1.3). Similarly for two ϕ-functions we have

$$(\phi_1 \cdot \phi_2)_P = \int_{E^2 - p^2 = P^2} \phi_1^*(\mathbf{p}, E)\phi_2(\mathbf{p}, E) d\tilde{p}, \quad . \quad . \quad . \quad (1.6)$$

where the integration extends over both sheets of the pseudo-sphere $E^2 - p^2 = P^2$. We call these integrals "*relativistic scalar products.*"

We consider particularly normalised plane waves

$$\frac{1}{h^{3/2}} e^{\frac{i}{h}S}, \qquad S = \mathbf{r} \cdot \mathbf{p} - Et, \quad . \quad . \quad . \quad . \quad (1.7)$$

and form the scalar products of these and other ψ- and ϕ-functions:

$$\left(\frac{1}{h^{3/2}} e^{\frac{i}{h}S} \cdot \psi\right)_R = \Phi^{(R)}, \qquad \left(\frac{1}{h^{3/2}} e^{-\frac{i}{h}S} \cdot \phi\right)_P = \Psi^{(P)}. \quad . \quad . \quad (1.8)$$

If we choose in particular $R = a$ and $P = \epsilon$, $\Phi^{(a)}$ and $\Psi^{(\epsilon)}$ are solutions of the wave equations (1.1).

The scalar products on the left-hand side of (1.8) are in general non-analytic functions. To make this clear we shall denote in $\mathbf{x}$, t-space

$$\left.\begin{array}{lll} \text{the region} & r^2 > t^2 & \text{by (I),} \\ \text{the region} & r^2 < t^2 & \text{by (II),} \end{array}\right\} \quad . \quad . \quad . \quad (1.9)$$

corresponding to the fact that the pseudo-sphere has one sheet in (I) and two sheets in (II). And correspondingly in $\mathbf{p}$, E-space. We shall show that for a certain class of functions $\Phi^{(R)}$ vanishes identically in (II) but not in (I). This makes the simple procedure suggested in Part I impossible, namely, to construct an integral equation by identifying ϕ with $\Phi^{(a)}$ (originally defined only on the pseudo-sphere in (II)) and ψ with $\Psi^{(\epsilon)}$.

We now define the Fourier coefficient on R of a function ψ by the scalar product as given in (1.8) *for the region (I) and by its analytical continuation in (II).*

For the ϕ-functions the situation is similar.

We have calculated the Fourier coefficients for certain proper functions and found that the Fourier coefficients of a ψ-function is indeed a ϕ-function of the same type, but not *vice versa*. We have not been able yet to understand this difference.

The wave equation satisfied by $\psi(\mathbf{r}, t)$ is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} - \frac{\epsilon^2}{h^2}\right)\psi = 0.$$

The most direct mathematical way of solving this equation is to introduce polar co-ordinates as defined in § 3 and then separate the variables. However, the separation of variables has a physical meaning. It is therefore more instructive to carry it through by introducing the angular momentum 6-vector (**M**, **N**), where the space vector **M** is the common angular momentum and **N** is a vector with the components $(tp_x - x\mathrm{E}, tp_y - y\mathrm{E}, tp_z - z\mathrm{E})$. This procedure will also facilitate the extension of our results to spinor wave functions. We shall show that the operators M_z, M^2, and $K^2 = M^2 - N^2$ possess simultaneous proper functions of integrable square with the proper values $\hbar m$, $\hbar^2 l(l+1)$, $\hbar^2 k(k+2)$, where m, l, k are integers and $|m| < l$, $0 < k < l-1$; and we develop in detail some of the properties of these operators.

We shall treat here only scalar wave functions, although in fact the wave functions of all elementary particles are spinors. This is done not only for the sake of simplicity, but also because the solution of the scalar problem is indispensable for the treatment of more involved cases (spinors of various ranks, which will be the subject of a later paper).

2. WAVE OPERATOR AND ANGULAR MOMENTUM.

We introduce in the usual way 4-dimensional co-ordinates:

$$\begin{array}{lll} t, x, y, z & \text{correspond to} & x^k, \\ E, p_x, p_y, p_z & \text{correspond to} & p^k; \end{array} \qquad k=0,1,2,3 \qquad (2.1)$$

the metric tensor is given by

$$\begin{array}{l} g_{kl}=0 \quad \text{for} \quad k \neq l, \\ g_{00}=-1;\ g_{kk}=1,\ k\neq 0. \end{array} \qquad g_k^l=\begin{Bmatrix} 1 & \text{for} & k=l \\ 0 & \text{for} & k\neq l \end{Bmatrix} \quad . \qquad (2.2)$$

Then, for instance, $-E$, p_x, p_y, p_z correspond to $p_k=g_{kl}p^l$.

We consider the x^k, p^k as operators satisfying the commutation laws

$$[p^k, x^l]=\frac{\hbar}{i}g^{kl}, \quad \text{or} \quad [p_k, x^l]=\frac{\hbar}{i}g_k^l. \quad . \quad . \quad . \qquad (2.3)$$

They can be represented

$$\left.\begin{array}{l} \text{in } x\text{-space by} \quad p_k=\dfrac{\hbar}{i}\dfrac{\partial}{\partial x^k}, \\ \text{in } p\text{-space by} \quad x^k=-\dfrac{\hbar}{i}\dfrac{\partial}{\partial p_k}. \end{array}\right\} \quad . \quad . \quad . \quad . \qquad (2.4)$$

We define the invariants

$$\left.\begin{array}{l} R^2=r^2-t^2=x_k x^k, \\ P^2=E^2-p^2=-p_k p^k, \\ S=\mathbf{r}\,.\,\mathbf{p}-tE=x^k p_k. \end{array}\right\} \quad . \quad . \quad . \quad . \qquad (2.5)$$

The first two of them are the left-hand terms of (A) and (B); the third is the quantity defined in (1.7) considered as operator.

We define now the operator "angular momentum" in 4 dimensions:

$$M^{kl}=x^k p^l - x^l p^k, \quad . \quad . \quad . \quad . \qquad (2.6)$$

which is a self-reciprocal 6-vector; three components, namely for $k, l=1, 2, 3$, are the 3-dimensional angular momentum

$$M_x=yp_z-zp_y=x^2p_3-x^3p_2, \ldots \quad . \quad . \quad . \qquad (2.7)$$

the other three can be considered as components of a 3-vector $\mathbf{N}$:

$$N_x=tp_x-xE=x^0p^1-x^1p^0, \ldots \quad . \quad . \quad . \qquad (2.8)$$

We consider the invariant

$$K^2 = M^2 - N^2 = \tfrac{1}{2}M_{kl}M^{kl} = \tfrac{1}{2}g_{km}g_{ln}M^{mn}M^{kl}. \qquad (2.9)$$

Using the definition (2.6) and the commutation law (2.3), we get with (2.5)

$$K^2 = M^2 - N^2 = -\left(R^2P^2 + S^2 + 2\frac{\hbar}{i}S\right). \qquad (2.10)$$

The equation

$$-R^2P^2 = S^2 + 2\frac{\hbar}{i}S + K^2 \qquad (2.11)$$

can be considered as the expression of the *wave operator*

$$P^2 = \hbar^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}\right)$$

in terms of the angular momentum.*

Taking the adjoint of (2.11), we get

$$-P^2R^2 = S^{\dagger 2} - \frac{2\hbar}{i}S^\dagger + K^2, \qquad (2.12)$$

where

$$S^\dagger = p_k x^k = \mathbf{p} \cdot \mathbf{r} - Et \qquad (2.13)$$

is the adjoint operator to S (P, R, **M**, **N** being self-adjoint). (2.12) can be considered as the expression of the wave operator in p-space

$$R^2 = -\hbar^2\left(\frac{\partial^2}{\partial p_x^2} + \frac{\partial^2}{\partial p_y^2} + \frac{\partial^2}{\partial p_z^2} - \frac{\partial^2}{\partial E^2}\right)$$

in terms of the angular momentum.

3. TRANSFORMATION TO 4-DIMENSIONAL POLAR CO-ORDINATES.

We write

$$\left.\begin{aligned} x^k &= R\rho^k, & \rho_k\rho^k &= 1, \\ p_k &= P\varpi_k, & \varpi_k\varpi^k &= -1. \end{aligned}\right\} \qquad (3.1)$$

Here ρ^k and ϖ_k are unit vectors in $\mathbf{x}$, t- and $\mathbf{p}$, E-space respectively.

Then

$$S = \frac{\hbar}{i}R\frac{\partial}{\partial R}. \qquad (3.2)$$

Observing that $x^k = -\hbar/i\ \partial/\partial p_k$, we find in the same way

$$S^\dagger = -\frac{\hbar}{i}P\frac{\partial}{\partial P}. \qquad (3.3)$$

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* This derivation holds for any number of dimensions, with the only modification that the factor of S in (2.11) depends on the number of dimensions n; instead of 2S (for $n = 4$) we have in general $(n-2)$S.

The two wave operators become now

$$\mathrm{P}^2 = \hbar^2\left(\frac{\partial^2}{\partial \mathrm{R}^2} + \frac{3}{\mathrm{R}}\frac{\partial}{\partial \mathrm{R}}\right) - \frac{\mathrm{K}^2}{\mathrm{R}^2}, \qquad (3{\cdot}4)$$

$$\mathrm{R}^2 = \hbar^2\left(\frac{\partial^2}{\partial \mathrm{P}^2} + \frac{3}{\mathrm{P}}\frac{\partial}{\partial \mathrm{P}}\right) - \frac{\mathrm{K}'^2}{\mathrm{P}^2}, \qquad (3{\cdot}5)$$

where $\mathrm{K}^2 = \mathrm{M}^2 - \mathrm{N}^2$ and $\mathrm{K}'^2 = \mathrm{M}'^2 - \mathrm{N}'^2$ are two different representations of the same operator, one as differential operator in $\mathbf{x}$, t-space, the other in $\mathbf{p}$, E-space.

We now introduce polar co-ordinates. We shall give all formulæ in $\mathbf{x}$, t-space in detail, and add at the end a remark about the corresponding expressions in $\mathbf{p}$, E-space. We put

$$\left.\begin{array}{ll} x^1 = r \sin\beta \cos\gamma, & \\ x^2 = r \sin\beta \sin\gamma, & r = \mathrm{R}\cosh\alpha, \\ & x^0 = \mathrm{R}\sinh\alpha. \\ x^3 = r\cos\beta, & \end{array}\right\} \qquad (3{\cdot}6)$$

Here β, γ are ordinary polar angles in the spatial part of space-time and so have the ranges of values

$$0 < \beta < \pi, \qquad 0 < \gamma < 2\pi.$$

For R and α we have the relations

$$\tanh\alpha = \frac{t}{r} = u, \qquad \mathrm{R} = \sqrt{r^2 - t^2} = r\sqrt{1 - u^2}. \qquad (3{\cdot}7)$$

In region (I) we have $r^2 > t^2$, and so R is real and never vanishes or changes sign. We suppose then that R is positive. Moreover, in (I), $r = \mathrm{R}\cosh\alpha$ is positive, but $t = \mathrm{R}\sinh\alpha$ can have either sign; hence α is real in (I).

But in (II) we have $r^2 < t^2$, and so R is purely imaginary; hence the values taken by $\cosh\alpha$ and $\sinh\alpha$ are also purely imaginary. To determine the appropriate ranges of values of α and R, we consider how α and R vary when the radius-vector in space-time rotates in a hyperplane containing the t-axis.

The region (II) consists of two separate regions: the region (II$^+$) interior to the light cone in which $t > 0$, and the region (II$^-$) interior to the light cone in which $t < 0$. The variable $u = t/r$ has the following ranges of values:—

$$\left.\begin{array}{ll} \text{in (II}^+) & 1 < u < \infty, \\ \text{in (I)} & -1 < u < 1, \\ \text{in (II}^-) & -\infty < u < -1. \end{array}\right\} \qquad (3{\cdot}8)$$

The equations $u = \pm 1$ represent the two sheets of the light cone. Consider now the conformal mapping defined by

$$\tanh \alpha = u, \qquad R = r\sqrt{1 - u^2},$$

where u, α, and R are complex variables. As u traces out the contour of fig. 1, α traces out that of fig. 2, and similarly for R. Hence we have

$$\left.\begin{array}{lll} \text{in } (\mathrm{II}^+) & 0 < iR < \infty, & 0 < \alpha - \tfrac{1}{2}\pi i < \infty, \\ \text{in } (\mathrm{I}) & 0 < R < \infty, & -\infty < \alpha < \infty, \\ \text{in } (\mathrm{II}^-) & -\infty < iR < 0, & -\infty < \alpha - \tfrac{1}{2}\pi i < 0. \end{array}\right\} \quad . \qquad (3.9)$$

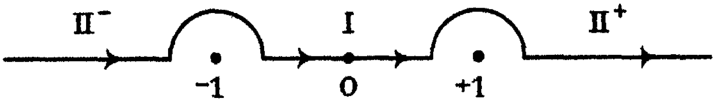

FIG. 1.

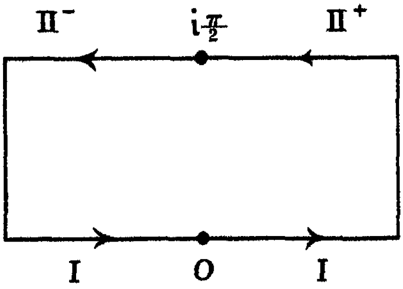

FIG. 2.

We shall have occasion to continue analytically certain functions of $u = \tanh \alpha$. Our choice of co-ordinates requires that in such analytical continuation *u must always lie in the upper half of the complex t-plane.*

From (3.6) we obtain the expressions of the differential operators $\partial/\partial x^k$ in terms of those in polar co-ordinates; we write these equations in matrix form:

$$\begin{pmatrix} \dfrac{\partial}{\partial x^1} \\ \dfrac{\partial}{\partial x^2} \\ \dfrac{\partial}{\partial x^3} \\ \dfrac{\partial}{\partial x^0} \end{pmatrix} = \begin{pmatrix} \cosh\alpha \sin\beta \cos\gamma & \dfrac{-\sinh\alpha \sin\beta \cos\gamma}{R} & \dfrac{\cos\beta \cos\gamma}{R \cosh\alpha} & \dfrac{-\sin\gamma}{R \cosh\alpha \sin\beta} \\ \cosh\alpha \sin\beta \sin\gamma & \dfrac{-\sinh\alpha \sin\beta \sin\gamma}{R} & \dfrac{\cos\beta \sin\gamma}{R \cosh\alpha} & \dfrac{\cos\gamma}{R \cosh\alpha \sin\beta} \\ \cosh\alpha \cos\beta & \dfrac{-\sinh\alpha \cos\beta}{R} & \dfrac{-\sin\beta}{R \cosh\alpha} & 0 \\ -\sinh\alpha & \dfrac{\cosh\alpha}{R} & 0 & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial R} \\ \dfrac{\partial}{\partial \alpha} \\ \dfrac{\partial}{\partial \beta} \\ \dfrac{\partial}{\partial \gamma} \end{pmatrix} \qquad (3.10)$$

From these relations we get the following expressions for the components of **M** and **N** in polar co-ordinates:—

$$\left.\begin{aligned} M_x &= \frac{\hbar}{i}\left(-\sin\gamma\frac{\partial}{\partial\beta} - \cot\beta\cos\gamma\frac{\partial}{\partial\gamma}\right),\\ M_y &= \frac{\hbar}{i}\left(\cos\gamma\frac{\partial}{\partial\beta} - \cot\beta\sin\gamma\frac{\partial}{\partial\gamma}\right),\\ M_z &= \frac{\hbar}{i}\frac{\partial}{\partial\gamma}; \end{aligned}\right\} \quad . \quad . \quad . \quad (3.11)$$

$$\left.\begin{aligned} N_x &= \frac{\hbar}{i}\left(\sin\beta\cos\gamma\frac{\partial}{\partial\alpha} + \tanh\alpha\cos\beta\cos\gamma\frac{\partial}{\partial\beta} - \tanh\alpha\frac{\sin\gamma}{\sin\beta}\frac{\partial}{\partial\gamma}\right),\\ N_y &= \frac{\hbar}{i}\left(\sin\beta\sin\gamma\frac{\partial}{\partial\alpha} + \tanh\alpha\cos\beta\sin\gamma\frac{\partial}{\partial\beta} + \tanh\alpha\frac{\cos\gamma}{\sin\beta}\frac{\partial}{\partial\gamma}\right),\\ N_z &= \frac{\hbar}{i}\left(\cos\beta\frac{\partial}{\partial\alpha} - \tanh\alpha\sin\beta\frac{\partial}{\partial\beta}\right). \end{aligned}\right\} \quad . \quad (3.12)$$

From (3.11) we obtain the well-known expression

$$M^2 = -\hbar^2\left(\frac{\partial^2}{\partial\beta^2} + \cot\beta\frac{\partial}{\partial\beta} + \frac{1}{\sin^2\beta}\frac{\partial^2}{\partial\gamma^2}\right), \quad . \quad . \quad . \quad (3.13)$$

and from (3.12):

$$N^2 = -\hbar^2\left(\frac{\partial^2}{\partial\alpha^2} + 2\tanh\alpha\frac{\partial}{\partial\alpha}\right) + \tanh^2\alpha\, M^2. \quad . \quad . \quad (3.14)$$

Finally, we obtain for $K^2 = M^2 - N^2$:

$$K^2 = \hbar^2\left(\frac{\partial^2}{\partial\alpha^2} + 2\tanh\alpha\frac{\partial}{\partial\alpha}\right) + \frac{M^2}{\cosh^2\alpha}. \quad . \quad . \quad . \quad (3.15)$$

We add the remark that the operators M_z, M^2, N^2, K^2 commute with one another.

We need later the 4-dimensional volume element, namely, the determinant of the matrix (3.9):

$$d\vec{r}dt = R^3\cosh^2\alpha\sin\beta\, dR\, d\alpha\, d\beta\, d\gamma. \quad . \quad . \quad . \quad (3.16)$$

We consider now briefly the **p**, E-space. Here we are more interested in the region (II), and therefore we introduce co-ordinates which are real in (II) but complex in (I); we put

$$\left.\begin{aligned} p_1 &= P\sinh\alpha'\sin\beta'\cos\gamma',\\ p_2 &= P\sinh\alpha'\sin\beta'\sin\gamma',\\ p_3 &= P\sinh\alpha'\cos\beta',\\ p_0 &= P\cosh\alpha', \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3.17)$$

where P and α' have the following ranges of values:—

$$\left.\begin{array}{lll} \text{in (II}^+\text{)} & 0 < P < \infty, & 0 < \alpha' < \infty, \\ \text{in (I)} & -\infty < iP < 0, & -\infty < \alpha' + i\frac{\pi}{2} < \infty, \\ \text{in (II}^-\text{)} & -\infty < P < 0, & -\infty < \alpha' < 0. \end{array}\right\} \quad . \quad (3.18)$$

The equations (3.17) can be obtained from (3.6) by replacing in the latter x^k by p_k and using the substitution

$$\alpha \to \alpha' + i\frac{\pi}{2}, \qquad R \to -iP, \qquad \beta \to \beta', \qquad \gamma \to \gamma'. \quad . \quad . \quad (3.19)$$

Therefore the matrix expressing $\partial/\partial p_k$ in terms of $\partial/\partial P$, $\partial/\partial\alpha'$, $\partial/\partial\beta'$, $\partial/\partial\gamma'$ is obtained from the matrix (3.10) by the substitution (3.19).

The components of $\mathbf{M}'$ (the dash indicates representation of these operators in $\mathbf{p}$, E-space) and the expressions for M'^2, N'^2, and K'^2 are obtained by the same substitution; but for the components of $\mathbf{N}'$ there occurs in addition a reversal of sign.

The volume element is

$$d\bar{p}dE = P^3 \sinh^2 \alpha' \sin \beta' dP d\alpha' d\beta' d\gamma'. \quad . \quad . \quad . \quad (3.20)$$

4. PSEUDO-HARMONICS IN 4-DIMENSIONAL SPACE.

We have now to determine the common proper functions of M_z, M^2, and K^2 in $\mathbf{x}$, t- and $\mathbf{p}$, E-space (it is sufficient to do this in $\mathbf{x}$-space). We postulate that these functions are regular and unique on the 2-dimensional sphere β, γ, and postpone the discussion of the behaviour in the variable α.

The proper functions of M_z and M^2 are the well-known spherical harmonics with integer indices l, m:

$$P_l^m (\cos \beta) e^{im\gamma} \quad \text{with} \quad \left\{\begin{array}{l} M_z : m\hbar, \\ M^2 : \hbar^2 l(l+1). \end{array}\right\} \quad . \quad . \quad . \quad (4.1)$$

We introduce the proper value of M^2 in (3.15) and obtain the equation

$$K^2 F = \lambda F \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.2)$$

for the determination of the proper functions of K^2. Written explicitly this equation is

$$\frac{d^2F}{d\alpha^2} + 2 \tanh \alpha \frac{dF}{d\alpha} + \left(\frac{l(l+1)}{\cosh^2 \alpha} - \frac{\lambda}{\hbar^2}\right) F = 0. \quad . \quad . \quad . \quad (4.3)$$

The substitution

$$u = \tanh \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.4)$$

gives

$$(1 - u^2)\frac{d^2F}{du^2} + \left[l(l+1)(1 - u^2) - \frac{\lambda}{\hbar^2}\right] F = 0. \quad . \quad . \quad . \quad (4.5)$$

Substituting

$$F(u) = \sqrt{1-u^2}G(u), \qquad (4.6)$$

we obtain for $G(u)$ the differential equation of an associated Legendre function:

$$(1-u^2)\frac{d^2G}{du^2} - 2u\frac{dG}{du} + \left[l(l+1) - \frac{\lambda+1}{1-u^2}\right]G = 0. \qquad (4.7)$$

Putting $\lambda = k(k+2)$, the solution is an associated Legendre function with indices l and $k+1$. We shall see below that we have to take the associated Legendre functions of the first kind. Then the common proper solutions of M_z, M^2, and K^2 are

$$\Pi_k^l(\tanh \alpha)P_l^m(\cos \beta)e^{im\gamma}, \qquad (4.8)$$

where

$$\Pi_k^l(u) = \sqrt{u^2-1}P_l^k(u), \qquad (4.9)$$

if $u > 1$; for other values of u, Π_k^l is defined by analytical continuation. They belong to the proper values

$$\hbar m, \qquad \hbar^2 l(l+1), \qquad \hbar^2 k(k+2). \qquad (4.10)$$

So far no restrictions have been placed on the values of k. These depend on the demands made on the proper functions as regards their analytical behaviour. According to (3.8) u ranges from $-\infty$ to $+\infty$. Now the associated Legendre functions have in general singularities for

$$u = \pm 1 \quad (\textit{i.e.} \text{ on the light cone})$$

and for

$$u = \pm\infty \ (\textit{i.e.} \text{ at the spatial origin } r = 0).$$

Only in exceptional cases both types of singularities can be made to vanish and we have therefore to decide which functions should be used: those with no singularities for $u = \pm 1$ or those with no singularities for $u = \pm\infty$. It is as yet too early to formulate a satisfactory principle for deciding this question and we shall, therefore, be led by the consideration that *the functions shall admit a Fourier transformation*. Then they must vanish on the (2-dimensional) infinite sphere bounding our pseudo-spheres at infinity, which lies on the light cone.

We demand, therefore, that the functions vanish for $u = \pm 1$. It then follows that we have to use the associated Legendre functions of the first kind with integer k and

$$-1 < k < l-1. \qquad (4.11)$$

The lower limit for k is explained by the fact that the associated Legendre functions with negative values of $k+1$ are identical with those having positive values and therefore give no independent solutions.

It can easily be shown that these functions are quadratically integrable on the pseudo-sphere in (I)-space ($-1 < u < 1$), with the exception of $k = -1$, but not in (II)-space ($-\infty < u < -1$; $1 < u < \infty$).

The proper functions in **p**, E-space are obtained by the substitution (3.19); hence they are

$$\Pi_k^l(\coth \alpha')P_l^m(\cos \beta')e^{im\gamma'}. \qquad (4.12)$$

5. THE SOLUTION OF THE WAVE EQUATION.

The wave equations (1.1) may be written with the help of the operators P^2 and R^2 defined by (2.5) in the form

$$P^2\psi(R, \alpha, \beta, \gamma) = \epsilon^2\psi(R, \alpha, \beta, \gamma), \qquad (5.1)$$

$$R^2\phi(P, \alpha', \beta', \gamma') = a^2\phi(P, \alpha', \beta', \gamma'), \qquad (5.2)$$

or with (3.4) and (3.5),

$$\left[k^2\left(\frac{\partial^2}{\partial R^2} + \frac{3}{R}\frac{\partial}{\partial R}\right) - \frac{K^2}{R^2} - \epsilon^2\right]\psi = 0, \qquad (5.3)$$

$$\left[k^2\left(\frac{\partial^2}{\partial P^2} + \frac{3}{P}\frac{\partial}{\partial P}\right) - \frac{K'^2}{P^2} - a^2\right]\phi = 0. \qquad (5.4)$$

Introducing the proper values (4.10) for K^2 and K'^2, we obtain a differential equation leading to a Bessel function Z_{k+1}. The solutions of the wave equation are therefore

$$\psi_{klm} = \frac{1}{R}Z_{k+1}\left(i\frac{\epsilon R}{k}\right)\Pi_k^l(\tanh \alpha)P_l^m(\cos \beta)e^{im\gamma}, \qquad (5.5)$$

$$\phi_{klm} = \frac{1}{P}Z_{k+1}\left(i\frac{aP}{k}\right)\Pi_k^l(\coth \alpha')P_l^m(\cos \beta')e^{im\gamma'}. \qquad (5.6)$$

In accordance with the remark in § 3, we have to take $\Pi_k^l(u)$ on the upper side of the cross-cut from $-\infty$ to $+1$. It then follows easily that the functions (5.5), (5.6) are analytical as we pass over the light cone from the region (I) to (II$^+$) or (II$^-$).

We obtain the correct analytical continuation if we put in **x**, *t*-space

$$\sqrt{u^2 - 1} = \frac{i}{\cosh \alpha} \quad \text{in (I)}, \qquad (5.7)$$

and in **p**, E-space

$$\sqrt{u^2 - 1} = \frac{1}{\sinh \alpha} \quad \text{in (II)}. \qquad (5.8)$$

We remark further that

$$\left.\begin{aligned} \psi_{klm}(-t, r) &= (-1)^{k+l+1}\psi_{klm}(t, r), \\ \phi_{klm}(-E, p) &= (-1)^{k+l+1}\phi_{klm}(E, p), \end{aligned}\right\} \qquad (5.9)$$

which follows easily from (5.5) and (5.6).

6. ADJOINT OPERATORS REFERRED TO RELATIVISTIC SCALAR PRODUCTS AND APPLICATION TO THE OPERATORS **M**, **N**.

In the introduction we defined the scalar product of two wave functions in analogy to the scalar product of non-relativistic quantum mechanics (*cf.* (1.5)). In terms of the pseudo-spherical co-ordinates we have

$$(\psi_1 \cdot \psi_2) = \mathrm{R}^3 \iiint \psi_1^* \psi_2 \cosh^2 \alpha \sin \beta \, d\alpha \, d\beta \, d\gamma \qquad \text{in } x\text{-space}, \quad (6.1)$$

$$(\phi_1 \cdot \phi_2) = |\,\mathrm{P}\,|^3 \iiint \phi_1^* \phi_2 \sinh^2 \alpha' \sin \beta' \, d\alpha' \, d\beta' \, d\gamma' \quad \text{in } p\text{-space}. \quad (6.2)$$

The integral in (6.1) is an integral in (I)-space over the pseudo-sphere $r^2 - t^2 = \mathrm{R}^2$. The integral (6.2) extends over the two sheets of the pseudo-sphere $\mathrm{E}^2 - p^2 = \mathrm{P}^2$ in p-space.

We further define the adjoint of an operator A by the equation

$$(\psi_1 \cdot \mathrm{A}\psi_2) = (\mathrm{A}^\dagger \psi_1 \cdot \psi_2) \quad . \quad . \quad . \quad . \quad . \quad (6.3)$$

or

$$(\phi_1 \cdot \mathrm{A}\phi_2) = (\mathrm{A}^\dagger \phi_1 \cdot \phi_2). \quad . \quad . \quad . \quad . \quad . \quad (6.4)$$

The two definitions (6.3) and (6.4) are not necessarily the same; but in the following we shall only deal with operators for which the two definitions coincide so that we need not distinguish between them.

The operators **M** and **N** are self-adjoint:

$$\mathbf{M}^\dagger = \mathbf{M}, \qquad \mathbf{N}^\dagger = \mathbf{N}. \quad . \quad . \quad . \quad . \quad . \quad (6.5)$$

This can easily be proved by partial integration. It follows that

$$(\mathrm{M}^2)^\dagger = \mathrm{M}^2, \qquad (\mathrm{N}^2)^\dagger = \mathrm{N}^2. \quad . \quad . \quad . \quad . \quad . \quad (6.6)$$

It should be noted that the proof by partial integration is correct only if the surface integrals, arising from the partial integration over α, vanish.

We consider now the function S defined by (2.5), but here we shall take it not as an operator, but as a function of the two sets of variables x^k and p_k, which are therefore both c-numbers:

$$\mathrm{S} = x^k p_k = \mathbf{r} \cdot \mathbf{p} - t\mathrm{E}. \quad . \quad . \quad . \quad . \quad . \quad (6.7)$$

If we consider S as a function in x-space and apply the operator **M** on S, we obtain

$$\mathrm{M}_1 \mathrm{S} = \frac{\hbar}{i}\left(x^2 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^2}\right) x^k p_k = \frac{\hbar}{i}(x^2 p_3 - x^3 p_2).$$

On the other hand, if S is considered as a function in p-space,

$$\mathrm{M}_1' \mathrm{S} = \frac{\hbar}{i}\left(p_2 \frac{\partial}{\partial p_3} - p^3 \frac{\partial}{\partial p_2}\right) x^k p_k = \frac{\hbar}{i}(p^2 x_3 - p^3 x_2).$$

Corresponding equations hold for the other components.

We conclude

$$\mathbf{M}\mathrm{S} = -\mathbf{M}'\mathrm{S}. \qquad (6.8)$$

Similarly, we find

$$\left.\begin{aligned} \mathrm{M}^2\mathrm{S} &= \mathrm{M}'^2\mathrm{S}, \\ \mathbf{N}\mathrm{S} &= -\mathbf{N}'\mathrm{S}, \\ \mathrm{N}^2\mathrm{S} &= \mathrm{N}'^2\mathrm{S}. \end{aligned}\right\} \qquad (6.9)$$

Obviously we obtain the same equations if S is replaced by any function of S.

We remark that the equations (6.8) and (6.9) remain correct if, in S, R and P are considered to be constants; for **M** and **N** do not operate on R and P. This fact allows us to use these equations under the integral sign of the scalar products (6.1), (6.2).

We shall now turn to the Fourier transform of a wave function defined by (*cf.* (1.8))

$$\Phi = \frac{1}{\hbar^{3/2}}\left(e^{\frac{i}{\hbar}\mathrm{S}} . \psi\right) \qquad (6.10)$$

or

$$\Psi = \frac{1}{\hbar^{3/2}}\left(e^{-\frac{i}{\hbar}\mathrm{S}} . \phi\right), \qquad (6.11)$$

and operate with **N** or **M**. Since $\mathbf{N}^* = -\mathbf{N}$, $\mathbf{N}'^* = -\mathbf{N}'$, we find with (6.9) and (6.5),

$$\mathbf{N}'\Phi = \frac{1}{\hbar^{3/2}}\left(e^{\frac{i}{\hbar}\mathrm{S}} . \mathbf{N}\psi\right), \qquad \mathbf{N}\Psi = \frac{1}{\hbar^{3/2}}\left(e^{-\frac{i}{\hbar}\mathrm{S}} . \mathbf{N}'\phi\right), \qquad (6.12)$$

$$\mathbf{M}'\Phi = \frac{1}{\hbar^{3/2}}\left(e^{\frac{i}{\hbar}\mathrm{S}} . \mathbf{M}\psi\right), \qquad \mathbf{M}\Psi = \frac{1}{\hbar^{3/2}}\left(e^{-\frac{i}{\hbar}\mathrm{S}} . \mathbf{M}'\phi\right), \qquad (6.13)$$

and

$$\left.\begin{aligned} \mathrm{N}'^2\Phi &= \frac{1}{\hbar^{3/2}}\left(e^{\frac{i}{\hbar}\mathrm{S}} . \mathrm{N}^2\psi\right), \qquad & \mathrm{N}^2\Psi &= \frac{1}{\hbar^{3/2}}\left(e^{-\frac{i}{\hbar}\mathrm{S}} . \mathrm{N}'^2\phi\right), \\ \mathrm{M}'^2\Phi &= \frac{1}{\hbar^{3/2}}\left(e^{\frac{i}{\hbar}\mathrm{S}} . \mathrm{M}^2\psi\right), \qquad & \mathrm{M}^2\Psi &= \frac{1}{\hbar^{3/2}}\left(e^{-\frac{i}{\hbar}\mathrm{S}} . \mathrm{M}'^2\phi\right). \end{aligned}\right\} \qquad (6.14)$$

These equations show that the Fourier transform of a proper function (ψ or ϕ) of the operators **M**, **N**, M^2, N^2, and hence also of K^2 taken in x- or p-space is again a proper function of the same operators. Each such Φ must therefore be a linear combination of the proper functions ϕ, and each Ψ one of the ψ. But this holds only provided that the boundary integrals vanish. This can be shown for the simultaneous proper functions of M_z, M^2, K^2 in x-space but not in p-space. But we shall not reproduce the proof as we have in any case to determine the Ψ, Φ explicitly for these proper functions.

7. FOURIER TRANSFORMATION OF THE PROPER FUNCTIONS.

We shall consider now the Fourier transformation of the proper functions ψ_{klm} and ϕ_{klm} given by (5.5) and (5.6), namely

$$\Phi_{klm}^{(\mathrm{R})} = \frac{1}{h^{3/2}}\left(e^{\frac{i}{h}\mathrm{S}} \,.\, \psi_{klm}\right)_{\mathrm{R}}, \qquad . \quad . \quad . \quad . \quad (7.1)$$

$$\Psi_{klm}^{(\mathrm{P})} = \frac{1}{h^{3/2}}\left(e^{-\frac{i}{h}\mathrm{S}} \,.\, \phi_{klm}\right)_{\mathrm{P}}, \qquad . \quad . \quad . \quad . \quad (7.2)$$

where

$$\left.\begin{aligned} \mathrm{S} = \mathbf{r}\,.\,\mathbf{p} - \mathrm{E}t = rp\cos\Gamma - \mathrm{E}t = rp\cos\Gamma + x^0 p_0, \\ \cos\Gamma = \cos\beta\cos\beta' + \sin\beta\sin\beta'\cos(\gamma - \gamma'). \end{aligned}\right\} \quad . \quad . \quad (7.3)$$

The integration over β, γ can be carried out easily with the help of the well-known expansion of the exponential function

$$e^{\pm i\sigma\cos\Gamma} = \sum_{l=0}^{\infty}\sum_{m=0}^{l}(2\pi)^{3/2}(\pm i)^l \frac{\mathrm{J}_{l+\frac{1}{2}}(\sigma)}{\sqrt{\sigma}}\mathbf{P}_l^m(\cos\beta)\mathbf{P}_l^m(\cos\beta')\cos m(\gamma - \gamma'). \quad (7.4)$$

Here the spherical harmonics $\mathbf{P}_l^m$ are normalised and $\mathrm{J}(\sigma)$ is the Bessel function of the first kind.

Substituting (7.4) in the integrals (7.1) and (7.2), we see that the integration over β, γ reproduces the spherical harmonics in β', γ'; hence we have

$$\Phi_{klm}^{(\mathrm{R})} = \frac{(-i)^l}{h^{3/2}}\mathrm{R}^3 \frac{\mathrm{Z}_{k+1}\left(i\frac{\epsilon\mathrm{R}}{h}\right)}{\mathrm{R}}\mathrm{P}_l^m(\cos\beta')e^{im\gamma'}\Phi_{kl}, \quad . \quad . \quad . \quad (7.5)$$

$$\Psi_{klm}^{(\mathrm{P})} = \frac{i^l}{h^{3/2}}|\,\mathrm{P}\,|^3 \frac{\mathrm{Z}_{k+1}\left(i\frac{a\,|\,\mathrm{P}\,|}{h}\right)}{|\,\mathrm{P}\,|}\mathrm{P}_l^m(\cos\beta)e^{im\gamma}\Psi_{kl}, \quad . \quad . \quad (7.6)$$

where

$$\Phi_{kl} = \int_{-\infty}^{+\infty} e^{-\frac{i}{h}p_0 x^0}\frac{\mathrm{J}_{l+\frac{1}{2}}\left(\frac{pr}{h}\right)}{\sqrt{pr/h}}\Pi_k^l(\tanh\alpha)\cosh^2\alpha\, d\alpha, \quad . \quad . \quad (7.7)$$

and (*cf.* (5.9))

$$\Psi_{kl} = \int_0^{\infty}\frac{\mathrm{J}_{l+\frac{1}{2}}\left(\frac{pr}{h}\right)}{\sqrt{pr/h}}\Pi_k^l(\coth\alpha')\left[e^{\frac{i}{h}p_0x^0} + (-1)^{k+l+1}e^{-\frac{i}{h}p_0x^0}\right]\sinh^2\alpha'\, d\alpha'. \quad (7.8)$$

We substitute in (7.7),

$$\left.\begin{aligned} x^0 &= \mathrm{R}\sinh\alpha, & r &= \mathrm{R}\cosh\alpha, \\ p_0 &= \mathrm{P}\varpi_0, & p &= \mathrm{P}\varpi, \\ \varpi_0 &= \cosh\alpha', & \varpi &= \sinh\alpha'. \end{aligned}\right\} \quad . \quad . \quad . \quad (7.9)$$

Then (7.7) becomes

$$\Phi_{kl}=\int_{-\infty}^{+\infty}e^{-\frac{i}{h}\mathrm{PR}\bar{\omega}_0\sinh a}\frac{\mathrm{J}_{l+\frac{1}{2}}\left(\frac{\mathrm{PR}}{h}\varpi\cosh a\right)}{\sqrt{\mathrm{PR}\varpi\cosh a/h}}\Pi_k^l(\tanh a)\cosh^2 a\,da. \quad . \quad (7.10)$$

We owe to Professor E. T. Whittaker a method of evaluating this integral, with the help of a theorem of Gegenbauer (*cf.* Watson, 1922, p. 379); we replace the Gegenbauer function by a Legendre function which we have to take, according to the remark following formula (5.6), as $\mathrm{P}_l^{k+1}(\cos\delta+i\,.\,0)$, and obtain

$$\int_0^{\pi}e^{iz\cos\delta\cos\psi}\mathrm{J}_{k+1}(z\sin\delta\sin\psi)\mathrm{P}_l^{k+1}(\cos\delta)\sin\delta d\delta$$
$$=\sqrt{2\pi}i^{l-k-1}\mathrm{P}_l^{k+1}(\cos\psi)\frac{\mathrm{J}_{l+\frac{1}{2}}(z)}{\sqrt{z}}. \quad (7.11)$$

We put

$$\cos\psi=\tanh a; \qquad z=\frac{\mathrm{PR}}{h}\cosh a\varpi,$$

and use the definition (4.9) of Π_k^l, namely (*cf.* also (5.7)),

$$\Pi_k^l(\tanh a)=\frac{i}{\cosh a}\mathrm{P}_l^{k+1}(\tanh a).$$

Then

$$\frac{\mathrm{J}_{l+\frac{1}{2}}\left(\frac{\mathrm{PR}}{h}\varpi\cosh a\right)}{\sqrt{\frac{\mathrm{PR}}{h}\varpi\cosh a}}\Pi_k^l(\tanh a)=\mathrm{G}(\sinh a), \quad . \quad . \quad (7.12)$$

where

$$\left.\begin{aligned}\mathrm{G}(\xi)&=\int_0^{\pi}e^{i\frac{\mathrm{PR}}{h}\bar{\omega}\xi\cos\delta}\mathrm{J}_{k+1}\left(\frac{\mathrm{PR}}{h}\varpi\sin\delta\right)\mathrm{P}_l^{k+1}(\cos\delta)\frac{i^{k-l+2}}{\sqrt{2\pi}}\sin\delta d\delta\\&=\int_{-1}^{+1}e^{i\frac{\mathrm{PR}}{h}\bar{\omega}\xi\eta}\mathrm{F}(\eta)d\eta,\end{aligned}\right\} \quad . \quad (7.13)$$

and

$$\mathrm{F}(\eta)=\mathrm{J}_{k+1}\left(\frac{\mathrm{PR}}{h}\varpi\sqrt{1-\eta^2}\right)\mathrm{P}_l^{k+1}(\eta)\frac{i^{k-l+2}}{\sqrt{2\pi}}. \quad . \quad . \quad . \quad (7.14)$$

We insert (7.12) into (7.10) and put $\sinh a=\xi$:

$$\Phi_{kl}=\int_{-\infty}^{+\infty}e^{-i\frac{\mathrm{PR}\bar{\omega}}{h}\frac{\bar{\omega}_0}{\bar{\omega}}\xi}\mathrm{G}(\xi)d\xi. \quad . \quad . \quad . \quad . \quad (7.15)$$

Now (7.13) can be regarded as the Fourier transformation of the function which is equal to $\mathrm{F}(\eta)$ for $|\eta|<1$ and vanishes for $|\eta|>1$. Then by the Fourier theorem (7.15) is essentially the same function for the argument ϖ_0/ϖ:

$$\Phi_{kl}=\begin{cases}\dfrac{2\pi k}{\mathrm{PR}\varpi}\mathrm{F}\left(\dfrac{\varpi_0}{\varpi}\right), & \left|\dfrac{\varpi_0}{\omega}\right|<1,\\[2ex] 0, & \left|\dfrac{\varpi_0}{\omega}\right|>1,\end{cases}$$

or with (7.14) and (7.9),

$$\Phi_{kl}=\begin{cases}-\dfrac{\sqrt{2\pi}\,i^{k-l+1}k}{\mathrm{PR}\cosh\alpha'}\mathrm{J}_{k+1}\left(-i\dfrac{\mathrm{PR}}{k}\right)\mathrm{P}_l^{k+1}(\coth\alpha') & \text{in (I)},\\ 0 & \text{in (II)}.\end{cases}$$

Introducing this expression in (7.5), we obtain

$$\Phi^{(\mathrm{R})}_{klm}=\left\{\begin{matrix}\dfrac{i^{-k-2l}\sqrt{2\pi}}{k^{1/2}}\mathrm{R}\mathrm{Z}_{k+1}\left(i\dfrac{\epsilon\mathrm{R}}{k}\right)\dfrac{\mathrm{J}_{k+1}\left(i\dfrac{\mathrm{RP}}{k}\right)}{\mathrm{P}}\Pi_k^l(\coth\alpha')\mathrm{P}_l^m(\cos\beta')e^{im\gamma'} & \text{in (I)},\\ 0 & \text{in (II)}.\end{matrix}\right\}\quad(7.16)$$

The evaluation of the integral Ψ'_{kl} (7.8) is rather complicated. We owe the proof to Professor E. T. Copson. Since the result obtained so far is rather negative from the physical point of view, we shall omit the proof. The integral $\Psi'^{(\mathrm{P})}_{klm}$ consists of two parts:

$$\Psi'^{(\mathrm{P})}_{klm}=\Psi'^{(1)}_{klm}+\Psi'^{(2)}_{klm}.\quad . \quad . \quad . \quad . \quad (7.17)$$

The first part is non-analytical and corresponds exactly to the result for $\Phi^{(\mathrm{R})}_{klm}$:

$$\Psi'^{(1)}_{klm}=\left\{\begin{matrix}\dfrac{(-1)^l i^{k+1}\sqrt{2\pi}\,|\mathrm{P}|}{k^{1/2}}\mathrm{Z}_{k+1}\left(i\dfrac{a|\mathrm{P}|}{k}\right)\dfrac{\mathrm{J}_{k+1}\left(\dfrac{i}{k}|\mathrm{P}|\mathrm{R}\right)}{\mathrm{R}}\Pi_k^l(\tanh\alpha)\mathrm{P}_l^m(\cos\beta)e^{im\gamma} & \text{in (I)},\\ 0 & \text{in (II)}.\end{matrix}\right\}\quad(7.18)$$

The second part is:

$$\Psi'^{(2)}_{klm}=\sum_{k'=l}^{\infty}\mathrm{A}_{k'}\frac{\mathrm{J}_{k'+1}\left(i\dfrac{|\mathrm{P}|\mathrm{R}}{k}\right)}{\mathrm{R}}\cdot\frac{\mathrm{Q}_l^{k'+1}(\tanh\alpha)}{\cosh\alpha}\mathrm{P}_l^m(\cos\beta)e^{im\gamma},\quad . \quad (7.19)$$

where

$$\mathrm{A}_{k'}=\frac{(-1)^{k'+1}i^{k'}}{k^{1/2}}2\sqrt{\frac{2}{\pi}}\,\frac{\{1+(-1)^{k+k'+1}\}}{(k+k'+2)(k'-k)}\,\frac{k'+1}{(k'-l)!\,(l-k-1)!}\,|\mathrm{P}|\,\mathrm{Z}_{k+1}\left(i\frac{a|\mathrm{P}|}{k}\right).\quad(7.20)$$

The result obtained is rather negative from the standpoint of reciprocity. If Ψ'_{klm} consisted only of the part $\Psi'^{(1)}_{klm}$ our problem would be solved; putting $\psi_{klm}=\Psi'^{(\epsilon)}_{klm}$, $\phi_{klm}=\Phi^{(a)}_{klm}$, we should obtain a determination of the Bessel functions Z_{k+1} in (5.5) and (5.6), and at the same time a transcendental equation for $\mu=a\epsilon/k$. But the existence of the part $\Psi^{(2)}_{klm}$ makes this impossible. If we could show that $\Psi^{(2)}_{klm}$ is a linear combination of the l proper functions $\psi_{k'lm}$ ($k'=0, 1, \ldots l-1$) a similar procedure would be possible. We have, however, not succeeded in deciding this point.

8. SUMMARY.

The theory suggested in Part I is improved by taking into account the different character of the 4-vectors $\mathbf{x}^k$ and $\mathbf{p}_k$ (the former being space-like, the latter time-like), by using an invariant element of integration and by introducing terms corresponding to negative values of energy and retardation time. A new formulation of the integral equation of Part I is given in terms of "relativistic Fourier coefficients" defined with the help of a new definition of "invariant scalar products." The wave equation in $\mathbf{x}$, t- and $\mathbf{p}$, E-space is transformed into 4-dimensional polar co-ordinates and solved by a product of functions each depending on one co-ordinate only. It is shown that these functions consist of a Bessel function of the 4-dimensional distance multiplied by a pseudo-spherical harmonic which is closely connected to the 6-vector representing the angular momentum in 4-dimensional space. This harmonic depends, apart from the usual quantum numbers l and m, on a new quantum number k, which assumes the values $0, 1, 2, \ldots l-1$; the eigenvalues of the norm of the 6-vector angular momentum are $\hbar^2 k(k+2)$. These solutions are quadratically integrable in $\mathbf{x}$, t-space, but not in $\mathbf{p}$, E-space; they have relativistic Fourier coefficients in both spaces.

It is proved that the Fourier coefficient of ψ_{klm} in $\mathbf{x}$, t-space is proportional to the reciprocal function ϕ_{klm} in $\mathbf{p}$, E-space, but not *vice versa*.

We have to express our sincerest gratitude to Professor E. T. Whittaker, Professor E. T. Copson, and Dr A. Erdélyi for giving us their mathematical assistance.

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IX.—Harmonic Riemannian Spaces. By E. T. Copson and H. S. Ruse.

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1. INTRODUCTION.

IN this paper we consider a new class of Riemannian spaces which arise in the theory of the solution of the tensor generalisation of Laplace's equation $\nabla^2 V=0$. To obtain this generalisation Beltrami's second differential parameter is defined in terms of the metric

$$d\sigma^2=g_{ij}dx^i dx^j \qquad (1.1)$$

of the associated n-dimensional Riemannian space by the usual formulæ

$$\Delta_2 V \equiv \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(g^{ij}\sqrt{g}\frac{\partial V}{\partial x^j}\right) \qquad (1.2)$$

$$\equiv g^{ij}\left[\frac{\partial^2 V}{\partial x^i \partial x^j}-\Gamma^k_{ij}\frac{\partial V}{\partial x^k}\right], \qquad (1.3)$$

where Γ^k_{ij} denotes the Christoffel symbol $\left\{ {k \atop ij} \right\}$. The generalised Laplace's equation is then $\Delta_2 V=0$. For simplicity the quadratic differential form (1.1) is taken to be positive definite, which involves no essential loss of generality.

Let s be the geodesic distance from a fixed point O, $(x_0{}^i)$, to a variable point P, (x^i), in the Riemannian space V_n with metric (1.1). Then V_n will be said to be *harmonic with respect to the base-point O if*

$$\Delta_2 s=\chi(s), \qquad (1.4)$$

where $\chi(s)$ is a function of s alone; it does not involve the co-ordinates x^i of the variable point P explicitly, but the co-ordinates $x_0{}^i$ of the base-point O may appear as parameters.

If the equation (1.4) holds for one base-point only, the space will be called *centrally harmonic*; if it holds for all choices of the base-point, the space will be called *completely harmonic*. For example, flat spaces and spaces of constant curvature are completely harmonic, but the space of metric

$$d\sigma^2 = \phi(x^2+y^2+z^2)(dx^2+dy^2+dz^2)$$

is in general only centrally harmonic (with respect to the origin). There is also another type of space which is harmonic with respect to a set of points which form a continuum but do not fill the space; thus

$$d\sigma^2 = dx^2 + x^2dy^2 + x^2dz^2$$

is harmonic with respect to every point on $x=0$ but not with respect to any other base-point.

In §§ 2 and 3 we show how the idea of harmonic spaces arises out of the theory of Laplace's equation. The subsequent sections are devoted to an investigation of the nature of these spaces. In particular we prove (§ 6) that a Schur space is centrally harmonic, but that, when the number of dimensions exceeds two, the most general centrally harmonic space is not necessarily a Schur space. In § 7 we obtain the curvature conditions satisfied by a completely harmonic space, and we deduce in § 8 that all completely harmonic spaces are Einstein spaces, and that, for $n=2$ and $n=3$, they are spaces of constant curvature. The question whether every completely harmonic space is of constant curvature remains at present unanswered.

Since this paper was written* we have noticed that a similar but simpler problem has been discussed recently by Thomas and Titt (1939). They have shown that the Hadamard elementary solution of $\Delta_2 V=0$ is of the classical form

$$\begin{aligned} V &= 1/s^{n-2} \qquad (n>2) \\ V &= \log s \qquad (n=2) \end{aligned}$$

only when the Riemannian space is flat.

2. HADAMARD'S ELEMENTARY SOLUTION OF LAPLACE'S EQUATION.

In an earlier paper (Ruse, 1930) one of us claimed to have proved that the elementary solution (Hadamard, 1930, pp. 70 *et seq.*) of Laplace's equation in a Riemannian space V_n of any number of dimensions is

$$V = A\int_a^s \frac{J}{\sqrt{g}\sqrt{g_0}}\frac{ds}{s^{n-1}}. \qquad (2.1)$$

In this formula A and a are constants, s is the geodesic distance from the singularity O, (x_0^i), to the variable point P, (x^i), g is the determinant of the coefficients g_{ij}, g_0 the value of g at O, and J is the determinant whose typical element is

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* Added 28th February 1940.

$$\frac{\partial^2(\frac{1}{2}s^2)}{\partial x^i \partial x_0^j}. \qquad (2.2)$$

It was thought that this gave the elementary solution having an infinity of the order $1/s^{n-2}$ at O and on all bicharacteristics (null geodesics) through O, no matter what the nature of the space. It is from a closer discussion of the conditions for the validity of this formula that the present paper arose.

The formula evidently holds in a flat space. It also gives the elementary solution in a space of constant curvature K, that is when

$$R_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk}),$$

where the Riemann-Christoffel tensor R_{ijkl} is taken with the classical sign adopted by most writers (see, for example, Eisenhart, 1926, p. 20). In such a space the formula (2.1) reduces to

$$V = C\int_a^s \operatorname{cosec}^{n-1}(s\sqrt{K})\,ds,$$

where C is a constant. That this function is a solution of Laplace's equation of the required type may be easily verified by direct calculation. The occurrence of a logarithmic term when n is even—a fact of importance in the theory of the elementary solution—is obvious.

An examination of Ruse's work shows that his proof of the formula (2.1) is incomplete, since he makes the implicit assumption that Laplace's equation always possesses a solution which depends only on the geodesic distance s of the point P, (x^i), from the singularity O, (x_0^i). If $J/\sqrt{g}$ is a function of s alone, this assumption is true and the formula holds. But if $J/\sqrt{g}$ depends on the co-ordinates x^i in a form which cannot be expressed in terms of s alone, the integral (2.1) is indeterminate; its value depends not only on the co-ordinates of P, but also on the choice of the path of integration.

It has been stated by van Mieghem (1932) that Ruse's formula will give a solution of Laplace's equation in any Riemannian space if the path of integration is an arc of the geodesic through O and P terminating at P. This is not the case, as we shall show.

Van Mieghem's assertion would be true if the equation

$$V = \int_C \frac{J}{\sqrt{g}\sqrt{g_0}} \frac{ds}{s^{n-1}}, \qquad (2.3)$$

where C is part * of the geodesic arc OP terminating at P, implied that

$$\frac{\partial V}{\partial x^i}=\frac{J}{\sqrt{g}\sqrt{g_0}}s^{1-n}\frac{\partial s}{\partial x^i}. \qquad (2.4)$$

This is in general not the case. For when we vary the position of P, we alter the path of integration C in (2.3) and the differential dV has to be calculated by the methods of the calculus of variations. It turns out that equation (2.4) should have an additional term on the right-hand side, namely, another integral along the arc C. The calculation of $\Delta_2 V$ from (2.3) by this method would be laborious and is unnecessary since we can give a simple direct example of the failure of van Mieghem's modification of Ruse's formula.

Let us consider the three-dimensional space of metric

$$d\sigma^2=dx^2+x^2dy^2+x^2dz^2, \qquad (2.5)$$

which is not a space of constant curvature. We take an arbitrary base-point (x_0, y_0, z_0), where x_0 is not zero, and introduce new co-ordinates (s, θ, ϕ) defined by the equations

$$x=(x_0^2+2sx_0\cos\theta+s^2)^{1/2}, \qquad y=y_0+r\cos\phi, \qquad z=z_0+r\sin\phi,$$

where

$$\tan r=\frac{s\sin\theta}{x_0+s\cos\theta}.$$

The metric (2.5) then becomes

$$d\sigma^2=ds^2+s^2d\theta^2+x^2r^2d\phi^2, \qquad (2.6)$$

which shows that the equations of the geodesics through the base-point $(s=0)$ are $\theta=$constant, $\phi=$constant, and that s is the geodesic distance from the base-point to the point (x, y, z).

The geodesic distance s is given explicitly by

$$s^2=x^2-2xx_0\cos r+x_0^2,$$

where

$$r^2=(y-y_0)^2+(z-z_0)^2.$$

From this it follows that, apart from a constant multiplier, van Mieghem's function (2.3) is

$$V=\int_a^s\frac{\sin\rho}{\rho}\frac{du}{u^2}, \qquad (2.7)$$

where a is a constant, (s, θ, ϕ) are the co-ordinates of P, and

$$\tan\rho=\frac{u\sin\theta}{x_0+u\cos\theta}.$$

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* There is one obvious slight error in van Mieghem's work. He integrates along the whole geodesic arc OP, ignoring the fact that the integral is divergent at O.

In this formula u is the geodesic distance from O to a typical point of the geodesic arc OP. The integrand in (2.7) depends, not only on u, but also on the parameter θ of the path of integration.

Although it is not possible to evaluate the integral (2.7) in a simple form, we can calculate from it the value of $\Delta_2 V$ by using the form (2.6) of the metric and the definition (1.2) of Beltrami's second differential parameter. It readily follows that van Mieghem's function (2.3) is not a solution of Laplace's equation.

We assumed in this example that the base-point did not lie on the surface $x=0$. We shall see later that the form of the metric (2.5) indicates that the space is harmonic with respect to every base-point on $x=0$, and that for such a base-point Ruse's original formula holds.

3. CENTRALLY HARMONIC SPACES.

Let us suppose that, in a space with metric

$$d\sigma^2 = g_{ij}dx^i dx^j, \qquad (3.1)$$

Ruse's formula (2.1) gives a solution of Laplace's equation which has a certain base-point O as singularity, no matter what the path of integration may be. This means that $\Delta_2 V=0$ has a solution of the form $V=f(s)$, where s is the geodesic distance from O. But, by (1.2),

$$\Delta_2 f(s) = f'(s)\Delta_2 s + f''(s)\Delta_1 s = f'(s)\Delta_2(s) + f''(s),$$

since, as is well known, $\Delta_1 s$ has the value unity. Hence if $V=f(s)$ satisfies Laplace's equation, we have

$$\Delta_2 s = -f''(s)/f'(s), \qquad (3.2)$$

and so the space is harmonic with respect to the base-point O.

Conversely, if the space is centrally harmonic, so that $\Delta_2 s$ satisfies the equation

$$\Delta_2 s = \chi(s),$$

the differential equation (3.2) becomes

$$\frac{f''(s)}{f'(s)} = -\chi(s),$$

whence

$$f(s) = A\int \exp\left(-\int \chi ds\right)ds + B.$$

This, as van Mieghem has shown, is equivalent to Ruse's formula (2.1). Thus in a centrally harmonic space Laplace's equation has a solution which depends on s alone, and this solution is given by Ruse's formula.

4. THE GEOMETRICAL AND ANALYTICAL CHARACTERISATION OF HARMONIC SPACES.

A natural generalisation to Riemannian geometry of the circle and sphere of Euclidean geometry is the *distance-hypersphere*, which is the locus of a point at constant geodesic distance from a fixed point. In a general Riemannian space V_n the distance-hypersphere has no simple property corresponding to the constant-curvature property of Euclidean circles and spheres, and it is therefore of interest to observe that *in a centrally harmonic Riemannian V_n with base-point O the distance-hyperspheres with O as centre have constant mean curvature.* For if s is the geodesic distance from the base-point O, the mean curvature of the distance-hypersphere $s=$constant is (Weatherburn, 1938, p. 140, (52))

$$\begin{aligned} M &= -\left[\frac{\Delta_2 s}{(\Delta_1 s)^{1/2}} + \Delta_1\left(s, \frac{1}{(\Delta_1 s)^{1/2}}\right)\right] \\ &= -[\Delta_2 s + \Delta_1(s, 1)] \\ &= -\chi(s) \end{aligned}$$

by (1.4), and the result follows.

In the case $n=2$ the distance-hypersphere is a *distance-circle*, and the mean curvature M reduces to the ordinary geodesic curvature $1/\rho_g$. In a completely harmonic V_2 all distance-circles are therefore of constant geodesic curvature, and it follows by a well-known theorem (Blaschke, 1924, p. 100) that *a completely harmonic V_2 is of constant curvature.*

A centrally harmonic V_n may also be characterised by the form its metric takes when the geodesic distance s from the base-point is taken as one of the co-ordinates. A geodesic through the base-point is specified by $n-1$ parameters, $\theta^1, \theta^2, \ldots, \theta^{n-1}$, say, which may be taken as co-ordinates having the same general character as the angular polar co-ordinates in ordinary Euclidean space. In terms of the geodesic-polar system of co-ordinates (s, θ^α), the metric of any V_n has the form

$$d\sigma^2 = ds^2 + a_{\alpha\beta} d\theta^\alpha d\theta^\beta, \quad . \quad . \quad . \quad . \quad (4.1)$$

where the coefficients $a_{\alpha\beta}$ vanish with s and the Greek suffixes run from 1 to $n-1$ (see McConnell, 1931, p. 177; *cf.* Eisenhart, 1926, p. 57; Weatherburn, 1938, p. 80). Hence

$$\Delta_2 V = \frac{1}{\sqrt{a}} \frac{\partial}{\partial s}\left(\sqrt{a}\frac{\partial V}{\partial s}\right) + \frac{1}{\sqrt{a}} \frac{\partial}{\partial \theta^\alpha}\left(a^{\alpha\beta}\sqrt{a}\frac{\partial V}{\partial \theta^\beta}\right),$$

where $(a^{\alpha\beta})$ is the matrix reciprocal to $(a_{\alpha\beta})$ and a is the determinant of the matrix $(a_{\alpha\beta})$.

Now suppose that the space with metric (4.1) is harmonic with respect to the base-point $s=0$. Then, by (1.4),

$$\chi(s)=\Delta_2 s=\frac{1}{\sqrt{a}}\frac{\partial\sqrt{a}}{\partial s}=\frac{1}{2}\frac{\partial \log a}{\partial s},$$

and so

$$a=\psi(\theta)\exp\int 2\chi(s)ds, \quad . \quad . \quad . \quad . \quad . \quad (4.2)$$

where $\psi(\theta)$ depends on the co-ordinates θ^α alone. And conversely, *if* a is of the form (4.2) (that is, is a product of a function of s alone and a function of the co-ordinates θ^α alone), the space of metric (4.1) is centrally harmonic. Hence *if the metric of a Riemannian* V_n *referred to geodesic-polar co-ordinates* (s, θ^α) *is*

$$d\sigma^2=ds^2+a_{\alpha\beta}d\theta^\alpha d\theta^\beta, \quad . \quad . \quad . \quad . \quad (4.3)$$

the space is harmonic with respect to the base-point $(s=0)$ *if and only if the determinant of the coefficients* $a_{\alpha\beta}$ *is the product of two functions, one depending on s alone, the other independent of s.*

The co-ordinates (s, θ, ϕ) introduced into the metric (2.5), namely,

$$d\sigma^2=dx^2+x^2dy^2+x^2dz^2,$$

are geodesic-polar co-ordinates with base-point (x_0, y_0, z_0), and the resulting form of the metric was

$$d\sigma^2=ds^2+s^2d\theta^2+x^2r^2d\phi^2.$$

The theorem we have just proved shows that this space is harmonic with respect to base-points on $x=0$ but not with respect to any other base-point.

It may be noted that the angular variables θ^α in a centrally harmonic space with metric (4.3) can be chosen so that the determinant of the coefficients depends on s alone. For if we substitute

$$\theta^\alpha=\theta^\alpha(\theta'^1, \theta'^2, \ldots, \theta'^{n-1})$$

in (5.3), choosing the functions $\theta^\alpha(\theta')$ so that their Jacobian is equal to $1/\sqrt{\psi(\theta)}$, the determinant a transforms into

$$a'=a/\psi(\theta)=\exp\int 2\chi(s)ds$$

by (4.2).

5. THE FORM OF A CENTRALLY HARMONIC V_2.

It follows from § 4 that the metric of a centrally harmonic V_2 is

$$d\sigma^2 = ds^2 + a\,d\theta^2 \quad . \quad . \quad . \quad . \quad . \quad (5.1)$$

in geodesic-polar co-ordinates with the base-point as pole, a being a function of s alone. This implies that *a centrally harmonic V_2 immersed in a three-dimensional flat space is a surface applicable to a surface of revolution* (Eisenhart, 1909, p. 108).

It is known (Duschek-Mayer, vol. i, 1930, p. 201) that in any V_2 geodesic-polar co-ordinates can be chosen so that the metric is

$$d\sigma^2 = ds^2 + a\,d\theta^2,$$

where

$$\sqrt{a} = s + O(s^3)$$

when s is small. Hence if a is a function of s alone, the function

$$r = s \exp\left\{\int_0^s \left(\frac{1}{\sqrt{a}} - \frac{1}{s}\right) ds\right\}$$

satisfies the equation $dr/ds = r/\sqrt{a}$ and vanishes with s. In terms of this new variable the metric (5.1) of the centrally harmonic V_2 becomes

$$d\sigma^2 = \lambda^2(dr^2 + r^2 d\theta^2),$$

where λ depends on r alone and is of the form $1 + O(r^2)$ when r is small. If we introduce co-ordinates

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

we obtain finally

$$d\sigma^2 = \lambda^2(dx^2 + dy^2), \quad . \quad . \quad . \quad . \quad (5.2)$$

where λ depends on $x^2 + y^2$ alone. Hence *a V_2 harmonic with respect to a base-point O can be mapped conformally on a Euclidean plane in such a way that O is mapped on the origin and the magnification factor λ depends only on the Euclidean distance from the origin.*

This result enables us to show that *a V_2 which is harmonic with respect to two base-points is a space of constant curvature and so is completely harmonic.* If the two base-points are A and B, it is possible to map the V_2 conformally on a Euclidean plane in two ways: in the first mapping A is mapped on the origin and

$$d\sigma^2 = \lambda^2(dx^2 + dy^2),$$

where λ depends on $x^2 + y^2$ alone, whereas in the second B is mapped on the origin and

$$d\sigma^2 = \Lambda^2(dX^2 + dY^2),$$

where Λ depends on X^2+Y^2 alone. This implies that it is possible to map a Euclidean plane conformally on itself in such a way that

$$\lambda^2(dx^2+dy^2)=\Lambda^2(dX^2+dY^2).$$

Now the most general conformal mapping of a Euclidean plane on itself is compounded out of translations, rotations, magnifications, reflections, and inversions. An examination of these possibilities shows that either λ is constant or else

$$\lambda=\frac{1}{\alpha+\beta(x^2+y^2)},$$

where α and β are constants. In the former case the space is flat, in the latter of constant curvature. Hence the result.

6. SCHUR SPACES.*

The natural generalisation of the centrally harmonic V_2 with metric (5.2) is the space of metric

$$d\sigma^2=\lambda^2\sum_{i=1}^{n}(dx_i)^2, \qquad . \quad . \quad . \quad . \quad . \quad (6.1)$$

where λ depends on

$$r^2=\sum_{i=1}^{n}x_i^2 \qquad . \quad . \quad . \quad . \quad . \quad (6\cdot2)$$

alone and

$$\lambda=1+O(r^2)$$

when r is small. Such a space turns out to be a Schur space with centre at the origin (see Schur, 1886, p. 157; Duschek-Mayer, vol. ii, 1930, p. 168). We prove below that *a Schur space is harmonic with respect to the origin*, but that *when $n>2$, not every centrally harmonic space is a Schur space.*

For simplicity we consider the case $n=3$. If we introduce polar co-ordinates r, θ, ϕ defined by

$$x_1=r\sin\theta\cos\phi, \qquad x_2=r\sin\theta\sin\phi, \qquad x_3=r\cos\theta,$$

which satisfy (6.2) identically, the metric (6.1) becomes

$$d\sigma^2=\lambda^2dr^2+\lambda^2r^2d\theta^2+\lambda^2r^2\sin^2\theta d\phi^2.$$

Hence if we write

$$s=\int_0^r\lambda dr=r+O(r^3),$$

we have

$$d\sigma^2=ds^2+\gamma(s)s^2(d\theta^2+\sin^2\theta d\phi^2), \qquad . \quad . \quad . \quad (6.3)$$

---

[*] In this section it is convenient to drop the convention of writing the index of a co-ordinate in the contravariant position.

where

$$\gamma(s) = \frac{\lambda^2 r^2}{s^2} = 1 + O(s^2)$$

when s is small.

The form of the metric (6.3) shows that the equations of a geodesic through the origin are $\theta = \text{constant}$, $\phi = \text{constant}$, and that s is the geodesic distance from the origin. The determinant of the coefficients in (6.3) is

$$\{s^2\gamma(s)\sin\theta\}^2;$$

hence, by § 4, the three-dimensional Schur space of metric (6.1) is harmonic with respect to the origin. Similarly for general values of n.

If we write

$$z_1 = s\sin\theta\cos\phi, \qquad z_2 = s\sin\theta\sin\phi, \qquad z_3 = s\cos\theta,$$

the parametric equations of a geodesic through the origin are

$$z_i = a_i s,$$

where the coefficients a_i are constants. The co-ordinates (z_i) then form a system of normal co-ordinates (see Veblen, 1933, p. 85), in terms of which the metric (6.3) becomes

$$d\sigma^2 = a_{ik}dz_i dz_k,$$

where

$$a_{ik} = \gamma(s)\delta_{ik} + \frac{z_i z_k}{s}\{1 - \gamma(s)\}, \quad . \quad . \quad . \quad . \quad (6.4)$$

δ_{ik} being a Kronecker symbol. The equation (6.4) is that usually given for the coefficients of the metric of a Schur space (see Duschek-Mayer, *loc. cit.*).

We note in passing that the formula for an n-dimensional Schur space corresponding to (6.3) is

$$d\sigma^2 = ds^2 + \gamma(s)s^2 d\omega^2,$$

where $d\omega$ is the element of length on a hypersphere of unit radius in an n-dimensional Euclidean space. Thus, apart from the sign of the coefficient $\gamma(s)$, *the Schur space is the generalisation to n dimensions of the expanding universe of general relativity* (*cf.* Tolman, 1934, p. 371, equations (149.9) and (149.10)).

When $n = 2$, the most general centrally harmonic space is a Schur space. This result is untrue when $n > 2$. For consider the space of metric

$$d\sigma^2 = ds^2 + s^2\left[\frac{\psi(s)\cos\theta}{\sqrt{\{\psi(s) - \sin^2\theta\}}}d\theta^2 + \frac{\sqrt{\{\psi(s) - \sin^2\theta\}}}{\cos\theta}\sin^2\theta\, d\phi^2\right], \qquad (6.5)$$

where $\psi(s)$ is any function of s. The geodesics through the origin $s = 0$ are

$\theta=$constant, $\phi=$constant, and s is the geodesic distance from the origin. Since the determinant of the coefficients is $s^4\psi(s)\sin^2\theta$, the space is centrally harmonic. Now as before, the co-ordinates

$$y_1=s\sin\theta\cos\phi,\qquad y_2=s\sin\theta\sin\phi,\qquad y_3=s\cos\theta$$

form a system of normal co-ordinates. In terms of them (6.5) becomes

$$d\sigma^2=g_{ik}dy_idy_k, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6.6)$$

where

$$g_{11}=\frac{y_1^2}{s^2}+\frac{\psi(s)y_1^2y_3^3}{As^2(y_1^2+y_2^2)}+\frac{Ay_2^2}{y_3(y_1^2+y_2^2)},$$

$$g_{22}=\frac{y_2^2}{s^2}+\frac{\psi(s)y_2^2y_3^3}{As^2(y_1^2+y_2^2)}+\frac{Ay_1^2}{y_3(y_1^2+y_2^2)},$$

$$g_{33}=\frac{y_3^2}{s^2}+\frac{\psi(s)y_3(y_1^2+y_2^2)}{As^2},$$

$$g_{23}=\frac{y_2y_3}{s^2}-\frac{\psi(s)y_2y_3^2}{As^2},$$

$$g_{31}=\frac{y_3y_1}{s^2}-\frac{\psi(s)y_1y_3^2}{As^2},$$

$$g_{12}=\frac{y_1y_2}{s^2}+\frac{\psi(s)y_1y_2y_3^3}{As^2(y_1^2+y_2^2)}-\frac{Ay_1y_2}{y_3(y_1^2+y_2^2)},$$

where

$$s^2=y_1^2+y_2^2+y_3^2,\qquad A=\sqrt{\{s^2\psi(s)-y_1^2-y_2^2\}}.$$

But if (6.6) is a Schur space, it must be possible, as we have already seen, to find another system of normal co-ordinates (z_i) with the same origin, in terms of which

$$d\sigma^2=a_{ik}dz_idz_k,$$

where a_{ik} is given by (6.4). Since any normal co-ordinate-system is connected with every other normal co-ordinate-system having the same origin by a linear homogeneous transformation (Veblen, 1933, p. 85), it is obvious that (6.6) is not the metric of a Schur space; for no linear transformation of the variables (y_i) would eliminate the irrational function A. This establishes the result that *there exist centrally harmonic spaces which are not Schur spaces.*

It would, perhaps, be natural to conjecture that a completely harmonic space would be of constant curvature. If every centrally harmonic space were a Schur space, this result would follow at once, because a Schur space with more than one centre is of constant curvature (Duschek-Mayer, vol. ii, 1930, p. 176).

7. CURVATURE CONDITIONS FOR A CENTRALLY HARMONIC SPACE.

We now show that, at the base-point of a centrally harmonic space, the values taken by the curvature tensor R_{ijkl} and its covariant derivatives are not independent but satisfy an infinite sequence of relations. In a completely harmonic space these relations are satisfied everywhere and so form a system of differential equations for the coefficients g_{ij} of the metric.

The condition that a Riemannian space be centrally harmonic is

$$\Delta_2 s = \chi(s),$$

where s is the distance from the base-point x_0^i. The function $\chi(s)$ will in general involve the co-ordinates of the base-point, and so we now write the condition in the form

$$\Delta_2 s = \chi(s\,;x_0). \qquad (7.1)$$

It is often advantageous in the theory of partial differential equations to use, instead of s, the function

$$\Omega = \tfrac{1}{2}s^2, \qquad (7.2)$$

which is, so to speak, the integrated form of the scalar differential $\tfrac{1}{2}ds^2$ which defines the metric (Hadamard, 1923, p. 89; Ruse, 1939). Using this notation, we have

$$\Delta_2\Omega = s\Delta_2 s + \Delta_1 s = s\chi(s\,;x_0) + 1,$$

or, say,

$$\Delta_2\Omega = f(\Omega\,;x_0). \qquad (7.3)$$

From now on we work with equation (7.3) instead of (7.1).

Let us denote the covariant derivatives of Ω by the simple addition of suffixes, and those of the curvature tensor by suffixes preceded by a comma. Then if we write

$$\lim_{x\to x_0} \Omega = (\Omega)_0,$$

$$\lim_{x\to x_0} \Omega_i = (\Omega_i)_0,$$

etc., it is known (Synge, 1931, p. 241) that

$$(\Omega)_0 = 0, \qquad (7.4)$$

$$(\Omega_i)_0 = 0, \qquad (7.5)$$

$$(\Omega_{ij})_0 = (g_{ij})_0, \qquad (7.6)$$

$$(\Omega_{ijk})_0 = 0, \qquad (7.7)$$

$$(\Omega_{ijkl})_0 = -\tfrac{1}{3}(R_{ikjl} + R_{iljk})_0, \qquad (7.8)$$

$$(\Omega_{ijklm})_0 = -\tfrac{1}{4}(R_{ikjl,m} + R_{imjl,k} + R_{iljm,k} + R_{ikjm,l} + R_{imjk,l} + R_{iljk,m})_0, \quad (7.9)$$

$$\begin{aligned}(\Omega_{ijklmn})_0 = [& -\tfrac{4}{15}(R_{ikjl,mn} + \text{eleven similar terms}) \\ & + \tfrac{1}{15}(R_{ikjl,nm} + \text{eleven similar terms}) \\ & - \tfrac{1}{9}\{g^{pq}(2R_{ikpj} - R_{ijpk})(R_{lmqn} + R_{lnqm}) + \text{three similar terms}\} \\ & - \tfrac{1}{45}\{g^{pq}(R_{ikpl} + R_{ilpk})(R_{jmqn} + R_{jnqm}) + \text{five similar terms}\}]_0, \quad (7.10)\end{aligned}$$

the expressions on the right-hand sides being all evaluated at the base-point $(x_0{}^i)$. Formula (7.10) is very long and complicated but is given in full by Synge; the expressions for the limits of the seventh and higher covariant derivatives of Ω would no doubt be more complicated still.

Now equation (7.3) may be written

$$g^{ij}\Omega_{ij} = f(\Omega; x_0), \quad . \quad . \quad . \quad . \quad (7.11)$$

from which we deduce that

$$n = f(0; x_0)$$

by making $x \to x_0$ and using (7.4) and (7.6).

We next form the successive covariant derivatives of (7.11) and then make $x \to x_0$. The first differentiation gives

$$g^{ij}\Omega_{ijk} = f'(\Omega; x_0)\Omega_k,$$

where the accent denotes partial differentiation with respect to Ω; but this leads only to the trivial identity $0 = 0$. However, a further differentiation gives

$$g^{ij}\Omega_{ijkl} = f''(\Omega; x_0)\Omega_k\Omega_l + f'(\Omega; x_0)\Omega_{kl},$$

whence, by (7.8),

$$-\tfrac{1}{3}[g^{ij}(R_{ikjl} + R_{iljk})]_0 = (g_{kl})_0 f'(0; x_0). \quad . \quad . \quad . \quad (7.12)$$

We denote the Ricci tensor $g^{ij}R_{iklj}$ by R_{kl} and the scalar curvature $g^{kl}R_{kl}$ by R. Equation (7.12) is then

$$\tfrac{2}{3}(R_{kl})_0 = (g_{kl})_0 f'(0; x_0), \quad . \quad . \quad . \quad . \quad (7.13)$$

whence, by contraction,

$$\tfrac{2}{3}(R)_0 = n f'(0; x_0).$$

Finally, elimination of $f'(0; x_0)$ gives

$$\left(R_{kl} - \frac{R}{n}g_{kl}\right)_0 = 0. \quad . \quad . \quad . \quad . \quad . \quad (7.14)$$

Hence *at the base-point of a centrally harmonic space the curvature tensor satisfies the Einstein equations.* Geometrically this means that *the Ricci principal directions are indeterminate at the base-point* (Eisenhart, 1926, p. 114).

The third differentiation of (7.11) gives

$$g^{ij}\Omega_{ijklm} = f'''(\Omega; x_0)\Omega_k\Omega_l\Omega_m + f''(\Omega; x_0)(\Omega_{kl}\Omega_m + \Omega_{lm}\Omega_k + \Omega_{mk}\Omega_l) + f'(\Omega; x_0)\Omega_{klm}. \quad (7.15)$$

Making $x \rightarrow x_0$ and using Synge's equations, we obtain

$$(R_{kl,m} + R_{lm,k} + R_{mk,l})_0 = 0, \quad . \quad . \quad . \quad . \quad (7.16)$$

where $R_{kl,m}$ denotes the covariant derivative of the Ricci tensor. Now

$$g^{kl} R_{kl,m} = (g^{kl} R_{kl})_m = R_{,m},$$

and also

$$g^{kl} R_{lm,k} \equiv \tfrac{1}{2} R_{,m}$$

(Eisenhart, 1926, p. 82, equation 26.4), so that it follows from (7.16) that

$$(R_{,m})_0 = 0. \quad . \quad . \quad . \quad . \quad . \quad (7.17)$$

Since the covariant derivative of a scalar is its ordinary partial derivative, (7.17) means that *at the base-point of a centrally harmonic space the scalar curvature R has a stationary value.* This result is connected with the fact that in a centrally harmonic V_2 the distance-circles at the base-point have constant geodesic curvature, as was proved in § 4 (*cf.* Blaschke, 1924, p. 101).

If we differentiate (7.15) covariantly and make $x \rightarrow x_0$, using Synge's relations and the equation (7.13), we obtain the identity

$$\begin{aligned} &\tfrac{8}{15}(R_{kl,mn} + R_{km,ln} + R_{kn,lm} + R_{mn,kl} + R_{ln,km} + R_{lm,kn})_0 \\ &- \tfrac{2}{15}(R_{kl,nm} + R_{km,nl} + R_{kn,ml} + R_{mn,lk} + R_{ln,mk} + R_{lm,nk})_0 \\ &- \tfrac{2}{45}[g^{pq}\{(R^i{}_{kpl} + R^i{}_{lpk})(R_{imqn} + R_{inqm}) + (R^i{}_{kpm} + R^i{}_{mpk})(R_{inql} + R_{ilqn}) \\ &\qquad + (R^i{}_{kpn} + R^i{}_{npk})(R_{ilqm} + R_{imql})\}]_0 \\ &= f''(0; x_0)(g_{kl}g_{mn} + g_{km}g_{nl} + g_{kn}g_{lm})_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.18) \end{aligned}$$

where R_{klmn} denotes the curvature tensor and $R_{kl,mn}$ the second covariant derivative of the Ricci tensor. From this $f''(0; x_0)$ may be evaluated by multiplying by $(g^{kl}g^{mn})_0$ and summing for k, l, m, and n; the substitution of this value in (7.18) leads to a complicated relation satisfied at the base-point of a centrally harmonic space by the curvature tensor, its covariant derivatives and the fundamental tensor.

The nature of the infinite sequence of relations to be satisfied at the base-point of a centrally harmonic space is more clearly seen when they are expressed in terms of the normal tensors (Veblen, 1933, pp. 89 *et seq.*). By transferring to normal co-ordinates with the base-point as origin it can be shown that the conditions to be satisfied *at the base-point*, in terms of the normal tensors $A^i{}_{jkl}$, $A^i{}_{jklm}$, . . ., are

$$A^i{}_{ijk} + A^i{}_{ikj} = \psi_1(x) g_{jk}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.19)$$

$$A^i{}_{ijkl} + A^i{}_{iklj} + A^i{}_{iljk} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.20)$$

$$A^i{}_{ijklm} + \ldots = \psi_2(x)(g_{jk}g_{lm} + g_{jl}g_{km} + g_{jm}g_{kl}), \quad . \quad . \quad . \quad (7.21)$$

$$A^i{}_{ijklmn} + \ldots = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.22)$$

$$A^i{}_{ijklmnp} + \ldots = \psi_3(x)(g_{jk}g_{lm}g_{np} + \text{fourteen similar terms}), \quad . \quad . \quad (7.23)$$

and so on, where the dots on the left indicate the terms obtained by cyclic permutation of the free suffixes. The law of formation of these relations is evident.

The functions $\psi_1(x)$, $\psi_2(x)$, $\psi_3(x)$, . . . are scalars which may be eliminated, in a manner similar to that indicated for the elimination of $f''(0;x_0)$ from (7.18), to give relations homogeneous in the normal tensors and the coefficients g_{ij} of the metric.

The curvature conditions in terms of normal tensors are necessarily the same as those obtained in terms of the curvature tensor. In particular (7.19) is exactly equivalent to (7.13) and (7.20) to (7.16). The advantage of the normal tensor method is that it gives at once the full set of conditions for a harmonic space.

8. THE CURVATURE TENSOR IN A COMPLETELY HARMONIC SPACE.

In a completely harmonic space the formulæ found in § 7 hold at all points $(x_0{}^i)$ and determine successively the coefficients in the expansion of $f(\Omega;x_0)$ as a Taylor series in Ω. Dropping the zero suffix, we have, in particular,

$$f(0;x)=n,$$
$$f'(0;x)=\frac{3}{2n}R,$$

at every point (x^i) of a completely harmonic space.

When, however, we turn to the converse problem of determining the conditions under which a space is completely harmonic, we must eliminate the scalars $f(0;x)$, $f'(0;x)$, $f''(0;x)$, . . . since they have no immediate geometrical significance. The first equation obtained in this way is (7.14), viz.

$$R_{kl}-\frac{1}{n}g_{kl}R=0. \qquad (8.1)$$

Here *every completely harmonic space is an Einstein space.*

The equations (8.1) give no information when $n=2$ since they are satisfied by every V_2. When $n=3$, on the other hand, they imply that the space is of constant curvature (Eisenhart, 1926, p. 92, Ex. 2). No such deduction can be made for $n>3$. Combining this result with that of § 4, we have: *All completely harmonic spaces of two and three dimensions are spaces of constant curvature.*

Hereafter we assume $n>3$. The next equation to be considered is (7.16), viz.

$$R_{kl,m}+R_{lm,k}+R_{mk,l}=0.$$

This gives no new information since it is a consequence of (8.1); for in an Einstein space the scalar curvature R is constant and so

$$R_{kl,\,m}=0. \qquad (8.2)$$

From (8.2) we obtain

$$R_{kl,\,mn}=0$$

by covariant differentiation. The first two groups of terms on the left-hand side of (7.18) then disappear. After elimination of $f''(0;x)$ in the manner previously indicated, we obtain

$$g^{pq}g^{rs}\{(R_{rkpl}+R_{rlpk})(R_{smqn}+R_{snqm})+\ .\ .\ .\}$$
$$=\frac{4}{n(n+2)}(g_{kl}g_{mn}+g_{km}g_{nl}+g_{kn}g_{lm})\left(\frac{1}{n}R^2+R^{pqrs}R_{pqrs}+R^{pqrs}R_{psrq}\right). \qquad (8.3)$$

To obtain any further conditions for a completely harmonic space by using Synge's relations is impracticable, but the whole set, infinite in number though not necessarily independent, may be obtained from (7.19), (7.20), . . .

Equation (8.3), like all of those of the infinite set of conditions, is certainly satisfied by a space of constant curvature, but does not appear to provide a sufficient condition for such a space. We have not succeeded in determining whether a completely harmonic space is necessarily of constant curvature when $n>3$; that it may be so therefore remains at present a pure conjecture. If it is not so, completely harmonic spaces constitute a class less general than Einstein spaces but more general than spaces of constant curvature.

9. SUMMARY.

In this paper, two classes of Riemannian spaces arising from the theory of the solution of Laplace's equation are discussed, namely centrally harmonic spaces and completely harmonic spaces. It is proved that

(i) a Schur space is centrally harmonic;
(ii) a space of constant curvature is completely harmonic;
(iii) a completely harmonic space is an Einstein space;
(iv) a completely harmonic V_n is of constant curvature when $n=2$ or 3.

A sequence of differential equations to be satisfied by the curvature tensor in a completely harmonic space is found.

The question whether every completely harmonic space is of constant curvature remains unanswered.

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X.—Interpolated Derivatives. By B. Spain, Ph.D., Mathematical Institute, University of Edinburgh. *Communicated by* Professor E. T. Whittaker, F.R.S.

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1. VARIOUS writers (Ferrar, 1927) have started out with different definitions of generalised derivatives. Essentially, the problem of the generalised derivative is a problem in interpolation. The values of the derivatives are known for all integer values of n; for all positive integers, being the ordinary derivatives; for zero, being the function itself; for negative integers, being repeated integrals. Any function of n which has the above values at the integers (*i.e.* any cotabular function) is a solution of the problem. Out of the infinite number of cotabular functions, there exists one discovered by E. T. Whittaker (Whittaker, 1915; Ferrar, 1925; Whittaker, 1935), called the cardinal function, possessing rather remarkable properties. In particular, if the cardinal series defining the cardinal function is convergent, then it is equivalent to the Newton-Gauss formula of interpolation.

It seems desirable to interpolate the derivatives by means of the unique cardinal function, and call the cardinal function so obtained the generalised derivative.

It turns out, however (§ 8), that this definition is unsatisfactory, owing to the impossibility of repeating the operation of generalised differentiation. Instead, the usual definition of the generalised derivative can be obtained starting from the cardinal series for the function divided by e^{kn} and allowing k to tend to infinity.

2. The cardinal function corresponding to any function $F(n)$ with given values at the arguments 0, ± 1, ± 2, etc., is

$$F(n)=\frac{\sin \pi n}{\pi}\sum_{r=-\infty}^{+\infty}\frac{(-1)^r F(r)}{n-r}. \qquad (2.1)$$

Here we are considering the nth derivative as a function of n, so let us put for positive integer values of r (including $r=0$),

$$F(r)=f^{(r)}(x). \qquad (r=0, 1, 2, \ldots)$$

For negative integer values of r, $F(r)$ will represent the rth repeated integral, and it is easy to see that

$$F(r)=\frac{1}{(-r-1)!}\int_a^x (x-u)^{-r-1}\,.\,f(u)du,\quad (r=-1,\ -2,\ \ldots)$$

where we have chosen the same arbitrary constant for the lower limit of each integration.

Substituting in (2.1), we obtain

$$\frac{\pi F(n)}{\sin \pi n}=\sum_{r=0}^{\infty}\frac{(-1)^r f^{(r)}(x)}{n-r}+\sum_{r=1}^{\infty}\frac{(-1)^r}{n+r}\cdot\frac{1}{(r-1)!}\int_a^x (x-u)^{r-1}\,.\,f(u)du. \qquad (2.2)$$

3. In order that the cardinal function should exist, it is of course necessary that the series for $F(n)$ should converge. As will be seen in § 5, the second series in (2.2) introduces no difficulties as regards convergence. The first series can obviously diverge unless $f^{(r)}(x)$ satisfies certain conditions.

Now $\sum_{r=0}^{\infty}(-1)^r f^{(r)}(x)/(n-r)$ converges and diverges together with $\sum_{r=1}^{\infty}(-1)^r f^{(r)}(x)/r$. This series converges if $\sum_{r=1}^{\infty}|f^{(r)}(x)|/r$ converges. A sufficient condition for the convergence of this series is

$$|f^{(r)}(x)|<A(x)\,.\,r^{-\epsilon},\qquad \epsilon>0,\ r\geq R \qquad (3.1)$$

where R is a sufficiently large, but finite integer.

From this inequality it follows immediately that $f(x)$ is an integral function.

In what follows we shall assume (3.1) holds.

4. Now consider

$$\sum_{r=0}^{\infty}\frac{(-1)^r f^{(r)}(x)}{n-r}=-\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\int_0^{\infty}e^{-v}\,.\,v^r dv\,.\,f^{(r)}(x)\int_1^{\infty}t^{n-r-1}\,.\,dt. \qquad (n<0) \qquad (4.1)$$

We are now going to apply the extension of theorem § 176 (B) of Bromwich (Bromwich, 1926) to repeated integrals in order to justify the inversion of summation and integration.

Firstly, we note that t^{n-1} is continuous for all finite values of t greater than or equal to unity, and e^{-v} is continuous for all finite values of v. Secondly, since $f(x)$ is an integral function, the series $\sum_{r=0}^{\infty}(-1)^r(v/t)^r f^{(r)}(x)/r!$ converges uniformly in any finite interval, being in fact $f(x-v/t)$. Thirdly, we have

$$\sum_{r=R}^{\infty}\int_0^{\infty}dv\int_1^{\infty}dt\left|\frac{(-1)^r}{r!}e^{-v}\,.\,v^r\,.\,f^{(r)}(x)\,.\,t^{n-r-1}\right| \quad . \quad . \quad (4.2)$$

$$=\sum_{r=R}^{\infty}\int_0^{\infty}dv\int_1^{\infty}dt\,.\,\frac{1}{r!}e^{-v}\,.\,v^r\,|f^{(r)}(x)|\,.\,t^{n-r-1}$$

$$=\sum_{r=R}^{\infty}\frac{|f^{(r)}(x)|}{n-r}.$$

This latter series converges because of (3.1); thus (4.2) converges, and thus we can reverse the order of summation and integration, obtaining

$$\sum_{r=0}^{\infty}\frac{(-1)^r f^{(r)}(x)}{n-r}=-\int_0^{\infty}dve^{-v}\int_1^{\infty}dt\,.\,t^{n-1}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{v}{t}\right)^r f^{(r)}(x)$$

$$=-\int_0^{\infty}dve^{-v}\int_1^{\infty}dt\,.\,t^{n-1}f\left(x-\frac{v}{t}\right),$$

i.e.

$$\sum_{r=0}^{\infty}\frac{(-1)^r f^{(r)}(x)}{n-r}=\int_x^{-\infty}duf(u)\int_1^{\infty}t^n e^{-t(x-u)}\,.\,dt$$

by the substitution $x-v/t=u$.

5. Consider

$$\xi^{-n-1}\gamma(n+1,\,\xi)=\int_0^1 t^n e^{-t\xi}\,.\,dt, \qquad (n>-1) \quad . \quad (5.1)$$

where γ is the incomplete Gamma-function (Whittaker and Watson, 1927). $\xi^{-n-1}\gamma(n+1,\,\xi)$ is a continuous function of ξ, having no singularities. Now

$$\xi^{-n-1}\gamma(n+1,\,\xi)=\int_0^1 t^n\sum_{r=0}^{\infty}\frac{(-1)^r(t\xi)^r}{r!}dt.$$

Since $\sum_{r=0}^{\infty}(-1)^r(t\xi)^r/r!$ is uniformly convergent in $0\leq t\leq 1$, we can perform the integration before the summation, obtaining

$$\xi^{-n-1}\gamma(n+1,\,\xi)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\xi^r\int_0^1 t^{n+r}\,.\,dt$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\xi^r\frac{1}{n+r+1}$$

$$=-\sum_{r=1}^{\infty}\frac{(-1)^r}{(r-1)!}\,\frac{\xi^{r-1}}{n+r}.$$

This series is obviously uniformly convergent in any finite interval $0\leq y\leq Y$. Thus in the second series on the right-hand side of (2.2) we can reverse the order of integration and summation, giving

$$\sum_{=1}\frac{(-1)^r}{n+r}\cdot\frac{1}{(r-1)!}\int_a^x (x-u)^{r-1}\,.\,f(u)du = -\int_a^x duf(u)\int_0^1 t^n e^{-t(x-u)}\,.\,dt. \quad (n > -1) \quad (5.2)$$

6. Substituting from (4.2) and (5.2) into (2.2), we obtain for $0 > n > -1$ that

$$\frac{\pi F(n)}{\sin \pi n} = \int_x^{-\infty} duf(u)\int_1^{\infty} t^n e^{-t(x-u)}\,.\,dt - \int_a^x duf(u)\int_0^1 t^n e^{-t(x-u)}\,.\,dt$$

$$= \int_x^a duf(u)\int_0^{\infty} t^n e^{-t(x-u)}\,.\,dt + \int_a^{-\infty} duf(u)\int_1^{\infty} t^n e^{-t(x-u)}\,.\,dt$$

$$= \int_x^a \frac{f(u)}{(x-u)^{n+1}}\,.\,\Gamma(n+1)\,.\,du + \int_a^{-\infty} duf(u)\int_1^{\infty} t^n e^{-t(x-u)}\,.\,dt. \quad (x > a)$$

Using the well-known identity

$$-\Gamma(n+1)\Gamma(-n) = \pi/\sin \pi n, \quad . \quad . \quad . \quad . \quad (6.1)$$

we obtain

$$F(n) = \frac{1}{\Gamma(-n)}\left\{\int_a^x \frac{f(u)du}{(x-u)^{n+1}} + \int_{-\infty}^a duf(u)\int_1^{\infty}\frac{t^n e^{-t(x-u)}}{\Gamma(n+1)}dt\right\}. \quad (x > a) \quad (6.2)$$

So far we have only established this formula for $0 > n > -1$. Now $\int_a^x \frac{f(u)du}{(x-u)^{n+1}}$ is analytic for any negative value of $R(n)$, and by the method used by Copson (Copson, 1935, § 9.11), the second integral of $F(n)$ is an integral function of n. Thus we can continue analytically and state that (6.2) gives the definition of the generalised derivative for $R(n) < 0$. For values of n such that $R(n) > 0$, we select an integer l greater than n, and obviously the analytical continuation of $F(n)$ is

$$F(n) = \frac{d^l}{dx^l}\left\{\frac{1}{\Gamma(-n+l)}\int_a^x \frac{f(u)du}{(x-u)^{n+1}}\right\} + \int_{-\infty}^a duf(u)\int_1^{\infty}\frac{t^n e^{-t(x-u)}}{\Gamma(n+1)}dt.$$

Thus we have obtained the generalised derivative in the complete n-plane.

For the remainder of this paper, we shall always suppose $R(n) < 0$, which really involves no loss in generality.

7. It is interesting to note that $F(n)$ is composed of two parts. The first is none other than $D^n\,.\,f(x)$, where

$$D^n\,.\,f(x) = \frac{1}{\Gamma(-n)}\int_a^x \frac{f(u)du}{(x-u)^{n+1}}. \quad . \quad . \quad . \quad (7.1)$$

$D^n\,.\,f(x)$ is substantially Riemann's definition of the generalised derivative (Ferrar, 1927). The second part is

$$J(n) = \frac{1}{\Gamma(-n)}\int_{-\infty}^a duf(u)\int_1^{\infty}\frac{t^n e^{-t(x-u)}}{\Gamma(n+1)}dt.$$

J(n) bears a close resemblance to the complementary function as defined by Riemann, namely,

$$\frac{1}{\Gamma(-n)}\int_{u_1}^{u_2}\frac{f(u)du}{(x-u)^{n+1}}.$$

In fact, J(n) would equal this expression if the lower limit of the integration over t were zero instead of unity. We shall accordingly call J(n) the complementary function. This complementary function vanishes for $a=-\infty$, which indicates that $-\infty$ plays a peculiar rôle here. The complementary function also vanishes when n is any positive or negative integer, as it ought to.

The integration over t can be carried out explicitly by introducing Whittaker's function (Whittaker and Watson). The final result is:

$$\mathrm{J}(n)=-\frac{\sin\pi n}{\pi}\int_{-\infty}^{a}\frac{f(u)e^{-\frac{1}{2}(x-u)}}{(x-u)^{\frac{1}{2}n+1}}\mathrm{W}_{\frac{1}{2}n,\,\frac{1}{2}(n+1)}(x-u)du. \qquad (7.2)$$

8. Introduce the operator notation for generalised differentiation by

$$\mathrm{I}^{(n)}\,.\,f(x)=\mathrm{F}(n). \qquad (8.1)$$

Then it is natural to ask, Do the relations

$$\mathrm{I}^{(m)}\,.\,\mathrm{I}^{(n)}\,.\,f(x)=\mathrm{I}^{(n)}\,.\,\mathrm{I}^{(m)}\,.\,f(x)=\mathrm{I}^{(m+n)}\,.\,f(x) \qquad (8.2)$$

still hold good? To investigate this we note from (6.2) that F(n) is only defined for $x>a$, yet in the complementary function a knowledge of $f(u)$ for values of $x<a$ is demanded. Thus, if we wish to differentiate F(n) m times ($m<0$), we require the values of F(n) for $x<a$. As (6.2) only holds for $x>a$, we come to the conclusion that it is impossible to perform such an operation as $\mathrm{I}^{(m)}\,.\,\mathrm{I}^{(n)}$. Therefore there is no meaning at all in asking whether (8.2) holds or not.

It is interesting to note that $a=-\infty$ is an exceptional case, as then the complementary function vanishes, and as is well known and easily demonstrated (8.2) does hold.

Thus we come to the conclusion that the cardinal function does not give a satisfactory definition of the generalised derivative unless $a=-\infty$.

9. The cardinal function really represents the Newton-Gauss formula for interpolation. Consider the interpolation formula

$$\mathrm{F}_k(n)=\frac{\sin\pi n}{\pi}e^{kn}\left\{\sum_{r=-\infty}^{+\infty}\frac{(-1)^r\mathrm{F}(r)e^{-kr}}{n-r}\right\}. \qquad (k>0) \quad . \quad (9.1)$$

We easily see that this is a permissible interpolation formula. This interpolation formula is no longer equivalent to the Newton-Gauss formula for $F(n)$, but it is related to the Newton-Gauss formula for $F(n)e^{-kn}$. From (9.1) and the definitions of $F(r)$ we obtain

$$e^{-kn}\frac{\pi F_k(n)}{\sin \pi n} = \sum_{r=0}^{\infty}\frac{(-1)^r f^{(r)}(x)e^{-kr}}{n-r} + \sum_{r=1}^{\infty}\frac{(-1)^r e^{kr}}{n+r}\cdot\frac{1}{(r-1)!}\int_a^x (x-u)^{r-1}\,.\,f(u)\,.\,du. \qquad (9.2)$$

Since the series $\sum_{r=1}^{\infty}\frac{(-1)^r e^{k(r-1)}\,.\,(x-u)^{r-1}}{(n+r)(r-1)!}$ is uniformly convergent in any finite interval $0 \leq (x-u)e^k \leq K$, we can in a similar manner to the derivation of (5.2) now obtain

$$\sum_{r=1}^{\infty}\frac{(-1)^r e^{kr}}{n+r}\cdot\frac{1}{(r-1)!}\int_a^x (x-u)^{r-1}\,.\,f(u)du = -e^{-nk}\int_a^x du f(u)\int_0^{e^k} t^n e^{-t(x-u)}\,.\,dt. \qquad (n > -1) \quad (9.3)$$

Also $\sum_{r=0}^{\infty} | f^{(r)}(x) | e^{-kr}/(n-r)$ is now convergent under the less restrictive condition

$$| f^{(r)}(x) | < A(x)e^{kr}r^{-\epsilon}. \qquad (\epsilon > 0),\ (r \geq R)\,. \quad (9.4)$$

(This condition for finite k also restricts $f(x)$ to integral functions.) We can, therefore, apply the same process as in § 4, leading to

$$\sum_{r=0}^{\infty}\frac{(-1)^r f^{(r)}(x)e^{-kr}}{n-r} = e^{-nk}\int_x^{-\infty} du f(u)\int_{e^k}^{\infty} t^n e^{-t(x-u)}\,.\,dt. \qquad (n < 0) \quad (9.5)$$

Thus from (9.3), (9.5), (9.2), and (6.1) we obtain

$$F_k(n) = \frac{1}{\Gamma(-n)}\left\{\int_a^x \frac{f(u)du}{(x-u)^{n+1}} + \int_{-\infty}^a du f(u)\int_{e^k}^{\infty} t^n e^{-t(x-u)}\,.\,dt\right\}. \qquad (0 > n > -1) \quad (9.6)$$

This formula also, of course, only holds for $x > a$. It is interesting to note that the main part of the derivative is identical with that of (6.2), only the complementary function being affected by k.

Now let k tend to infinity and we obtain the final result:

$$D^n\,.\,f(x) = \lim_{k \to \infty} F_k(n) = \frac{1}{\Gamma(-n)}\int_a^x \frac{f(u)du}{(x-u)^{n+1}}. \qquad (0 > n > -1) \quad (9.7)$$

From (9.3) and (9.5) we see that the main contribution comes from the negative portion, the portion with r positive or zero giving a zero contribution.

The definition of the generalised derivative by (9.7) from the interpolation formula (9.1) with $k \to \infty$ does agree with the usual definition, and its analytic continuation for all values of n is at once evident. From (9.4) as $k \to \infty$, we see that in the limiting case $f(x)$ need no longer be an integral function.

My best thanks are due to Dr A. Erdélyi.

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XI.—Reciprocity. Part III: Reciprocal Wave Functions.

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1. INTRODUCTION.

THE attempt to find reciprocal wave functions made in Part II led to some difficulties. We believe that these were due to an incorrect definition of the relativistic scalar product and Fourier coefficient. We shall show in this paper that a modification of these definitions leads to simpler equations, which allow us to determine the reciprocal wave functions and to find a transcendental equation for $\mu = a\epsilon/k$, as was envisaged in Part I.

In Part II we started from the fundamental wave functions

$$\left.\begin{aligned}(E^2 - p^2)\psi &= \epsilon^2\psi,\\ (r^2 - t^2)\phi &= a^2\phi,\end{aligned}\right\} \qquad (1.1)$$

where ϵ, a were assumed to be an energy and a length associated with a particle. Consequently they were assumed to be real quantities.

But there is no reason why we should restrict the general theory *a priori* to positive proper values of the operators (1.1). This is clear if we consider $\mathbf{r}$, t-space: the values of $r^2 - t^2$ may be either positive or negative. Hence we shall discard this restriction. Thereby we gain sufficient elasticity of the theory to overcome the difficulties encountered in Part II.

The change in the point of view necessitates some changes of notation, which will be introduced presently.

2. NOTATION.

It will be useful to introduce a name for the three parts (I), (II$^+$), and (II$^-$) into which space-time (and momentum-energy) is divided by the light cone. We shall call the region (I) "present" ($-r < ct < r$), the region (II$^+$) "future" ($ct > r$), and the region (II$^-$) "past" ($ct < -r$).

We define

$$R = \sqrt{r^2 - t^2}, \qquad P = \sqrt{p^2 - E^2}, \qquad (2.1)$$

whereas in Part II P was defined as $\sqrt{E^2 - p^2}$. Also the introduction of polar co-ordinates will here be done in the same way in both $\mathbf{r}$, t- and $\mathbf{p}$, E-space, the only difference being that in the one case we use the contra-variant vector x^k, in the other the co-variant vector p_k:

$$\begin{aligned} \mathbf{r} &= \mathrm{R}\cosh\alpha\,\mathbf{e}, \\ x^0 &= \mathrm{R}\sinh\alpha, \end{aligned} \quad \mathbf{e} = \begin{cases} \sin\beta\cos\gamma; \\ \sin\beta\sin\gamma; \\ \cos\beta; \end{cases} \quad \begin{aligned} \mathbf{p} &= \mathrm{P}\cosh\alpha'\,\mathbf{e}', \\ p_0 &= \mathrm{P}\sinh\alpha', \end{aligned} \quad \mathbf{e}' = \left\{\begin{matrix} \sin\beta'\cos\gamma', \\ \sin\beta'\sin\gamma', \\ \cos\beta'. \end{matrix}\right\} \qquad (2.2)$$

Equations in $\mathbf{p}$, E-space are now obtained from those in $\mathbf{r}$, t-space by the substitution

$$\mathrm{R} \rightarrow \mathrm{P}, \qquad \alpha \rightarrow -\alpha', \qquad \beta \rightarrow \beta', \qquad \gamma \rightarrow \gamma'. \quad . \quad . \quad (2.3)$$

3. RELATIVISTIC FOURIER COEFFICIENTS.

The *relativistic scalar* product of two ψ-functions or two ϕ-functions is defined formally as in Part II, (1.5) and (1.6):

$$\left.\begin{aligned} (\psi_1 \cdot \psi_2)_{\mathrm{R}} &= \int_{(\mathrm{R})} \psi_1^*(\mathbf{r}, t)\psi_2(\mathbf{r}, t)d\tilde{r}, \\ (\phi_1 \cdot \phi_2)_{\mathrm{P}} &= \int_{(\mathrm{P})} \phi_1^*(\mathbf{p}, \mathrm{E})\phi_2(\mathbf{p}, \mathrm{E})d\tilde{p}, \end{aligned}\right\} \quad . \quad . \quad . \quad (3.1)$$

where the integrals extend over the surfaces of the pseudo-spheres $r^2 - t^2 = \mathrm{R}^2$ and $p^2 - \mathrm{E}^2 = \mathrm{P}^2$, respectively. These scalar products are not analytical functions of the parameters R and P. For, on changing from real to imaginary values of R or P, the region of integration changes from a one-sheet pseudo-sphere into a two-sheet pseudo-sphere.

In Part II we considered only real values of R and imaginary values of P; therefore the first integral was an integral over a one-sheet pseudo-sphere, the second over a two-sheet pseudo-sphere, and the question of analytical continuity did not arise. Here we consider real and imaginary values of R as well as of P, and we define the scalar product by the integrals (3.1) for real values of R and P (integrals over one-sheet pseudo-sphere) and by analytical continuation, if R or P are imaginary.

It should be noted that the general considerations about the properties of the operators M and N in § 6 of Part II now hold in general, and not only for scalar products in $\mathbf{r}$, t-space.

The Fourier coefficients are now defined as in Part II, (1.8), by

$$\Phi^{(\mathrm{R})}(\mathbf{p}, \mathrm{E}) = \left(\frac{1}{h^{3/2}}e^{i\mathrm{S}/\hbar} \cdot \psi\right)_{\mathrm{R}}, \qquad \Psi^{(\mathrm{P})}(\mathbf{r}, t) = \left(\frac{1}{h^{3/2}}e^{-i\mathrm{S}/\hbar} \cdot \phi\right)_{\mathrm{P}}, \quad . \quad (3.2)$$

with

$$\mathrm{S} = \mathbf{r} \cdot \mathbf{p} - t\mathrm{E}. \quad . \quad . \quad . \quad . \quad . \quad (3.3)$$

In view of our definition of the scalar product, these are analytical functions of the parameters R and P. But they are not analytical functions of the arguments (*i.e.* $\mathbf{p}$, E for Φ and $\mathbf{r}$, t for Ψ). This point has already been considered in Part II, and the following definition of the Fourier coefficients has been given: They are defined by the scalar products (3.2) for values of the argument in the "present," and by analytical continuation in the "past" and "future."

This definition will also be used here, though its meaning is not the same owing to the new definition of the scalar product.

4. THE WAVE EQUATION.

Instead of (1.1), we consider the wave equations

$$\left.\begin{aligned} P^2\psi = (p^2 - E^2)\psi = b^2\psi, \\ R^2\phi = (r^2 - t^2)\phi = a^2\phi. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4.1)$$

Since we consider positive as well as negative proper values of the operators R^2 and P^2, a and b are either real or imaginary. In the latter case we also introduce real quantities τ, ϵ by the definitions

$$a = i\tau, \qquad b = i\epsilon. \quad . \quad . \quad . \quad . \quad . \quad (4.2)$$

The quantities a, τ and b, ϵ are characteristic invariant constants of a particle: a length (radius) or time (of retardation), a momentum or energy.

The solutions of the wave equation are in our changed notation (*cf.* Part II, (5.5) and (5.6)):

$$\left.\begin{aligned} \psi^{(b)}_{klm}(R, \alpha, \beta, \gamma) = \frac{1}{R}Z^{(1)}_{k+1}(bR/\hbar)f_{klm}(\alpha, \beta, \gamma), \\ \phi^{(a)}_{klm}(P, \alpha', \beta', \gamma') = \frac{1}{P}Z^{(2)}_{k+1}(aP/\hbar)f_{klm}(\alpha', \beta', \gamma'). \end{aligned}\right\} \quad . \quad . \quad (4.3)$$

Here Z is a Bessel function and f_{klm} a pseudo-spherical harmonic in four dimensions (*cf.* Part II, (4.8)).

5. RECIPROCAL FUNCTIONS.

If $\psi^{(b)}$ and $\phi^{(a)}$ are solutions of the wave equations (4.1), the relativistic Fourier coefficients (3.2) have two indices:

$$\Phi^{(b, R)} = \left(\frac{1}{\hbar^{3/2}}e^{iS/\hbar} \cdot \psi^{(b)}\right)_R, \qquad \Psi^{(a, P)} = \left(\frac{1}{\hbar^{3/2}}e^{-iS/\hbar} \cdot \phi^{(a)}\right)_P. \quad . \quad (5.1)$$

We now define a pair of reciprocal functions by the equations

$$\Phi^{(b,a)}=\phi^{(a)}, \qquad \Psi^{(a,b)}=\psi^{(b)}. \quad . \quad . \quad . \quad . \quad . \quad (5.2)$$

We shall now show that under certain conditions $\psi^{(b)}_{klm}$ and $\phi^{(a)}_{klm}$ defined by (4.3) are reciprocal functions.

$\Phi^{(b,a)}_{klm}$ can be obtained from (7.16) of Part II; in the present notation we have

$$\Phi^{(b,a)}_{klm}=(-1)^{l+1}i^{(k+1)}\sqrt{\frac{2\pi}{k}}aZ^{(1)}_{k+1}(ba/k)\frac{J_{k+1}(aP/k)}{P}f_{klm}(\alpha', \beta', \gamma'). \qquad (5.3)$$

From this $\Psi^{(a,b)}_{klm}$ can be found by exchanging a, b and substituting $p_k \to -x^k$, or $P \to R$, $\alpha' \to -\alpha$, $\cos\beta' \to -\cos\beta$, $\gamma'=\gamma+\pi$, with the result

$$\Psi^{(a,b)}_{klm}=(-1)^{l+1}i^{-(k+1)}\sqrt{\frac{2\pi}{k}}bZ^{(2)}_{k+1}(ab/k)\frac{J_{k+1}(bR/k)}{R}f_{klm}(\alpha, \beta, \gamma). \qquad (5.4)$$

Inserting (5.3) and (5.4) into (5.2) and comparing with (4.3), we see that the functions (4.3) are reciprocal functions, if we choose the Bessel functions so that

$$Z^{(1)}_{k+1}=C_1J_{k+1}, \qquad Z^{(2)}_{k+1}=C_2J_{k+1}. \quad . \quad . \quad . \quad . \quad (5.5)$$

We then obtain for C_1 and C_2 the equations

$$\left.\begin{aligned}(-1)^{l+1}i^{(k+1)}\sqrt{\frac{2\pi}{k}}aC_1J_{k+1}(ab/k)=C_2,\\(-1)^{l+1}i^{-(k+1)}\sqrt{\frac{2\pi}{k}}bC_2J_{k+1}(ab/k)=C_1.\end{aligned}\right\} \quad . \quad . \quad . \quad (5.6)$$

It follows that

$$\mu=\frac{ab}{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.7)$$

must satisfy the equation

$$2\pi\mu J^2_{k+1}(\mu)=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.8)$$

a, b and hence μ are either real or imaginary. The equation (5.8) has an infinite number of real solutions for any given value of k, which will be denoted by μ_{kn}, $n=1, 2, \ldots$ But there are no imaginary solutions. Imaginary solutions can be obtained by a slight relaxation in the definition of reciprocal functions, allowing a factor of the absolute value 1 to be introduced in (5.2). Instead of (5.8) we then obtain

$$2\pi\,|\,\mu J^2_{k+1}(\mu)\,|=1. \quad . \quad . \quad . \quad . \quad . \quad (5.8a)$$

This equation has one imaginary solution μ_{k0} for a given value of k. The first few solutions are given in the following table (as far as they can be obtained from the tables of Jahnke and Emde, 1933):—

TABLE OF μ_{kn}.

n \ k	−1	0	1	2	3	4	5	6	7
0	·157i	·815i	1·48i	2·15i	2·81i	3·47i	4·14i	4·80i	
1	·161	·925	1·773	2·652	3·548	4·457	5·37	6·30	7·24
2	1·885	3·307	4·596	5·830	7·028	8·203	9·34		
3	2·925	4·362	5·677	6·932	8·150	9·342			
4	4·997	6·490	7·887	9·226					
5	6·043	7·541	8·947						
6	8·130	9·649							
7	9·176	10·698							
8	11·268	12·800							
9	12·315	13·848							
10	14·407								
11	15·454								

The asymptotic solutions for large values of n are ($n >> k$):

$$\mu_{k,\,2n+1}=\pi\left(\frac{2k+3}{4}+n-\tfrac{1}{8}\right),\qquad \mu_{k,\,2n+2}=\pi\left(\frac{2k+3}{4}+n+\tfrac{1}{8}\right). \quad . \quad (5{\cdot}9)$$

From (5.6) we obtain the following relation between C_1 and C_2:—

$$\frac{C_1}{C_2}=i^{-(k+1)}(-1)^{l+1}\sqrt{\frac{b}{a}}. \quad . \quad . \quad . \quad . \quad (5.10)$$

Thus we obtain the reciprocal functions

$$\left.\begin{aligned}\psi^{(b)}_{klm}&=Ci^{-(k+1)}(-1)^{l+1}\sqrt{b}\,\frac{J_{k+1}(bR/h)}{R}f_{klm}(\alpha,\beta,\gamma),\\ \phi^{(a)}_{klm}&=C\sqrt{a}\,\frac{J_{k+1}(aP/h)}{P}f_{klm}(\alpha',\beta',\gamma').\end{aligned}\right\} \quad . \quad . \quad (5.11)$$

These functions can easily be normalised. We assume that the f_{klm} are normalised pseudo-spherical harmonics. This normalisation has to be performed by means of the relativistic scalar products in the "present" and is, with the help of our new definitions, preserved in "past" and "future."

We have

$$(\psi\,.\,\psi)_a=CC^*\mid baJ^2_{k+1}(ba/h)\mid=1,$$
$$(\phi\,.\,\phi)_b=CC^*\mid baJ^2_{k+1}(ba/h)\mid=1;$$

with (5.8) (or (5.8*a*) for imaginary values of μ) we find from both equations

$$CC^*=\frac{2\pi}{h}=\frac{1}{\hbar}. \quad . \quad . \quad . \quad . \quad . \quad (5.12)$$

Thus we have the normalised reciprocal functions

$$\left.\begin{aligned} \psi^{(b)}_{klm} &= i^{-(k+1)}(-1)^{l+1}\sqrt{b/h}\,\frac{J_{k+1}(bR/h)}{R}f_{klm}(a, \beta, \gamma), \\ \phi^{(a)}_{klm} &= \sqrt{a/h}\,\frac{J_{k+1}(aP/h)}{P}f_{klm}(a', \beta', \gamma'). \end{aligned}\right\} \quad . \quad . \quad (5.13)$$

6. DISCUSSION.

With the help of the reciprocal functions (5.12) we can form other reciprocal functions:

$$\left.\begin{aligned} \psi^{(b)} &= \sum_{l,m} C_{l,m}\psi^{(b)}_{klm}, \\ \phi^{(a)} &= \sum_{l,m} C_{l,m}\phi^{(a)}_{klm}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (6.1)$$

However, the question of extending our considerations to general linear combinations (*i.e.* summation over k) needs further consideration.

We believe that such developments have to be taken as the starting-point of relativistic wave mechanics, treating the C as non-commuting quantities (second quantisation). Each term in such a development represents a kind of explosive process at the origin (arbitrary point in space-time), and the interpretation will probably follow lines similar to those in the ordinary theory of collisions where the incident wave is developed into a series of spherical waves.

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XII.—Reciprocity. Part IV: Spinor Wave Functions.

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THIS paper is the continuation of the papers by Born (1939, Part I) and by Born and myself (1940, Parts II and III). Here the theory is extended to the case of spinor wave functions satisfying the Dirac equation.

1. THE ANGULAR MOMENTUM OPERATOR.

Let us consider first the angular momentum operator in 4 dimensions defined in Part II:

$$M^{kl} = x^k p^l - p^l x^k. \quad . \quad . \quad . \quad . \quad . \quad (1.1)$$

Here $x^k = (t, x, y, z)$, $p_k = (-E, p_x, p_y, p_z)$, are operators satisfying the commutation relations

$$[p^k, x^l] = \frac{\hbar}{i} g^{kl}, \qquad [p_k, x^l] = \frac{\hbar}{i} g_k^l, \quad . \quad . \quad . \quad (1.2)$$

and g^{kl} is the metric tensor:

$$\left.\begin{array}{ll} g^{kl} = 0 \text{ if } k \neq l, & g_k^l = 0 \text{ if } k \neq l. \\ g^{kk} = 1 \text{ if } k \neq 0, \ g^{00} = -1, & g_k^l = 1 \text{ if } k = l. \end{array}\right\} \quad . \quad . \quad (1.3)$$

The operator (1.1) is a 6-vector. Three components form the 3-dimensional angular momentum vector **M**; the other three components form another 3-vector **N**:

$$\left.\begin{array}{lll} M_x = M^{23}, & M_y = M^{31}, & M_z = M^{12}. \\ N_x = M^{01}, & N_y = M^{02}, & N_z = M^{03}. \end{array}\right\} \quad . \quad . \quad . \quad (1.4)$$

They satisfy the commutation relations

$$\left.\begin{array}{ll} [M_x, M_y] = -\frac{\hbar}{i} M_z, & [N_x, N_y] = \frac{\hbar}{i} M_z, \\ [M_x, N_y] = -\frac{\hbar}{i} N_z, & [N_x, M_y] = -\frac{\hbar}{i} N_z, \\ \multicolumn{2}{c}{[N_x, M_x] = 0,} \end{array}\right\} \quad . \quad . \quad (1.5)$$

and similar equations obtained by cyclic interchange of x, y, z.

It will be convenient to introduce

$$M_{\pm} = M_x \pm i M_y, \qquad N_{\pm} = N_x \pm i N_y. \quad . \quad . \quad . \quad (1.6)$$

From (1.5) we obtain the commutation relations

$$\left.\begin{aligned} &[M_+, M_-] = 2\hbar M_z, && [M_\pm, M_z] = \mp \hbar M_\pm, \\ &[M_+, N_-] = [N_+, M_-] = 2\hbar N_z, && [M_\pm, N_z] = [N_\pm, M_z] = \mp \hbar N_\pm, \\ &[N_+, N_-] = -2\hbar M_z, && [N_\pm, N_z] = \pm \hbar M_\pm, \\ &[N_+, M_+] = [N_-, M_-] = [N_z, M_z] = 0, \end{aligned}\right\} \quad (1.7)$$

All three components of **M** commute with M^2; therefore

$$[M_\pm, M^2] = 0, \qquad [M_z, M^2] = 0. \quad . \quad . \quad . \quad (1.8)$$

For the commutation of **N** with M^2 (1.7) yields

$$\left.\begin{aligned} [N_\pm, M^2] &= \pm 2\hbar\{M_\pm N_z - M_z N_\pm\} + 2\hbar^2 N_\pm, \\ [N_z, M^2] &= \hbar\{M_- N_+ - M_+ N_-\} + 2\hbar^2 N_z. \end{aligned}\right\}. \quad . \quad . \quad (1.9)$$

Further, it follows easily that all components of **M** and **N** commute with the invariant

$$K^2 = M^2 - N^2, \quad . \quad . \quad . \quad . \quad . \quad (1.10)$$

so that

$$[\mathbf{M}, K^2] = [\mathbf{N}, K^2] = 0. \quad . \quad . \quad . \quad . \quad . \quad (1.11)$$

We shall later require also the commutation rules for the co-ordinates x^k and the angular momentum. Corresponding to (1.6), we introduce the combination

$$x^\pm = x \pm iy. \quad . \quad . \quad . \quad . \quad . \quad (1.12)$$

From (1.1) and (1.2) it follows easily that

$$\left.\begin{aligned} &[x^+, M_+] = [x^-, M_-] = [z, M_z] = 0, && [x^+, N_+] = [x^-, N_-] = [z, N_\pm] = 0, \\ &[x^+, M_-] = [M_+, x^-] = 2\hbar z, && [x^+, N_-] = [x^-, N_+] = -2\frac{\hbar}{i}x^0, \\ &[z, M_\pm] = [M_z, x^\pm] = \pm \hbar x^\pm, && [z, N_z] = -\frac{\hbar}{i}x^0, \quad [x^\pm, N_z] = 0, \\ &[x^0, \mathbf{M}] = 0. && [x^0, N_\pm] = -\frac{\hbar}{i}x^\pm, \quad [x^0, N_z] = -\frac{\hbar}{i}z. \end{aligned}\right\} \quad (1.13)$$

With these relations we obtain at once the commutation of x^k with M^2 and N^2; they give

$$\left.\begin{aligned} [x^\pm, M^2] &= \pm 2\hbar\{M_\pm z - M_z x^\pm\} + 2\hbar^2 x^\pm, \\ [z, M^2] &= 2\hbar\{M_- x^+ - M_+ x^-\} + 2\hbar^2 z, \\ [x^0, M^2] &= 0. \\ [x^\pm, N^2] &= -2\frac{\hbar}{i}N_\pm x^0 - \hbar^2 x^\pm, \\ [z, N^2] &= -2\frac{\hbar}{i}N_z x^0 - \hbar^2 z, \\ [x^0, N^2] &= -\frac{\hbar}{i}\{N_+ x^- + N_- x^+ + 2N_z z\} - 3\hbar^2 x^0. \end{aligned}\right\} \quad . \quad . \quad (1.14)$$

Corresponding relations hold for the momenta p^k; they are obtained from those for x^k by changing x^k into p^k.

2. THE ANGULAR MOMENTUM AND THE PROPER SOLUTIONS OF THE KLEIN-GORDON EQUATION.

With the help of the commutation relations established in the first section, we may consider the effect of the angular momentum operator on the proper solutions of the Klein-Gordon equation found in Part II. For this purpose we introduce polar co-ordinates (*cf.* Part II, (3.6)):

$$\left.\begin{aligned} x^1 &= \mathrm{R} \cosh \alpha \sin \beta \cos \gamma, \\ x^2 &= \mathrm{R} \cosh \alpha \sin \beta \sin \gamma, \\ x^3 &= \mathrm{R} \cosh \alpha \cos \beta, \\ x^0 &= \mathrm{R} \sinh \alpha, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2.1)$$

where R, α are real in the region denoted in Part II by (I), and complex in the region (II).

If $f_{k, l, m}$ denotes the angular dependent part of the proper solutions of the Klein-Gordon equation, we have (*cf.* Part II, (5.5))

$$f_{k, l, m}(\alpha, \beta, \gamma) = \Pi^l_k (\tanh \alpha) \mathrm{P}^m_l (\cos \beta) e^{im\gamma}. \quad . \quad . \quad (2.2)$$

We shall use for the ordinary spherical harmonics P^m_l the normalisation adopted by Kramers (1938, p. 171):

$$\mathrm{P}^m_l(\xi) = (1 - \xi^2)^{m/2} \frac{1}{2^l (l+m)!} \left(\frac{d}{d\xi}\right)^{l+m} (\xi^2 - 1)^l. \quad . \quad . \quad (2.3)$$

Then we have from the well-known formula for spherical harmonics,

$$f^*_{k, l, m} = (-1)^m f_{k, l, -m}. \quad . \quad . \quad . \quad . \quad (2.4)$$

The function Π^l_k is here defined as

$$\Pi^l_k(u) = (u^2 - 1)^{\frac{k+2}{2}} \frac{1}{2^l (l+k+1)!} \left(\frac{\partial}{\partial u}\right)^{l+k+1} (u^2 - 1)^l = (u^2 - 1)^{\frac{1}{2}} \mathrm{P}^{k+1}_l (u + i \,.\, 0). \quad (2.5)$$

The term $+i \,.\, 0$ indicates that for $u < 1$ the function is to be taken on the upper side of the real axis.

The effect of the common angular momentum operator, which operates only on the ordinary spherical harmonics, is well known (*cf.* Kramers, 1938, p. 174):

$$\left.\begin{aligned} \mathrm{M}_+ f_{k, l, m} &= -\hbar(l+m+1) f_{k, l, m+1}, \\ \mathrm{M}_- f_{k, l, m} &= -\hbar(l-m+1) f_{k, l, m-1}, \\ \mathrm{M}_z f_{k, l, m} &= \hbar m f_{k, l, m}. \end{aligned}\right\} \quad . \quad . \quad . \quad (2.6)$$

In order to obtain the effect of **N**, consider the three functions

$$f^{\pm}_{k,l,m} = \mathrm{N}_{\pm} f_{k,l,m}, \qquad f^{z}_{k,l,m} = \mathrm{N}_z f_{k,l,m}. \quad . \quad . \quad . \quad (2.7)$$

With (1.7) and (2.6) we find

$$\left.\begin{aligned} \mathrm{M}_z f^{+}_{k,l,m} &= \hbar(m+1) f^{+}_{k,l,m}, \\ \mathrm{M}_z f^{-}_{k,l,m} &= \hbar(m-1) f^{-}_{k,l,m}, \\ \mathrm{M}_z f^{z}_{k,l,m} &= \hbar m f^{z}_{k,l,m}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2.8)$$

If now the functions $f^{\pm}$, f^{z} are developed with respect to the proper functions $f_{k,l,m}$, it follows that there occur only the functions with $m \pm 1$ or m respectively. Furthermore, since **N** commutes with K^2, only functions with index k will be needed; thus

$$\left.\begin{aligned} f^{\pm}_{k,l,m} &= \sum_{l'} c^{\pm}_{l'} f_{k,l',m\pm 1}, \\ f^{z}_{k,l,m} &= \sum_{l'} c^{z}_{l'} f_{k,l',m}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2.9)$$

We next apply the operator M^2; using the commutation relations (1.9) and observing that the proper value of M^2 belonging to $f_{k,l,m}$ is $\hbar^2 l(l+1)$, we find

$$\left.\begin{aligned} \mathrm{M}^2 f^{\pm}_{k,l,m} &= \hbar^2\{l(l+1)-2\} f^{\pm}_{k,l,m} \mp 2\hbar\{\mathrm{M}_{\pm} f^{z}_{k,l,m} - \mathrm{M}_z f^{\pm}_{k,l,m}\}, \\ \mathrm{M}^2 f^{z}_{k,l,m} &= \hbar^2\{l(l+1)-2\} f^{z}_{k,l,m} - \hbar\{\mathrm{M}_{-} f^{+}_{k,l,m} - \mathrm{M}_{+} f^{-}_{k,l,m}\}. \end{aligned}\right\} \quad . \quad (2.10)$$

Inserting the expansion (2.9), applying (2.8), and comparing coefficients, we obtain

$$\left.\begin{aligned} (\tau \pm 2m + 2) c^{\pm}_{l'} \pm 2(l' \pm m + 1) c^{z}_{l'} &= 0, \\ \tau c^{z}_{l'} + (l' - m) c^{+}_{l'} - (l' + m) c^{-}_{l'} &= 0. \end{aligned}\right\} \quad . \quad . \quad . \quad (2.11)$$

Here we have put for abbreviation

$$\tau = l(l+1) - l'(l'+1) - 2. \quad . \quad . \quad . \quad . \quad (2.12)$$

The secular determinant of (2.11) has the solutions $\tau = -2,\ -2(l'+1),\ +2l'$. Therefore l' can only take the values l, $l \pm 1$. The ratio of the constants $c^{\pm}$, c^{z} can now be obtained from (2.11):

$$\left.\begin{aligned} c^{+}_{l} : c^{-}_{l} : c^{z}_{l} &= (l+m+1) : (l-m+1) : (-m), \\ c^{+}_{l+1} : c^{-}_{l+1} : c^{z}_{l+1} &= (l+m+1)(l+m+2) : -(l-m+1)(l-m+2) : (l+m+1)(l-m+1), \\ c^{+}_{l-1} : c^{-}_{l-1} : c^{z}_{l-1} &= -1 : 1 : 1. \end{aligned}\right\} \quad (2.13)$$

The absolute value of these constants depends on the normalisation of the proper functions and cannot therefore be determined by such simple considerations. We determine the constants c^{z} by operating explicitly with N_z on $f_{k,l,m}$. N_z is given by (*cf.* Part II, (3.12))

$$\mathrm{N}_z = \frac{\hbar}{i}\left(\cos\beta\frac{\partial}{\partial\alpha} - \tanh\alpha\sin\beta\frac{\partial}{\partial\beta}\right). \qquad (2.14)$$

The effect of this operator on $f_{k,l,m}$ can, therefore, easily be found by means of the theorems on associated Legendre functions, and the constants c^z, determined. $c^{\pm}$ are then given by (2.13); the result is

$$\left.\begin{aligned} \mathrm{N}_z f_{k,l,m} &= \frac{\hbar}{i}\left\{\frac{l^2}{2l+1} f_{k,l-1,m} - \frac{(l-k)(l+k+2)(l-m+1)(l+m+1)}{(2l+1)(l+1)^2} f_{k,l+1,m}\right\}, \\ \mathrm{N}_{\pm} f_{k,l,m} &= \mp\frac{\hbar}{i}\left\{\frac{l^2}{2l+1} f_{k,l-1,m\pm 1} + \frac{(l-k)(l+k+2)(l\pm m+1)(l\pm m+2)}{(2l+1)(l+1)^2} f_{k,l+1,m\pm 1}\right\}. \end{aligned}\right\} (2.15)$$

We shall further be interested in the effect of multiplying a proper solution $f_{k,l,m}$ with any of the co-ordinates x^k. We introduce for this purpose the functions

$$f^{(\pm)}_{k,l,m} = x^{\pm} f_{k,l,m}, \qquad f^{(z)}_{k,l,m} = z f_{k,l,m}, \qquad f^{(0)}_{k,l,m} = x^0 f_{k,l,m}. \qquad (2.16)$$

The development of these functions in terms of the proper functions can be carried out as before. Now, the commutation relations (1.7) and (1.9) of **N** with M_z, M^2 are identical with those of **x** with M_z, M^2, if **N** is replaced by **x**; therefore part of the calculation is identical, except that **x** does not commute with K^2. In this way we find immediately the expansion corresponding to (2.9):

$$\left.\begin{aligned} f^{(\pm)}_{k,l,m} &= \sum_{k'} c^{(\pm)}_{k',l+1} f_{k',l+1,m\pm 1} + c^{(\pm)}_{k',l} f_{k',l,m\pm 1} + c^{(\pm)}_{k',l-1} f_{k',l-1,m\pm 1}, \\ f^{(z)}_{k,l,m} &= \sum_{k'} c^{(z)}_{k',l+1} f_{k',l+1,m} + c^{(z)}_{k',l} f_{k',l,m} + c^{(z)}_{k',l-1} f_{k',l-1,m}, \end{aligned}\right\} \qquad (2.17)$$

where the constants $c^{(\pm)}$, $c^{(z)}$ satisfy the same relations (2.13) as $c^{\pm}$, c^z. It is therefore sufficient to consider $f^{(z)}$ and $f^{(0)}$ only. Furthermore, x^0 commutes with M^2 and M_z. Therefore $f^{(0)}$ can be expanded in the series

$$f^{(0)}_{k,l,m} = \sum_{k'} c^{(0)}_{k'} f_{k',l,m}. \qquad (2.18)$$

In order to determine the possible values of k' we operate with N^2 on $f^{(z)}$ and $f^{(0)}$. With (1.14) we obtain

$$\left.\begin{aligned} \mathrm{N}^2 f^{(z)}_{k,l,m} &= \hbar^2\{l(l+1) - k(k+2) + 1\} f^{(z)}_{k,l,m} + 2\frac{\hbar}{i}\mathrm{N}_z f^{(0)}_{k,l,m}, \\ \mathrm{N}^2 f^{(0)}_{k,l,m} &= \hbar^2\{l(l+1) - k(k+2) + 3\} f^{(0)}_{k,l,m} + \frac{\hbar}{i}\{\mathrm{N}_+ f^{(-)}_{k,l,m} + \mathrm{N}_- f^{(+)}_{k,l,m} + 2\mathrm{N}_z f^{(z)}_{k,l,m}\}. \end{aligned}\right\} \qquad (2.19)$$

Here we have used the fact that the proper value of $N^2 = M^2 - K^2$ is $\hbar^2\{l(l+1) - k(k+2)\}$. We now introduce the expansions (2.17) and (2.18) into the first of these equations and use (2.15). Comparing coefficients in the resultant equation, we find

$$\left.\begin{aligned} &c^{(s)}_{k',l} = 0, \qquad (\tau' + 2l + 1)c^{(s)}_{k',l+1} = 2\frac{(l-k')(l+k'+2)(l-m+1)(l+m+1)}{(2l+1)(l+1)^2}c^{(0)}_{k',l}, \\ &(\tau' - 2l - 1)c^{(s)}_{k',l-1} = -2\frac{l^2}{l+1}c^{(0)}_{k',l}, \qquad \tau' = k(k+2) - k'(k'+2). \end{aligned}\right\} \quad (2.20)$$

From the first of these follows that also the corresponding constants in the development of $f^{(\pm)}$ vanish:

$$c^{(+)}_{k',l} = c^{(-)}_{k',l} = 0. \quad . \quad . \quad . \quad . \quad . \quad (2.21)$$

If we treat the second equation (2.19) in the same way, the constants $c^{(\pm)}$ will also appear. These can, however, be eliminated by means of (2.21) and (2.13). The result is

$$(\tau' - 3)c^{(0)}_{k',l} + 2\frac{(l+1)^2}{(l+m+1)(l-m+1)}c^{(s)}_{k',l+1} - 2\frac{(l+k'+1)(l-k'-1)}{l}c^{(s)}_{k',l-1} = 0. \quad (2.22)$$

Elimination of $c^{(s)}$ by means of (2.20) yields the secular equation for τ', with the solutions $\tau' = 1,\ 2k'+3,\ -(2k'+1)$. Introducing these in (2.20), it follows that k' can take the values $k \pm 1$; the third value corresponding to $\tau' = 1$ is not integral and has therefore no significance. The ratio of $c^{(s)}$, $c^{(0)}$ is, from (2.20):

$$\left.\begin{aligned} c^{(s)}_{k-1,l+1} &= \frac{(l-k+1)(l-m+1)(l+m+1)}{(2l+1)(l+1)^2}c^{(0)}_{k-1,l}, \qquad c^{(s)}_{k-1,l-1} = \frac{l^2}{(2l+1)(l-k)}c^{(0)}_{k-1,l}, \\ c^{(s)}_{k+1,l+1} &= \frac{(l+k+3)(l-m+1)(l+m+1)}{(2l+1)(l+1)^2}c^{(0)}_{k+1,l}, \qquad c^{(s)}_{k+1,l-1} = \frac{l^2}{(2l+1)(l+k+2)}c^{(0)}_{k+1,l}. \end{aligned}\right\} \quad (2.23)$$

The absolute value of the constants for $f^{(0)}$ can very easily be determined. We have $x^0 = R \sinh \alpha = iRu/\sqrt{u^2 - 1}$, $u = \tanh \alpha$. A theorem on associated Legendre functions (*cf.*, for example, Jahnke and Emde, 1933, p. 180) gives, therefore,

$$x^0 f_{k,l,m} = iR\left\{\frac{l-k}{2(k+1)}f_{k-1,l,m} - \frac{(l+k+2)}{2(k+1)}f_{k+1,l,m}\right\}. \quad . \quad . \quad (2.24)$$

The constants for $f^{(s)}$ and $f^{(\pm)}$ are now determined by (2.20) and (2.13); the result is

$$\left.\begin{aligned}
zf_{k,l,m} = iR\Big\{&\frac{(l-k)(l-k+1)(l-m+1)(l+m+1)}{2(k+1)(2l+1)(l+1)^2}f_{k-1,l+1,m} \\
&\qquad+\frac{l^2}{2(k+1)(2l+1)}f_{k-1,l-1,m} \\
&-\frac{(l+k+2)(l+k+3)(l-m+1)(l+m+1)}{2(k+1)(2l+1)(l+1)^2}f_{k+1,l+1,m} \\
&\qquad-\frac{l^2}{2(k+1)(2l+1)}f_{k+1,l-1,m}\Big\}, \\
x^{\pm}f_{k,l,m} = \pm iR\Big\{&\frac{(l-k)(l-k+1)(l\pm m+1)(l\pm m+2)}{2(k+1)(2l+1)(l+1)^2}f_{k-1,l+1,m\pm1} \\
&\qquad-\frac{l^2}{2(k+1)(2l+1)}f_{k-1,l-1,m\pm1} \\
&-\frac{(l+k+2)(l+k+3)(l\pm m+1)(l\pm m+2)}{2(k+1)(2l+1)(l+1)^2}f_{k+1,l+1,m\pm1} \\
&\qquad+\frac{l^2}{2(k+1)(2l+1)}f_{k+1,l-1,m\pm1}\Big\}.
\end{aligned}\right\} \quad (2.25)$$

We indicate shortly the corresponding equations in p-space. In polar co-ordinates one has

$$\left.\begin{aligned}
p_1 &= P\sinh\alpha'\sin\beta'\cos\gamma', \\
p_2 &= P\sinh\alpha'\sin\beta'\sin\gamma', \\
p_3 &= P\sinh\alpha'\cos\beta', \\
p_0 &= P\cosh\alpha'.
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2.26)$$

It will readily be seen that these equations arise from (2.1) by the substitution

$$R \rightarrow -iP, \qquad \alpha \rightarrow \alpha' + i\frac{\pi}{2}, \qquad \beta \rightarrow \beta', \qquad \gamma \rightarrow \gamma', \quad . \quad . \quad (2.27)$$

or

$$u = \tanh\alpha \rightarrow u' = \coth\alpha'. \quad . \quad . \quad . \quad . \quad (2.28)$$

Corresponding to (2.2) the 4-dimensional spherical harmonics are (*cf.* Part II, (5.6))

$$f'_{k,l,m}(\alpha', \beta', \gamma') = \Pi^l_k(\coth\alpha')P^m_l(\cos\beta')e^{im\gamma'}. \quad . \quad . \quad (2.29)$$

As in Part II, we indicate the representation of the angular momentum operator in p-space by a dash. Since this operator is self-reciprocal, it might be expected that its representation in p-space is obtained from that in x-space by the substitution (2.27). However, it must be observed that the polar co-ordinates in x-space are connected with the contravariant vector x^k, whereas those in p-space are introduced through the co-variant vector p_k. This leads to a change of sign in **N**. We have therefore the following rule for finding reciprocal equations of relations in terms of the polar co-ordinates and angular momentum operators:—

The relations in p-space are obtained from those in x-space by means of the substitutions (2.27), (2.28), *and by changing* **M**, **N** *into* **M**′, −**N**′.

It can easily be verified that this rule is in agreement with the remark made at the end of § 1; it applies, therefore, to all equations in §§ 1 and 2.

3. THE DIRAC EQUATION AND THE ANGULAR MOMENTUM.

In Part II the angular dependent part of the Klein-Gordon equation was expressed in terms of the angular momentum, and it was shown that only the invariant K^2 was needed for this purpose. In a similar way it is possible to express the angular dependent part of the Dirac equation with the help of K^2. Let us take the Dirac equation in the form given by Cartan (1938, vol. ii, p. 71), which may be written as follows:—

$$\{g^k p_k - g\epsilon\}\psi = 0. \qquad (3.1)$$

Here the g^k are matrices of the type

$$g^k = \begin{pmatrix} 0 & \rho^k \\ \rho^{k*} & 0 \end{pmatrix}, \qquad (3.2)$$

satisfying the commutation relations

$$\frac{g^k g^l + g^l g^k}{2} = g^{kl}. \qquad (3.3)$$

The matrices ρ^k are

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho^2 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad \rho^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad (3.4)$$

The matrix g is a scalar

$$g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (3.5)$$

It anti-commutes with all g^k and its square is unity:

$$gg^k + g^k g = 0, \qquad g^2 = 1. \qquad (3.6)$$

From (3.3) and (3.6) follows at once that the iteration of the Dirac equation (3.1) leads to the Klein-Gordon equation.

In order to eliminate the operators p_k from (3.1), multiply the definition (1.1) of the angular momentum by x_k:

$$x_k M^{kl} = x_k x^k p^l - x_k x^l p^k.$$

Now we have (*cf.* Part II, (2.5), (3.2))

$$x_k x^k = R^2, \qquad x_k p^k = \frac{\hbar}{i} R \frac{\partial}{\partial R}. \quad . \quad . \quad . \quad . \quad (3.7)$$

Thus the momentum operators are

$$p^l = \frac{1}{R^2} x_k M^{kl} + \frac{\hbar}{i} \frac{x^l}{R} \frac{\partial}{\partial R}. \quad . \quad . \quad . \quad . \quad . \quad (3.8)$$

It is further possible to express $x_k M^{kl}$ in terms of x_k and K^2; with (1.1) and (1.2) one finds easily

$$x_l M^{lk} = -\frac{1}{2} \frac{i}{\hbar} [x^k, K^2] + \frac{3}{2} \frac{\hbar}{i} x^k. \quad . \quad . \quad . \quad (3.9)$$

Let us now introduce the matrix of a vector (*cf.* Cartan, 1938, vol. ii, p. 62):

$$[X] = g_l x^l = \begin{pmatrix} 0 & [x] \\ [x^\dagger] & 0 \end{pmatrix}, \quad . \quad . \quad . \quad . \quad . \quad (3.10)$$

where, according to (3.2) and (3.4),

$$[x] = \rho_l x^l = \begin{pmatrix} x^+ & z + x^0 \\ z - x^0 & -x^- \end{pmatrix}. \quad . \quad . \quad . \quad (3.11)$$

The matrix $[x^\dagger]$ is obtained from $[x]$ by replacing all elements by their adjoint. Since x^k is self-adjoint, this gives

$$[x^\dagger] = \rho_l^* x^l = \begin{pmatrix} x^- & z + x^0 \\ z - x^0 & -x^+ \end{pmatrix}. \quad . \quad . \quad . \quad (3.12)$$

If the x_k are ordinary c-numbers, we have obviously

$$[x^\dagger] = [x]^*.$$

But in p-space, where $x^k = -(\hbar/i)\partial/\partial p_k$, we have

$$[x^\dagger] = -[x]^*.$$

Now, substitute (3.9) and (3.8) into (3.1). Then the Dirac equation (3.1) is obtained in the form

$$\left\{ -\frac{1}{2R^2} \frac{i}{\hbar} [[X], K^2] + \frac{\hbar}{i} [X] \left(\frac{3}{2R^2} + \frac{1}{R} \frac{\partial}{\partial R} \right) - g\epsilon \right\} \psi = 0. \quad . \quad (3.13)$$

4. THE DIRAC EQUATION IN p-SPACE.

Corresponding to (3.1), we postulate in p-space the wave equation

$$\{g_k x^k - iag\} \phi = 0. \quad . \quad . \quad . \quad . \quad (4.1)$$

On iteration we find the fundamental wave equation (B) of Part II:

$$(x_k x^k) \phi = (r^2 - t^2) \phi = a^2 \phi. \quad . \quad . \quad . \quad . \quad (4.2)$$

It might be thought that we could avoid the imaginary unit in (4.1) by omitting instead the matrix g. Indeed, the wave equation

$$\{g_k x^k - a\}\phi = 0$$

also leads to the wave equation (4.2). However, it is not invariant against reflections, since the unit matrix in Cartan's notation is a pseudoscalar and changes sign on reflection, whereas g is a true scalar (*cf.* Cartan, 1938, vol. ii, p. 72).

We remark further that

$$g_k x^k = g^k x_k = [\mathrm{X}] \quad . \quad . \quad . \quad . \quad . \quad (4.3)$$

is exactly the matrix of a vector (3.10), which here of course is a matrix operator.

In the same way we introduce now the matrix of the momentum operator

$$g^k p_k = g_k p^k = [\mathrm{P}] = \begin{pmatrix} 0 & [p] \\ [p^\dagger] & 0 \end{pmatrix}, \quad . \quad . \quad . \quad . \quad . \quad (4.4)$$

where

$$[p] = \begin{pmatrix} p_1 + ip_2 & p_3 - p_0 \\ p_3 + p_0 & -p_1 + ip_2 \end{pmatrix}. \quad . \quad . \quad . \quad . \quad . \quad (4.5)$$

The matrix [P] is the operator occurring in the wave equation (3.1). Since the matrix of a vector (4.4) and (4.3) represent the contra-variant vectors x^k, p^k just as well as the co-variant vectors x_k, p_k, the two matrices [X] and [P] transform in the same way. Therefore the two spinors ψ, ϕ in x-space and p-space follow the same law of transformation.

x^k can be eliminated from (4.1) in the same way as p_k was eliminated from (3.1). The result is

$$\left\{-\frac{1}{2\mathrm{P}^2}\frac{i}{\hbar}[[\mathrm{P}], \mathrm{K}'^2] + \frac{\hbar}{i}[\mathrm{P}]\left(\frac{3}{2\mathrm{P}^2} + \frac{1}{\mathrm{P}}\frac{\partial}{\partial \mathrm{P}}\right) - ia\overset{*}{g}\right\}\phi = 0. \quad . \quad . \quad (4.6)$$

5. THE SPIN OPERATORS.

It is well known from the theory of the Dirac electron that the mechanical angular momentum **M** does not commute with the Hamiltonian. However, the sum of the mechanical angular momentum and the spin commute with the Hamiltonian. Furthermore, it has been noted that the spin forms only three components of a 6-vector, the other three components representing an imaginary electric momentum (*cf.*, for example, Dirac, 1935, p. 263).

Let us now consider this 6-vector, which may be defined as

$$s^{kl} = \frac{g^k g^l - g^l g^k}{2i}. \quad . \quad . \quad . \quad . \quad . \quad (5.1)$$

With (3.2) it is

$$s^{kl} = \begin{pmatrix} \sigma^{kl} & 0 \\ 0 & -(\sigma^{kl})^* \end{pmatrix}, \qquad \sigma^{kl} = \frac{\rho^k \rho^{l*} - \rho^l \rho^{k*}}{2i}. \qquad . \quad . \quad . \quad (5.2)$$

The components of (5.1) can be written in the form of a 6-vector:

$$\left\{\begin{aligned} s_x^{\mathrm{M}} &= s^{23}, & s_y^{\mathrm{M}} &= s^{31}, & s_z^{\mathrm{M}} &= s^{12}, \\ s_x^{\mathrm{N}} &= s^{01}, & s_y^{\mathrm{N}} &= s^{02}, & s_z^{\mathrm{N}} &= s^{03}. \end{aligned}\right\} \qquad . \quad . \quad . \quad (5.3)$$

These two vectors are given by the matrices

$$\mathbf{s}^{\mathrm{M}} = \begin{pmatrix} \boldsymbol{\sigma}^{\mathrm{M}} & 0 \\ 0 & -\boldsymbol{\sigma}^{\mathrm{M}*} \end{pmatrix}, \qquad \mathbf{s}^{\mathrm{N}} = \begin{pmatrix} \boldsymbol{\sigma}^{\mathrm{N}} & 0 \\ 0 & -\boldsymbol{\sigma}^{\mathrm{N}*} \end{pmatrix}, \quad . \quad . \quad . \quad (5.4)$$

where

$$\left.\begin{aligned} \sigma_x^{M} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y^{M} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_z^{M} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ \boldsymbol{\sigma}^{\mathrm{N}} = -i\boldsymbol{\sigma}^{\mathrm{M}}. \end{aligned}\right\} \quad . \quad . \quad (5.5)$$

The $\boldsymbol{\sigma}^{\mathrm{M}}$ are a representation of Pauli's spin matrices. The $\mathbf{s}^{\mathrm{M}}$, $\mathbf{s}^{\mathrm{N}}$ satisfy the commutation laws,

$$[s_x^{\mathrm{M}}, s_y^{\mathrm{M}}] = -[s_x^{\mathrm{N}}, s_y^{\mathrm{N}}] = 2is_z^{\mathrm{M}}, \qquad \frac{s_x^{\mathrm{M}} s_y^{\mathrm{M}} + s_y^{\mathrm{M}} s_x^{\mathrm{M}}}{2} = -\frac{s_x^{\mathrm{N}} s_y^{\mathrm{N}} + s_y^{\mathrm{N}} s_x^{\mathrm{N}}}{2} = \delta_{xy}, \quad . \quad (5.6)$$

which follow immediately from (3.3).

We now form the operators

$$\left.\begin{aligned} {}^{s}\mathbf{M} &= \mathbf{M} + \tfrac{1}{2}\hbar\mathbf{s}^{\mathrm{M}}, \\ {}^{s}\mathbf{N} &= \mathbf{N} + \tfrac{1}{2}\hbar\mathbf{s}^{\mathrm{N}}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (5.7)$$

and form their squares:

$$\left.\begin{aligned} {}^{s}\mathrm{M}^2 &= \mathrm{M}^2 + [\mathrm{M}] + \tfrac{3}{4}\hbar^2, \\ {}^{s}\mathrm{N}^2 &= \mathrm{N}^2 + [\mathrm{N}] - \tfrac{3}{4}\hbar^2. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (5.8)$$

Here

$$\left.\begin{aligned} [\mathrm{M}] &= \hbar(\mathbf{s}^{\mathrm{M}} \,.\, \mathbf{M}) = \begin{pmatrix} [m] & 0 \\ 0 & -[m^\dagger] \end{pmatrix}, \qquad [m] = \hbar(\boldsymbol{\sigma}^{\mathrm{M}} \,.\, \mathbf{M}) = \hbar\begin{pmatrix} -\mathrm{M}_z & \mathrm{M}_+ \\ \mathrm{M}_- & \mathrm{M}_z \end{pmatrix}, \\ [\mathrm{N}] &= \hbar(\mathbf{s}^{\mathrm{N}} \,.\, \mathbf{N}) = \begin{pmatrix} [n] & 0 \\ 0 & -[n^\dagger] \end{pmatrix}, \qquad [n] = \hbar(\boldsymbol{\sigma}^{\mathrm{N}} \,.\, \mathbf{N}) = \frac{\hbar}{i}\begin{pmatrix} -\mathrm{N}_z & \mathrm{N}_+ \\ \mathrm{N}_- & \mathrm{N}_z \end{pmatrix}. \end{aligned}\right\} \quad (5.9)$$

The operators ${}^{s}\mathrm{M}_z$, ${}^{s}\mathrm{M}^2$, and ${}^{s}\mathrm{N}^2$ commute with each other:

$$[{}^{s}\mathrm{M}_z, {}^{s}\mathrm{M}^2] = [{}^{s}\mathrm{M}_z, {}^{s}\mathrm{N}^2] = [{}^{s}\mathrm{M}^2, {}^{s}\mathrm{N}^2] = 0. \quad . \quad . \quad . \quad (5.10)$$

These relations follow from the commutation relations (1.5) and (5.6) after a short calculation.

Finally, let us introduce the invariant

$$ {}^{s}K^2 = {}^{s}M^2 - {}^{s}N^2, \quad . \quad . \quad . \quad . \quad . \quad (5.11) $$

which obviously commutes with ${}^{s}M^2$ and ${}^{s}M_z$.

The equations of this section are independent of the particular representation of the operators **M** and **N**, and hold therefore in x-space as well as in p-space.

6. THE PROPER FUNCTIONS OF ${}^{s}M_z$, ${}^{s}M^2$, AND ${}^{s}K^2$.

Since ${}^{s}M_z$, ${}^{s}M^2$, and ${}^{s}K^2$ commute with each other, they can be brought simultaneously into diagonal form. Since they are given by step matrices each equation in the determination of the proper values splits into two independent equations. Accordingly we write their simultaneous proper functions as

$$ \chi_{\kappa,\lambda,\mu} = \begin{pmatrix} \chi^1_{\kappa,\lambda,\mu} \\ \chi^2_{\kappa,\lambda,\mu} \end{pmatrix}, \quad . \quad . \quad . \quad . \quad . \quad (6.1) $$

where χ^1, χ^2 are two semi-spinors:

$$ \chi^1_{\kappa,\lambda,\mu} = \begin{pmatrix} \chi^{11}_{\kappa,\lambda,\mu} \\ \chi^{12}_{\kappa,\lambda,\mu} \end{pmatrix}, \qquad \chi^2_{\kappa,\lambda,\mu} = \begin{pmatrix} \chi^{21}_{\kappa,\lambda,\mu} \\ \chi^{22}_{\kappa,\lambda,\mu} \end{pmatrix}. \quad . \quad . \quad (6.2) $$

The indices κ, λ, μ refer to the proper values of the three operators, and they are defined by

$$ \left. \begin{aligned} {}^{s}M_z \chi_{\kappa,\lambda,\mu} &= \hbar\mu\chi_{\kappa,\lambda,\mu} \\ {}^{s}M^2 \chi_{\kappa,\lambda,\mu} &= \hbar^2\lambda(\lambda+1)\chi_{\kappa,\lambda,\mu} \\ {}^{s}K^2 \chi_{\kappa,\lambda,\mu} &= \hbar^2\{\kappa(\kappa+2)+\tfrac{1}{4}\}\chi_{\kappa,\lambda,\mu} \end{aligned} \right\} . \quad . \quad . \quad . \quad . \quad (6.3) $$

The $\frac{1}{4}$ in the third equation is introduced for convenience. We substitute the expressions (5.7) and (5.8) for the three operators and use (5.4) and (5.9) (observe that $M^{kl\dagger} = -M^{kl*}$):

$$ \left(M_z + \frac{\hbar}{2}\sigma^M_z\right)\chi^1_{\kappa,\lambda,\mu} = \hbar\mu\chi^1_{\kappa,\lambda,\mu}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.4) $$

$$ (M^2 + [m] + \tfrac{3}{4}\hbar^2)\chi^1_{\kappa,\lambda,\mu} = \hbar^2\lambda(\lambda+1)\chi^1_{\kappa,\lambda,\mu}, \quad . \quad . \quad . \quad . \quad . \quad (6.5) $$

$$ (N^2 + [n] - \tfrac{3}{4}\hbar^2)\chi^1_{\kappa,\lambda,\mu} = \hbar^2\{\lambda(\lambda+1) - \kappa(\kappa+2) - \tfrac{1}{4}\}\chi^1_{\kappa,\lambda,\mu}, \quad . \quad . \quad (6.6) $$

and similar equations for χ^2 differing only in the sign on the left-hand side in (6.4), and in that the operators are to be replaced by their conjugate complex. Therefore, if $\chi^1_{\kappa,\lambda,\mu}$ is a solution of the equations for χ^1, then

$$ \chi^2_{\kappa,\lambda,\mu} = C\chi^{1*}_{\kappa,\lambda,-\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.7) $$

is a solution of the equations for χ^2 and *vice versa*. It is therefore sufficient to consider χ^1 only.

The solutions of the equations (6.4) and (6.5) are well known from the theory of the spinning electron (*cf.*, for example, Kramers, 1938, p. 249; it should be noted that Kramers uses a different representation of Pauli's spin matrices (defined on p. 242), which amounts to an exchange of the two components of the semi-spinors; furthermore, the normalisation of the spherical harmonics used on p. 249 is different from ours, which coincides with that used by Kramers elsewhere in his book). There are two independent solutions belonging to the proper values $\hbar\mu$ and $\hbar^2\lambda(\lambda+1)$; they are

$$\begin{pmatrix} \mathrm{P}_{\lambda-\frac{1}{2}}^{\mu+\frac{1}{2}}(\cos\beta)e^{i(\mu+\frac{1}{2})\gamma} \\ -\mathrm{P}_{\lambda-\frac{1}{2}}^{\mu-\frac{1}{2}}(\cos\beta)e^{i(\mu-\frac{1}{2})\gamma} \end{pmatrix}, \qquad \begin{pmatrix} (\lambda+\mu+1)\mathrm{P}_{\lambda+\frac{1}{2}}^{\mu+\frac{1}{2}}(\cos\beta)e^{i(\mu+\frac{1}{2})\gamma} \\ (\lambda-\mu+1)\mathrm{P}_{\lambda+\frac{1}{2}}^{\mu-\frac{1}{2}}(\cos\beta)e^{i(\mu-\frac{1}{2})\gamma} \end{pmatrix}. \quad . \quad (6.8)$$

In order to determine the proper solutions of ${}^s\mathrm{K}^2$, we observe that K^2 commutes with the three operators, as follows from (5.7) and (1.11). Furthermore, since the iteration of the Dirac equation leads to the Klein-Gordon equation, all components of the spinor must be linear combinations of the solutions $f_{k,l,m}$ of this equation. Therefore χ^1 must be of the form

$$\chi^1_{\kappa,\lambda,\mu} = c_1\begin{pmatrix} f_{k,\lambda-\frac{1}{2},\mu+\frac{1}{2}} \\ -f_{k,\lambda-\frac{1}{2},\mu-\frac{1}{2}} \end{pmatrix} + c_2\begin{pmatrix} (\lambda+\mu+1)f_{k,\lambda+\frac{1}{2},\mu+\frac{1}{2}} \\ (\lambda-\mu+1)f_{k,\lambda+\frac{1}{2},\mu-\frac{1}{2}} \end{pmatrix}. \quad . \quad (6.9)$$

We substitute this expression in (6.6) and replace the matrix $[n]$ by its value (5.9). The resulting two equations may be written in one:

$$\left.\begin{aligned} &\pm c_1\left\{\left(\mathrm{N}^2 \mp \frac{\hbar}{i}\mathrm{N}_z\right)f_{k,\lambda-\frac{1}{2},\mu\pm\frac{1}{2}} - \frac{\hbar}{i}\mathrm{N}_\pm f_{k,\lambda-\frac{1}{2},\mu\mp\frac{1}{2}}\right\}, \\ &+c_2\left\{\left(\mathrm{N}^2 \mp \frac{\hbar}{i}\mathrm{N}_z\right)(\lambda\pm\mu+1)f_{k,\lambda+\frac{1}{2},\mu\pm\frac{1}{2}} + \frac{\hbar}{i}\mathrm{N}_\pm(\lambda\mp\mu+1)f_{k,\lambda+\frac{1}{2},\mu\mp\frac{1}{2}}\right\}, \\ &= \hbar^2\{\lambda(\lambda+1) - \kappa(\kappa+2) + \tfrac{1}{2}\}\{\pm c_1 f_{k,\lambda-\frac{1}{2},\mu+\frac{1}{2}} + c_2(\lambda\pm\mu+1)f_{k,\lambda+\frac{1}{2},\mu\pm\frac{1}{2}}\}. \end{aligned}\right\} \quad (6.10)$$

Eliminating the operators **N** by means of (2.15) and comparing coefficients, both equations give the same relations, namely,

$$\left.\begin{aligned} &c_1\{\kappa(\kappa+2) - k(k+2) - \lambda - \tfrac{3}{4}\} + c_2(\lambda+\tfrac{1}{2})^2 = 0, \\ &-c_1\frac{(\lambda-k-\frac{1}{2})(\lambda+k+\frac{3}{2})}{(\lambda+\frac{1}{2})^2} + c_2\{\kappa(\kappa+2) - k(k+2) + \lambda + \tfrac{1}{4}\} = 0. \end{aligned}\right\}. \quad . \quad (6.11)$$

The secular determinant has the two solutions $k = \kappa \pm \frac{1}{2}$, and the ratio of the constants c_1, c_2 is thus determined. Introducing it in (6.9), we find the two independent proper functions of the operators ${}^s\mathrm{M}_z$, ${}^s\mathrm{M}^2$, and ${}^s\mathrm{K}^2$:

$$\chi^{-1}_{\kappa,\lambda,\mu}=(\lambda+\tfrac{1}{2})^2\begin{pmatrix} f_{\kappa-\frac{1}{2},\lambda-\frac{1}{2},\mu+\frac{1}{2}} \\ -f_{\kappa-\frac{1}{2},\lambda-\frac{1}{2},\mu-\frac{1}{2}}\end{pmatrix}+(\lambda-\kappa)\begin{pmatrix}(\lambda+\mu+1)f_{\kappa-\frac{1}{2},\lambda+\frac{1}{2},\mu+\frac{1}{2}} \\ (\lambda-\mu+1)f_{\kappa-\frac{1}{2},\lambda+\frac{1}{2},\mu-\frac{1}{2}}\end{pmatrix} \qquad (6.12)$$

$$\chi^{+1}_{\kappa,\lambda,\mu}=(\lambda+\tfrac{1}{2})^2\begin{pmatrix} f_{\kappa+\frac{1}{2},\lambda-\frac{1}{2},\mu+\frac{1}{2}} \\ -f_{\kappa+\frac{1}{2},\lambda-\frac{1}{2},\mu-\frac{1}{2}}\end{pmatrix}+(\lambda+\kappa+2)\begin{pmatrix}(\lambda+\mu+1)f_{\kappa+\frac{1}{2},\lambda+\frac{1}{2},\mu+\frac{1}{2}} \\ (\lambda-\mu+1)f_{\kappa+\frac{1}{2},\lambda+\frac{1}{2},\mu-\frac{1}{2}}\end{pmatrix}. \qquad (6.13)$$

The two independent semi-spinors χ^2 are now given by (6.7); with (2.4) they are:

$$\chi^{-2}_{\kappa,\lambda,\mu}=(\lambda+\tfrac{1}{2})^2\begin{pmatrix} f_{\kappa-\frac{1}{2},\lambda-\frac{1}{2},\mu-\frac{1}{2}} \\ f_{\kappa-\frac{1}{2},\lambda-\frac{1}{2},\mu+\frac{1}{2}}\end{pmatrix}+(\lambda-\kappa)\begin{pmatrix}(\lambda-\mu+1)f_{\kappa-\frac{1}{2},\lambda+\frac{1}{2},\mu-\frac{1}{2}} \\ -(\lambda+\mu+1)f_{\kappa-\frac{1}{2},\lambda+\frac{1}{2},\mu+\frac{1}{2}}\end{pmatrix}. \qquad (6.14)$$

$$\chi^{+2}_{\kappa,\lambda,\mu}=(\lambda+\tfrac{1}{2})^2\begin{pmatrix} f_{\kappa+\frac{1}{2},\lambda-\frac{1}{2},\mu-\frac{1}{2}} \\ f_{\kappa+\frac{1}{2},\lambda-\frac{1}{2},\mu+\frac{1}{2}}\end{pmatrix}+(\lambda+\kappa+2)\begin{pmatrix}(\lambda-\mu+1)f_{\kappa+\frac{1}{2},\lambda+\frac{1}{2},\mu-\frac{1}{2}} \\ -(\lambda+\mu+1)f_{\kappa+\frac{1}{2},\lambda+\frac{1}{2},\mu-\frac{1}{2}}\end{pmatrix}. \qquad (6.15)$$

Here we have taken the arbitrary constant in (6.7) so that

$$\chi^{-2}_{\kappa,\lambda,\mu}=(-1)^{\mu-\frac{1}{2}}\chi^{-1*}_{\kappa,\lambda,-\mu}, \qquad \chi^{+2}_{\kappa,\lambda,\mu}=(-1)^{\mu-\frac{1}{2}}\chi^{+1*}_{\kappa,\lambda,-\mu}. \qquad (6.16)$$

Changing μ into $-\mu$ and taking the conjugate complex, we find

$$\chi^{-1}_{\kappa,\lambda,\mu}=(-1)^{\mu+\frac{1}{2}}\chi^{-2*}_{\kappa,\lambda,-\mu}, \qquad \chi^{+1}_{\kappa,\lambda,\mu}=(-1)^{\mu+\frac{1}{2}}\chi^{+2*}_{\kappa,\lambda,-\mu}. \qquad (6.17)$$

It should be noted that the equations for χ^1 and χ^2 are entirely independent. In order to form a proper solution we may take any linear combination of $\chi^{\pm 1}$ for the first semi-spinor and any other linear combination of $\chi^{\pm 2}$ for the second semi-spinor.

The corresponding proper functions in p-space differ slightly, owing to the change of sign as we pass from N to N′; they are

$$\omega^{\mp 1}_{\kappa,\lambda,\mu}=(\lambda+\tfrac{1}{2})^2\begin{pmatrix} f'_{\kappa\mp\frac{1}{2},\lambda-\frac{1}{2},\mu+\frac{1}{2}} \\ -f'_{\kappa\mp\frac{1}{2},\lambda-\frac{1}{2},\mu-\frac{1}{2}}\end{pmatrix}-(\lambda+1\mp\kappa\mp 1)\begin{pmatrix}(\lambda+\mu+1)f'_{\kappa\mp\frac{1}{2},\lambda+\frac{1}{2},\mu+\frac{1}{2}} \\ (\lambda-\mu+1)f'_{\kappa\mp\frac{1}{2},\lambda+\frac{1}{2},\mu-\frac{1}{2}}\end{pmatrix}. \qquad (6.18)$$

$$\omega^{\mp 2}_{\kappa,\lambda,\mu}=(\lambda+\tfrac{1}{2})^2\begin{pmatrix} f'_{\kappa\mp\frac{1}{2},\lambda-\frac{1}{2},\mu-\frac{1}{2}} \\ f'_{\kappa\mp\frac{1}{2},\lambda-\frac{1}{2},\mu+\frac{1}{2}}\end{pmatrix}-(\lambda+1\mp\kappa\mp 1)\begin{pmatrix}(\lambda-\mu+1)f'_{\kappa\mp\frac{1}{2},\lambda+\frac{1}{2},\mu-\frac{1}{2}} \\ -(\lambda+\mu+1)f'_{\kappa\mp\frac{1}{2},\lambda+\frac{1}{2},\mu+\frac{1}{2}}\end{pmatrix}. \qquad (6.19)$$

Here the dash indicates the substitution $\alpha\rightarrow\alpha'+i\frac{\pi}{2}$, $\beta\rightarrow\beta'$, $\lambda\rightarrow\lambda'$.

7. SOLUTION OF THE DIRAC EQUATION.

In order to solve the Dirac equation (3.13), let us first inquire into the effect of the matrix $[x]$ on the proper functions found in the last section. We shall prove that

$$\left.\begin{aligned} &[x]^*\chi^{-1}_{\kappa,\lambda,\mu}=i\mathrm{R}\chi^{+2}_{\kappa,\lambda,\mu}, \qquad &&[x]^*\chi^{+1}_{\kappa,\lambda,\mu}=-i\mathrm{R}\chi^{-2}_{\kappa,\lambda,\mu}, \\ &[x]\chi^{-2}_{\kappa,\lambda,\mu}=i\mathrm{R}\chi^{+1}_{\kappa,\lambda,\mu}, \qquad &&[x]\chi^{+2}_{\kappa,\lambda,\mu}=-i\mathrm{R}\chi^{-1}_{\kappa,\lambda,\mu}\end{aligned}\right\} . \qquad (7.1)$$

If one of these equations is proved, the others follow immediately. Assume, for example, that the first is correct; we change μ into $-\mu$ and take the conjugate complex. With (6.16) and (6.17) we find the third equation. If now we multiply the first and third with $[x]$ and $[x]^*$ respectively, and observe that $[x][x]^* = [x]^*[x] = \mathrm{R}^2$, the two remaining equations are obtained.

It remains to prove the first equation (7.1), the left-hand side of which is, according to (3.11),

$$\begin{pmatrix} x^- & z+x^0 \\ z-x^0 & x^+ \end{pmatrix}\begin{pmatrix} \chi^{-11}_{\kappa,\lambda,\mu} \\ \chi^{-12}_{\kappa,\lambda,\mu} \end{pmatrix} = \begin{pmatrix} x^-\chi^{-11}_{\kappa,\lambda,\mu} + (z-x^0)\chi^{-12}_{\kappa,\lambda,\mu} \\ (z-x^0)\chi^{-11}_{\kappa,\lambda,\mu} - x^+\chi^{-12}_{\kappa,\lambda,\mu} \end{pmatrix}.$$

Introducing (6.12), the two components of this spinor may be written in one:

$$\begin{aligned}(\lambda+\tfrac{1}{2})^2\{x^{\mp}f_{\kappa-\frac{1}{2},\lambda-\frac{1}{2},\mu\pm\frac{1}{2}} \mp (z\pm x^0)f_{\kappa-\frac{1}{2},\lambda-\frac{1}{2},\mu\mp\frac{1}{2}}\}& \\ +(\lambda-\kappa)\{\pm(\lambda\pm\mu+1)x^{\mp}f_{\kappa-\frac{1}{2},\lambda+\frac{1}{2},\mu\pm\frac{1}{2}} + (\lambda\mp\mu+1)(z\pm x^0)f_{\kappa-\frac{1}{2},\lambda+\frac{1}{2},\mu\mp\frac{1}{2}}\}&.\end{aligned}$$

Here the co-ordinates $\mathbf{x}$ can be eliminated by means of (2.25). The result reduces to the two components (6.15) of χ^{+2}. Thus (7.1) is proved.

We now turn to the Dirac equation (3.13). It may be regarded as the definition of the operator $\partial/\partial\mathrm{R}$ just as in the common quantum theory $\mathrm{H}-\mathrm{E}=0$ may be regarded as the definition of the operator $\mathrm{E}=-(\hbar/i)\partial/\partial t$. Multiplying (3.13) with [X], one finds

$$\frac{\hbar}{i}\mathrm{R}\frac{\partial}{\partial\mathrm{R}} = -\frac{\hbar}{i}\frac{3}{2}+\frac{i}{\hbar}\frac{1}{2\mathrm{R}^2}[\mathrm{X}][[\mathrm{X}],\ \mathrm{K}^2]+\epsilon[\mathrm{X}]g, \qquad (7.2)$$

since $[\mathrm{X}]^2=\mathrm{R}^2$. Now, from the definition (3.10) of [X] follows that g anti-commutes with [X], since it anti-commutes with all g_k. Thus (7.2) can also be written as

$$\frac{\hbar}{i}\mathrm{R}\frac{\partial}{\partial\mathrm{R}} = -\frac{\hbar}{i}\frac{3}{2}-\frac{i}{\hbar}\frac{1}{2\mathrm{R}^2}[\mathrm{X}]g[[\mathrm{X}]g,\ \mathrm{K}^2]+\epsilon[\mathrm{X}]g. \qquad (7.3)$$

It can now be shown that ${}^s\mathrm{M}_z$, ${}^s\mathrm{M}^2$, ${}^s\mathrm{K}^2$ commute with this operator. Since all three commute with K^2, it is only left to show that they also commute with $[\mathrm{X}]g$; this, however, follows immediately from (7.1), which proves that

$$\begin{pmatrix} \chi^{-1}_{\kappa,\lambda,\mu} \\ \chi^{+2}_{\kappa,\lambda,\mu} \end{pmatrix}, \qquad \begin{pmatrix} \chi^{+1}_{\kappa,\lambda,\mu} \\ \chi^{-2}_{\kappa,\lambda,\mu} \end{pmatrix}$$

are simultaneous proper functions of ${}^s\mathrm{M}_z$, ${}^s\mathrm{M}^2$, ${}^s\mathrm{K}^2$, and $[\mathrm{X}]g$.

The solutions of the Dirac equation must therefore be linear combinations of the proper functions (6.12) to (6.15). The equations (7.1), furthermore, suggest that we have to combine χ^{-1} with χ^{+2} and χ^{+1} with χ^{-2}. Each of these functions must of course be multiplied with a radial function, which, however, is already determined, since each component of the spinor must be a solution of the Klein-Gordon equation treated in Part II. They are Bessel functions of the order $k+1$ (*cf.* Part II, (5.5)), *i.e.* $\kappa+\frac{1}{2}$ or $\kappa+\frac{3}{2}$; each semi-spinor is further to be multiplied with a constant. The ratio of the two constants for the two semi-spinors is easily obtained, on substitution into the Dirac equation (3.13), by using the well-known relation for Bessel functions:

$$\frac{d}{dz}Z_p(z) = -\frac{p}{z}Z_p(z) + Z_{p-1}(z) = \frac{p}{z}Z_p(z) - Z_{p+1}(z).$$

The result is that there are the following two independent solutions of the Dirac equation, which are at the same time simultaneous proper functions of the operators ${}^sM_z$, ${}^sM^2$, ${}^sK^2$:—

$$\psi_{\kappa,\lambda,\mu} = \begin{pmatrix} \frac{i}{R} Z_{\kappa+\frac{1}{2}}\left(i\frac{R\epsilon}{\hbar}\right)\chi^{-1}_{\kappa,\lambda,\mu} \\ \frac{1}{R} Z_{\kappa+\frac{3}{2}}\left(i\frac{R\epsilon}{\hbar}\right)\chi^{+2}_{\kappa,\lambda,\mu} \end{pmatrix}. \quad . \quad . \quad . \quad (7.4)$$

$$\psi'_{\kappa,\lambda,\mu} = \begin{pmatrix} \frac{i}{R} Z_{\kappa+\frac{3}{2}}\left(i\frac{R\epsilon}{\hbar}\right)\chi^{+1}_{\kappa,\lambda,\mu} \\ \frac{1}{R} Z_{\kappa+\frac{1}{2}}\left(i\frac{R\epsilon}{\hbar}\right)\chi^{-2}_{\kappa,\lambda,\mu} \end{pmatrix}. \quad . \quad . \quad . \quad (7.5)$$

The solutions of the Dirac equation (4.6) in p-space are similarly

$$\phi_{\kappa,\lambda,\mu} = \begin{pmatrix} \frac{i}{P} Z_{\kappa+\frac{1}{2}}\left(i\frac{Pa}{\hbar}\right)\omega^{-1}_{\kappa,\lambda,\mu} \\ \frac{1}{P} Z_{\kappa+\frac{3}{2}}\left(i\frac{Pa}{\hbar}\right)\omega^{+2}_{\kappa,\lambda,\mu} \end{pmatrix}. \quad . \quad . \quad . \quad (7.6)$$

$$\phi'_{\kappa,\lambda,\mu} = \begin{pmatrix} \frac{i}{P} Z_{\kappa+\frac{3}{2}}\left(i\frac{Pa}{\hbar}\right)\omega^{+1}_{\kappa,\lambda,\mu} \\ \frac{1}{P} Z_{\kappa+\frac{1}{2}}\left(i\frac{Pa}{\hbar}\right)\omega^{-2}_{\kappa,\lambda,\mu} \end{pmatrix}. \quad . \quad . \quad . \quad (7.7)$$

SUMMARY.

Solutions of the Dirac equation are found which are quadratically integrable on the hyperboloid $r^2 - t^2 = a^2$. If $\mathbf{M}$, $\mathbf{N}$ is the generalised 6-vector representing the angular momentum, $\mathbf{s}^M$, $\mathbf{s}^N$ the matrices representing the magnetic and electric moment of the electron, and ${}^s\mathbf{M} = \mathbf{M} + \hbar\mathbf{s}^M/2$, ${}^s\mathbf{N} = \mathbf{N} + \hbar\mathbf{s}^N/2$, then the solutions of the Dirac equation can be classified with respect to the proper values $\hbar\mu$, $\hbar^2\lambda(\lambda+1)$, $\hbar^2\{\kappa(\kappa+2)+\frac{1}{2}\}$ of the operators ${}^s\mathrm{M}_z$, ${}^s\mathrm{M}^2$, ${}^s\mathrm{M}^2 - {}^s\mathrm{N}^2$. Here κ, λ, μ are half-integers.

I wish to express my gratitude to Professor Max Born for many valuable discussions.

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XIII.—Tests of Carcinogenic Substances in Relation to the Production of Mutations in *Drosophila melanogaster*.

By **Charlotte Auerbach,** Ph.D., Institute of Animal Genetics, University of Edinburgh. *Communicated by* Dr A. W. GREENWOOD.

(MS. received February 9, 1940. Revised MS. received March 21, 1940. Read May 6, 1940.)

THE PROBLEM.

THE theory has often been put forward that cancer is a somatic mutation. The changes which differentiate the cancerous cell from the normal are reproduced in many or all of the cells derived from the original malignant cell, *i.e.* they show the properties by which we define a mutation. Thus, to a geneticist, the possibility which most readily suggests itself is (1) that of a mutation in the genetic material of the somatic cell. Other possible mechanisms that might simulate the results of (1) are (*2a*) the introduction into the cell of a malignant virus; (*2b*) the "activation" of some such virus pre-existing in the cell in an inactive state, a situation which would imply mutation in the virus since the change becomes reproduced; (3) the production of some autocatalytic substance other than gene or virus, inducing malignant changes in the cell which contains a certain amount of it.

Possibility (*2a*) has been found to hold in a certain number of cases, *e.g.* the Rous sarcoma of the fowl. On the other hand, there is no reason to suppose that the same changes which can be brought about by a virus may not in other instances be the result of a mutation in the germinal material proper to the cell itself. If we visualise the virus as a free gene in the protoplasm (Muller, 1926), these two causes, (1) and (2), would not in the last analysis be intrinsically different.

The third possibility, that of an autocatalytic substance, appears to have least to recommend it. It seems difficult to conceive of a substance of this nature able to produce, in the same kind of individual, presumably containing cells with the same genotypes, tumours of such widely differing, self-perpetuating types as are known, and it is likewise difficult to conceive of the existence of many different substances of this kind, if we exclude the possibility of their being genes (either chromosomal or virus genes), which of course are of manifold sort and capable of mutation. And certainly the self-perpetuating changes which sometimes supervene in a tumour that is long cultivated or transplanted, altering its capacity to take in certain hosts (Strong, 1926), must be conceived of as mutations. Thus, on the whole, the somatic mutation theory, in one form or another, seems to offer the most plausible explanation for the phenomena of cancer, and to be the one which requires the least in the way of additional hypotheses.

To design experimental tests to distinguish clearly between the various possibilities mentioned above is very difficult, and has been attempted but rarely and then without decisive results. For example, Curtis, Dunning, and Bullock (1933), and Dunning, Curtis, and Bullock (1937), proved that under the conditions of their experiments—numerous independent foci of irritation—the change of a normal cell into a malignant one is a random event like mutation, and they regarded this as evidence for the mutation theory of cancer. Their data, however, might be equally well interpreted on the other conceptions of cancer, so long as it is assumed that the change into a cancerous cell under the influence of the given stimulus is a threshold affair, and that the irritating agent at each focus has a given chance of rising above this threshold, independently of what happens at the other foci—as is in all probability the case.

The above authors also found that age had no enhancing effect on the production of these tumours other than by increasing the time of exposure, which again they regard as evidence favouring the mutation theory, but this result too would seem as likely under the other conceptions. Likewise, the well-known fact that the incidence of mammalian tumours increases with age is only of doubtful corroborative value for the mutation conception, inasmuch as the increase is very much steeper than would be expected on the mere basis of the increased time allowing more opportunity for the event to occur and so must, at least partly, be attributed to special effects of age itself. Somewhat similar special effects of age have been reported for mutation rate in the case of seeds of plants (Navashin, 1933; Stubbe, 1935), but not yet in animals.

The main support for the mutation theory of cancer is still to be found in the consideration, first pointed out by Muller (1927; see also 1937) for X-rays, that the agent which has so far been found most effective in producing mutations, namely high-energy radiation (such as X- and radium-rays), is effective in producing cancer also. We may here point out that modern work with ultra-violet light now seems to permit the important extension of this parallelism in effect on mutations and cancer respectively to the latter agency. The effectiveness of ultra-violet light in causing mutations has been proved by Altenburg (1930) and later workers in *Drosophila*, and by Noethling and Stubbe (1934), Stadler and Sprague (1936), and others in plant material. As very recently shown by Stadler and Uber (1938), and by Muller and Mackenzie (1939; also Mackenzie and Muller, in press), the mutations produced by ultra-violet light, unlike those from higher energy radiation, are all or in the vast majority of cases mutations of individual genes, not mutations of whole chromosomes or chromosome sections. Thus if the cancer said to be produced by ultra-violet light is of mutational origin, we must conclude it to be caused by gene mutations—a conception better fitting the expectation of modern geneticists—rather than by the gross chromosome changes to which it has often been attributed.

In the case of ultra-violet radiation, there is an opportunity of obtaining further light on the correctness of the mutation theory of cancer by a study of the relative effectiveness of different wave-lengths in the production of cancers as compared with mutations. It has been found in the work on plant material (Stubbe and Noethling, 1937; Stadler and Uber, 1938; and others) that, when differences of penetration of the rays are allowed for, there is a maximum of mutation-producing efficiency in the ultra-violet in the region near 265 $m\mu$, which corresponds with a region of maximum absorption by nucleic acid, and therefore by chromosomes, and disagrees with the absorption maxima for proteins as well as for the great majority of other cell constituents. If a similar maximum could be found for the cancer-producing effect, this would constitute strong evidence in favour of the mutation theory of cancer. As yet, sufficiently exact studies of this question have not been made.

If now it could be shown that there is also a parallelism between the induction of mutations under the influence of chemical substances and the induction of malignant tumours by such substances, the mutation theory of cancer would gain important further support. Unfortunately, all attempts hitherto made to induce mutations by chemical means have given negative or indecisive results. None of the experiments published up to the time our work began has, however, been concerned with substances known to be effective in producing cancer. It has therefore seemed worth while to try the effect of carcinogenic substances on the mutation rate of *Drosophila melanogaster*, the most suitable animal for studying mutation. Such experiments had in fact been projected several years ago by Dr Alexander Weinstein, and independently by Muller and others, with these considerations in mind, but had not, to our knowledge, ever been carried out on a sufficient scale for the results to be reported.

TECHNIQUE.

The object of the various techniques tried was to expose proliferating gonadic tissue to the action of powerful carcinogenic substances. After a preliminary experiment on a small scale, involving the feeding of carcinogenic substances, had given a negative result, injection was used on a large scale and in divers variations. It was supplemented by a small number of insertions of crystals into the abdomen. The substances used were (1) 1 : 2 : 5 : 6-dibenzanthracene in 0·03 per cent. colloidal solution, (2) 9 : 10-dimethyl-1 : 2-benzanthracene in 0·06 per cent. colloidal solution, both suspended in a medium consisting of Ringer's solution as given by Ephrussi and Beadle (1936) minus calcium plus 0·5 per cent. gelatine. This medium by itself was used for the controls. The third substance, methyl-cholanthrene, was introduced in crystal form.

Injections were done by means of an ordinary pipette drawn out into a fine capillary and sharpened at its end in the way described by Ephrussi and Beadle (1936). Adults as well as larvæ were injected. The latter proved by far the more difficult. This was because the high viscosity of the colloidal solutions and their tendency to coagulate required the capillary to be so wide and the pressure used in injection to be so great that the majority of the injected larvæ died through subsequent oozing or infection of the wound. Injection of adults, on the other hand, was found to be extremely easy. Wounds, even when comparatively large, heal readily, and consequently the mortality is very low. The only danger appears to be an excess of fluid in the abdomen. The border-line between a dose mechanically tolerated and a lethal or sublethal one seems very distinct: whereas amounts between 0·2 and 0·3 c.mm. were easily tolerated (3rd and 4th series), amounts between 0·3 and 0·4 c.mm. killed 50 per cent. of experimental as well as control flies (5th and 6th series), and 0·5 c.mm. were almost lethal (2nd series). The amount of fluid injected was estimated from the diameter of the drop when an equal amount was suspended from the point of the pipette over a micrometer scale. Checks with the micro-burette of Fox-Wingfield showed this rough estimate to be sufficiently accurate.

Experimental and control flies were injected with the same amount of fluid and in the same spot. Injections of larvæ were done dorsally or laterally near the site of the gonads. Adults were injected either mid-dorsally or laterally after etherising and clipping of the wings. The medium was found to contain a considerable bacterial flora after only two days, and the injections were then fatal to the flies, so that the medium for the control series had to be made up fresh for every injected batch. Medium containing carcinogenic substances, however, remained free from this type of noxious bacteria for many weeks, and in the first series was used repeatedly with intervals up to a fortnight between injections. But in order to avoid the suspicion that some undetected type of bacteria affecting the carcinogenic substances might have been present in the older solutions, series 5 and 6 were carried out with solutions made up on the day of the injection.

Crystals were inserted into adult females mid-dorsally behind the fourth abdominal ring. With a blunt glass needle they were pushed into a hole pierced previously by a pointed glass needle. Their size varied considerably; the majority were thin needles about 0·5 mm. long and 0·05 mm. in diameter. Dissections of some of the treated flies after a few days showed the crystal between the ovaries, sometimes even touching one of them.

Injected larvæ were allowed to pupate on moist filter-paper, and collecting of offspring was carried out from the hatching of the imago to its death. Injected males were kept with other females for a fortnight before mating them to the females whose offspring were used for the test. This was to induce active functioning of the testes and ensure the test being conducted on gonadic tissue in a proliferating condition at the time of treatment. Similarly, the collection of eggs from injected females was started 10–14 days after injection, during which interim they were kept with males, and the collection was continued up to 50 days. Treated females were divided into several batches and kept under differing conditions of temperature and food, a parallel series of control females being kept for each set of conditions. Control flies were taken from the same stock, and, when possible, from the same culture as the experimental flies. Care was taken to keep the number of treated and control animals in each group reasonably high in order to exclude the possibility that the results might be falsified by the presence of individual differences in the spontaneous mutation rate

The incidence of sex-linked lethals was taken as an estimate of mutation rate. Mutations occurring in the males were detected by the usual ClB method. Females to be injected were heterozygous for sex-linked markers and for inversions preventing the appearance of cross-overs, so that their sex chromosomes could be recovered intact from their daughters, when the latter were tested for the presence of sex-linked mutations received from the treated generation. The presence of a sex-linked lethal in an injected female antecedent to treatment was avoided in each case by determining that she produced both expected types of sons. The genetic formulæ of the crosses are shown below.

P_1 female $\dfrac{sc^8 w^a bb}{sc^8 w^a bb}$ × male $sc\,\delta 49\ v\ od\ ca$.

F_1 female $\dfrac{sc^8 w^a bb}{sc\,\delta 49\ v\ od\ ca}$ (injected) × male vB^{M1}.

F_2 two types of females:

(1) $\dfrac{sc^8 w^a bb}{vB^{M1}}$ (appearance moderate bar eyes).

(2) $\dfrac{sc\,\delta 49\ v\ od\ ca}{vB^{M1}}$ (appearance vermilion eyes, moderate bar).

sc^8 = scute 8, bristles missing; long inversion.
w^a = apricot, eye colour.
bb = bobbed, bristle shape.
B^{M1} = Bar M1, eye shape; inversion extending from B to "inert region."
v = vermilion, eye colour.
od = outstretched, wings.
ca = carnation, eye colour.
$\delta 49$ = inversion of moderate size in middle of X.

The F_2 females were mated in separate vials to brothers, and their progeny examined for w^a sons in the case of type (1)♀♀, and for *od* sons in the case of type (2)♀♀. The absence of such sons was taken as evidence of a lethal in the X-chromosome of the mother, *i.e.* the F_2 female. Since this X-chromosome came from the treated F_1 female, the lethal must have arisen in one of her germ cells through mutation. Thus the number of fertile F_2 cultures can be taken to represent the total number of tested chromosomes of the F_1 female, and the number of F_2 cultures with a lethal to represent the number of lethals which have arisen among these chromosomes.

RESULTS.

1st Series (preliminary).—1 : 2 : 5 : 6-dibenzanthracene. Females. Injections given laterally, into or very near the ovary, alternately left and right, from 2 to 6 times, with several days interval between injections. Total amount of substance injected ranging from about 0·25 c.mm. to 0·75 c.mm., corresponding to 0·08 γ up to 0·24 γ of substance. No controls. No lethals among 241 chromosomes tested.

2nd Series.—1 : 2 : 5 : 6-dibenzanthracene. Females. One medio-dorsal injection of 0·5 c.mm., corresponding to 0·15 γ of substance. Of 59 injected females only 4 survived. No lethal in 382 chromosomes, and no lethal in 265 control chromosomes.

3rd Series.—9 : 10-dimethyl-1 : 2-benzanthracene. Females. Treatment once (83 females surviving) or twice (46 females) with medio-dorsal or simultaneous right and left injections, each time 0·2–0·3 c.mm., corresponding to about 0·15 γ–0·2 γ. Various sub-series in respect of conditions following injection. 3 lethals, all from separate females, among 2919 examined X-chromosomes. Controls: 1 lethal among 1132 chromosomes, *i.e.* sensibly the same mutation rate.

4th Series.—9 : 10-dimethyl-1 : 2-benzanthracene. Males. One medio-dorsal injection of 0·2–0·3 c.mm. containing 0·15 γ–0·2 γ of substance. 15 treated, 19 control males. 3 lethals, all from different males, among 685 tested chromosomes. Controls: 2 lethals from different males, among 445 chromosomes, *i.e.* no difference between treated flies and controls.

5th Series.—Repetition of third series with media made up the day of injection. 9 : 10-dimethyl-1 : 2-benzanthracene, injection once medio-dorsally with 0·3 c.mm., corresponding to 0·2 γ of substance. The amount of fluid seemed to be near the threshold of mechanical toleration, since of 61 treated females and 56 control females only 25 survived in each batch, 19 of which were used for testing. No lethal was detected in 934 treated and 867 control chromosomes.

6th Series.—Repetition of fourth series with fresh media. 9 : 10-dimethyl-1 : 2-benzanthracene, 0·3 c.mm., corresponding to 0·2 γ of substance injected once medio-dorsally. 26 survivors out of 60 treated males, and 27 out of 57 controls. 20 used in each batch for producing offspring. In 1125 treated chromosomes there were 8 lethals, 7 of these from different paternal chromosomes. In 814 control chromosomes there were 7 lethals, 5 from separate males. No difference in mutation rate was observed.

7th Series.—Methyl-cholanthrene. Crystals inserted into adult females. 12 females treated. 1 lethal in 734 chromosomes. No controls, but as the mutation rate appears as low as commonly found in untreated material, further testing of this method was not considered worth while.

8th Series.—9 : 10-dimethyl-1 : 2-benzanthracene. 9 female larvæ injected with about 0·3 c.mm., corresponding to 0·2 γ of substance. No lethal in 749 tested chromosomes.

DISCUSSION.

None of the substances and techniques used gave a detectable increase in mutation rate over the controls. The amount which could be injected was of course very small, somewhat less than 0·2 γ on the average. But as the average weight of *Drosophila* is not over 1 mg., the amount of injected substance taken in proportion to body-weight corresponds to 40 mg. for a rat of 200 g., a dose which in the rat is more than sufficient for tumour induction with any of these substances. Conceivably, however, in the rat the substance will, by precipitation or otherwise, become concentrated in a relatively small area so that a direct comparison of the injected amounts taken in relation to the body-weights might not be feasible. A second reason for the failure to cause germ-cell mutations in the fly may be that an insufficient time was allowed for action, as offspring from treated flies could be collected for several weeks only, whereas in the rat several months are usually required for the detection of the tumour (the time of detection, however, being considerably later than the time of the primary cellular change). Other possible causes for the negative result are difficulties of penetration through the thin mesodermal membrane surrounding the ovarioles or through the even thicker testis sheath, differences in susceptibility between germ cells and somatic cells or between mammalian and insect cells, and differences in other factors, of an anatomical or chemical nature, between mammal and insect.

According to a paper by Sacharov (1938), which appeared after our own work had begun, this investigator obtained an apparently significant increase in mutation rate (18 lethals in 2921 treated, or 0·62 per cent. as compared with 87 lethals in 33,975 control chromosomes, or 0·25 per cent.) by treating *Drosophila* eggs with a solution of methylcholanthrene. Since, however, differences in mutation rate of this order may arise spontaneously from unknown causes when genetical and environmental conditions have not been very exactly controlled, and since similar differences are reported in the same paper for a considerable number of treatments with different chemicals, it does not seem possible to assess the importance of this result before the details concerning the conduct of the experiment are published. It would appear, however, to give some encouragement to further attempts along these lines, using varying types of technique.

An incidental result of the present study was the pronounced sex difference in spontaneous mutation rate in the stocks used (compare series 3 with 4, and 5 with 6). Experiments to follow up this rather surprising finding have been started.

SUMMARY.

Attempts to increase the mutation rate in *Drosophila melanogaster* by introduction into the body of carcinogenic substances as colloidal solutions or crystals have given a negative result.

ACKNOWLEDGMENTS.

The work formed part of a programme carried out under the direction of Dr H. J. Muller, and financed by the Scottish Cancer Control Organisation. Thanks are hereby expressed to the above, and also to Dr C. M. Scott and Professor J. W. Cook, who provided the carcinogenic substances and gave helpful advice concerning their use. Thanks are also given to Professor F. A. E. Crew for his kind interest and encouragement and for providing facilities for the work.

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XIV.—The Genetical and Mechanical Properties of the Sex Chromosome. VI. *Hexacentrus mundus*.

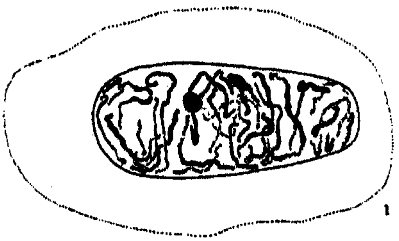
By **P. C. Koller,** D.Sc., Ph.D., Institute of Animal Genetics, University of Edinburgh. (With Fifteen Text-figures.)

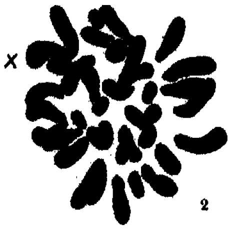
(MS. received March 15, 1940. Revised MS. received April 4, 1940.
Read May 6, 1940.)

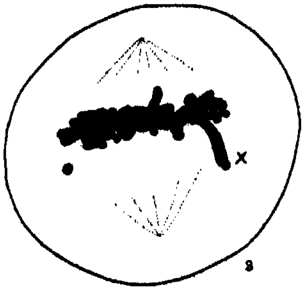
CYTOLOGICAL studies carried out by McClung (1905, 1914) on various species of *Orthoptera* have shown that the male is the heterogametic sex. The male has only one X-chromosome, whereas the female has two. During spermatogenesis two kinds of gametes are produced, one with the X-chromosome and the other without it. It was also found that the segregation of the single sex chromosome takes place at first meiotic anaphase. The present paper describes the sex chromosome of the male *Hexacentrus mundus* Walker, from India. During the meiotic division this chromosome exhibits peculiarities which it is believed have not hitherto been seen in any species of the Orthopteræ.

The diploid chromosome number in the male sex of *Hexacentrus mundus* is 31, representing 15 pairs of autosomes and one X-chromosome (Asana, Makino, and Niiyama, 1938). The position of the centromere in the autosomes is nearly terminal; these chromosomes are composed of two arms, one of which is very short and resembles the satellites observed in plant-chromosomes (fig. 2). Such a position of the centromere is similar to that seen in the autosomes of *Locusta* and *Schistocerca* (White, 1933, 1935). Very often, because of inadequate technique, the short arms of the autosomes cannot be seen, and this is responsible for the common belief that the chromosomes are "telomitic," *i.e.* that they have a terminal centromere. The autosomes fall into three groups, according to size; the lengths of the chromosomes are 7 μ, 4–5 μ, and 2 μ respectively. The X-chromosome is 12 μ long, and is the largest. During mitotic prophase the sex chromosome cannot be distinguished from the autosomes, since it exhibits the same degree of contraction and condensation as they do (fig. 1). The ball-shaped nucleolus is a characteristic constituent of the prophase nuclei; no chromosomes are found attached to the nucleolus.

At mitotic metaphase of the spermatogonium the large X-chromosome usually lies on the periphery of the spindle. In 63 spermatogonia only 7 were found in which the X-chromosome was lying across the spindle with both chromosome arms stretched out. The centromere of the X is on the equatorial plate, while the longer arm is usually carried away by the cytoplasmic currents (fig. 3). The contours or outlines of the X-chromosome are smooth, which suggests that the contraction of the X-chromosome is complete at mitotic metaphase. In the various species of Acrididæ the X was seen to have a rough outline, indicating a delay in contraction (Asana, Makino, and Niiyama, 1938).







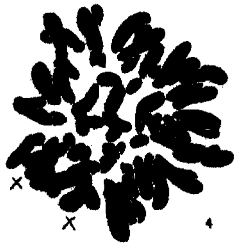


FIG. 1.—Prophase of mitosis in a spermatogonium. The nucleolus is shown and the double structure of chromosome threads can be seen. All chromosomes exhibit the same degree of contraction. × 2400.

FIG. 2.—Mitotic metaphase plate. In some chromosomes the short arm can be seen. × 2400.

FIG. 3.—Mitotic anaphase showing the long arm of the X-chromosome. × 2400.

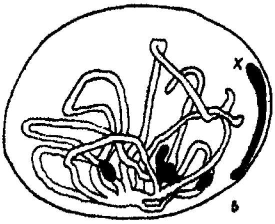
FIG. 4.—Mitotic metaphase plate with tetraploid chromosome number. × 2400.

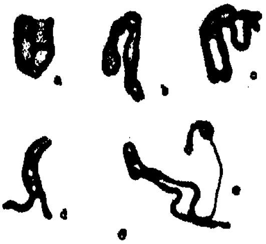
Testes of *Hexacentrus* were fixed in medium Flemming, sections were cut at 20–22 μ, and stained with gentian violet and iron hæmatoxylin. The drawings were made with the aid of a Zeiss camera lucida.

In three spermatogonia the X-chromosome and a few autosomes were found to be under-stained, while the other chromosomes were deeply stained. It is interesting to note that all these under-stained autosomes were of the same size and that they always occupied the centre of the spindle. Though a Feulgen test was not made, it is very probable that under-staining is caused by the lack of nucleic acid in these particular chromosomes (Caspersson, 1939).

A number of tetraploid spermatogonia were found (fig. 4). Polyploid cells are particularly numerous in the fat tissue surrounding the gonads. The absence of binucleate cells suggests that polyploid cells originate not through syndiploidy but through the failure of the daughter chromosomes to segregate towards the opposite poles at mitotic anaphase. The chromosomes are crowded in the metaphase plate of the tetraploid cell; the smaller chromosomes lie at the centre, while the larger ones are arranged at the periphery of the spindle. Chromosomes of the same length lie adjacent to each other, an arrangement which resembles the somatic pairing of homologous chromosomes.

The nucleus of the primary spermatocytes during leptotene and zygotene of meiosis is filled with fine threads, which represent the chromosomes. In the midst of them lies a large body of irregular shape, which is identified as the "heteropycnotic" X-chromosome. This is composed of a diffuse, slightly stained portion and a deeply stained region. Various changes occur in this nucleolus-like body during zygotene and early pachytene. First of all the deeply stained region forms itself into a thread, which appears to be coiled within the diffuse portion. This part rapidly decreases in volume through the breaking off of smaller fragments. These loose fragments stain as deeply as the paired chromosome threads, and they are seen to be scattered in the pachytene nucleus (fig. 5). A small quantity of the diffuse part of the nucleolus is sometimes left attached to the X-chromosome for a long period (fig. 6, *e*). The thread-like appearance of the X becomes more apparent as more of the diffuse part of the nucleolus is cast off (fig. 6, *a*). The double structure of the thread can easily be seen when the diffuse substance is greatly diminished (fig. 6, *b*). The two threads are not close together, but lie apart and are held together by means of a network of fine filaments. Occasionally one of the threads is shorter than the other (fig. 6, *c*, *d*). One may assume that the two threads represent the two unequal arms of the X-chromosome, and the apparently double structure is brought about by the uncoiled sex chromosome being bent double in such a manner that the two limbs are lying alongside one another. This condition is very common in Orthopteræ, and several primary spermatocytes were encountered in *Hexacentrus* in which the structure of the X during pachytene can be satisfactorily explained by this interpretation. Further analysis, however, suggests that the sex chromosome may have divided in such a manner that not two chromatids but two chromosomes are formed, each with its own centromere. The two sister chromosomes lie side by side, and the difference in length occasionally observed may be due to a difference in contraction within the two threads, suggesting that they are independent of each other. If our interpretation is correct such behaviour of the X-chromosome has not hitherto been recorded in any organism. This interpretation assumes that, though the centromere is divided, the two sister chromosomes remain associated till the second meiotic anaphase, when their segregation takes place. In most cases analysed the structure of the X strongly suggests that our interpretation, which assumes a complete division of the X-chromosome during the first meiotic prophase, is justified.





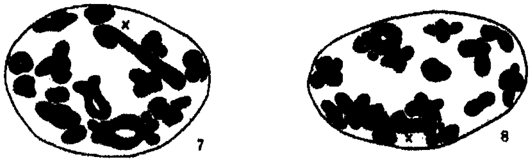


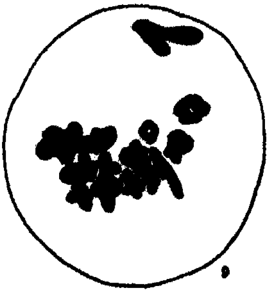
FIG. 5.—Late pachytene of meiosis in a primary spermatocyte. The autosomal chromosomes are paired and polarised. The deeply stained fragments are assumed to be the remnants of the diffuse part of the chromosome nucleolus. The sex chromosome appears as a single thick rod. × 2400.

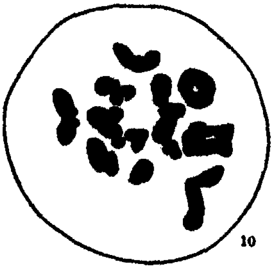
FIG. 6, *a*, *b*, *c*, *d*, *e*.—Various stages in the development of the heteropycnotic X-chromosome. The division into two independent chromosomes is clearly shown. The two chromosomes are held together by a network of fine filaments. × 2400.

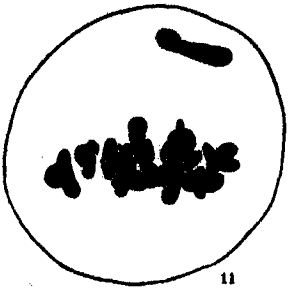
FIGS. 7, 8.—Early (7) and late (8) diakinesis. The compound X-chromosome appears as a single unit, and it is strongly contracted. × 2400.

At late pachytene, when the paired autosomal bivalents show polarisation, the two sister chromosomes of the X lie very close together and appear as a single thick rod, which is pressed against the nuclear membrane (fig. 5). During the following diplotene and diakinesis the apparently single X-chromosome becomes thicker and shorter, indicating further contraction (figs. 7, 8). Here the two arms of the X can be identified. In some primary spermatocytes the short arm is contracted to a greater extent than in others. After pachytene the compound X behaves as a single unit, and its double structure cannot be seen.

At first meiotic metaphase the autosomal bivalents orientate themselves on the equatorial plate, while the "monosome" X lies off the plate (figs. 9, 11). In spite of the fact that the X-chromosome usually lies outside the spindle, its centromere is directed towards the pole. In a few primary spermatocytes the X-chromosome is seen to be lying at the same level as the autosomal bivalents. In these cases the X-chromosome is always outside the spindle (fig. 10). This position is very different from that of the X-chromosome in the closely related family of Acrididæ, where the single X-chromosome always stands perpendicular to the equatorial plate and is included amongst the autosomal bivalents (Asana, Makino, and Niiyama, 1939). The contraction is greatest at the end of first metaphase, and the short arm of the X was found to be less than half the length of the long arm (fig. 12). During first anaphase the X moves with the segregating members of the autosomal bivalents to the







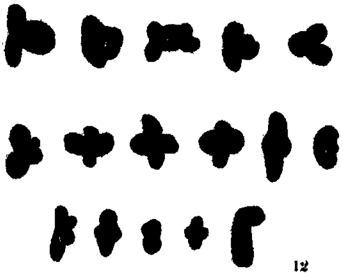


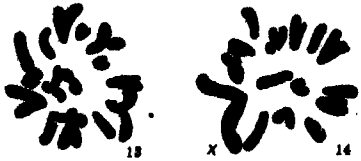
FIG. 9.—First meiotic metaphase. The X-chromosome is lying off the equatorial plate. × 2400.

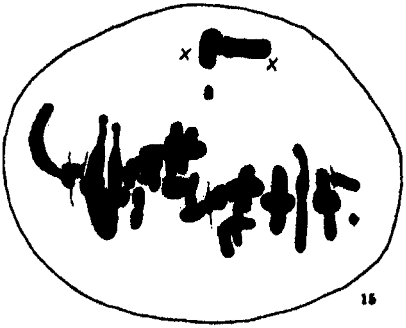
FIG. 10.—Polar view of first meiotic metaphase. The X-chromosome is on the same plane as the autosomal bivalents, but it lies outside the spindle. × 2400.

FIG. 11.—Side view of first meiotic metaphase. The short arm of the X-chromosome exhibits a stronger contraction than in fig. 9. × 2400.

FIG. 12.—The 15 bivalents and the X-chromosome of a meiotic metaphase. The X-chromosome shows a great degree of contraction. × 2400.

pole (figs. 13, 14). In the interkinetic nucleus the compound X shows heteropycnosis. White found a similar behaviour in the X-chromosomes of *Mecostethus* and *Callimantis* (1936, 1938). The double structure of the heteropycnotic X-chromosome in *Hexacentrus* can be seen clearly during interkinesis. The separation of the two daughter chromosomes of the compound X takes place at second meiotic anaphase. This separation sometimes precedes that of the daughter chromosomes of the autosomes.





FIGS. 13, 14.—Second meiotic metaphase without (13) and with (14) the X-chromosome. Division of the compound X takes place during second meiotic anaphase. × 2400.

FIG. 15.—Meiotic metaphase in a tetraploid primary spermatocyte. Multiple chromosome associations and univalents are frequent in such polyploid cells. The two X's are left unpaired. × 2400.

The meiotic division was studied in primary spermatocytes which have a tetraploid chromosome number. Besides the univalents and bivalents, multiple chromosome associations also were observed in these spermatocytes. The two X-chromosomes were also identified; these did not pair, though in some instances they were seen to be lying close together (fig. 15). A similar behaviour of the X was reported by Wilson (1932) in *Archimerus*, by White (1933) in *Schistocerca*, and by Makino (1939) in *Podisma*. It is assumed that the heteropycnosis of the two X-chromosomes during the early prophase of meiosis in the tetraploid spermatocytes is responsible for their failure to associate. Our observations also contradict the suggestion, put forward by Mohr (1916), that the heteropycnosis of the X in the heterogametic sex is due to the absence of a homologue.

ACKNOWLEDGMENT.

The author wishes to express his thanks to Dr J. J. Asana, Gujarat College, Ahmedabad, India, who kindly supplied him with the material.

SUMMARY.

1. The male *Hexacentrus mundus* Walker has one X-chromosome which is strongly heteropycnotic at early prophase of meiosis. It is suggested that during the pachytene stage the single X-chromosome divides into two independent chromosomes which, however, remain associated and form a compound X. The separation of the two sister sex chromosomes takes place at the second meiotic anaphase.

2. In tetraploid spermatocytes two X's are present. These are always unpaired during the first meiotic division. It is suggested that the strong heteropycnosis of the sex chromosomes at early prophase is responsible for their failure to pair.

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XV.—A Theoretical Atomic Distribution Curve for Liquid Argon at 90° K.

By **G. S. Rushbrooke,** University College, Dundee. *Communicated by* Professor E. T. Copson, M.A., D.Sc. (With Two Text-figures.)

(MS. received February 24, 1940. Read May 6, 1940.)

1. *Introduction.*—The radial distribution of the atoms (or molecules) of liquids, as of solids, can be found from X-ray scattering photographs (Zernike and Prins, 1927; Warren and Gingrich, 1934), and in this way many such distributions have been determined experimentally (Harvey, 1938, 1939; Gingrich, 1940; references in Coulson and Rushbrooke, 1939). The results are generally given as distribution curves showing $\rho(r)$ or $r^2\rho(r)$ as a function of r, where $4\pi r^2\rho(r)dr$ measures the probability that two arbitrarily selected atoms (or molecules) of the liquid are distant r to $r+dr$ apart. Such distribution curves show a sequence of peaks—for $\rho(r)$ these are usually of diminishing amplitude—and for atomic liquids successive peaks may conveniently be ascribed to successive co-ordination shells (of atoms) about any (arbitrarily selected) central atom. The position and precise shape of the peaks depends, of course, upon the temperature as well as the substance of the liquid.

There have been fewer attempts to calculate such atomic distribution curves theoretically. Very recently Kirkwood (1939) has shown how distribution curves for non-polar liquids can be derived theoretically, and the method may be capable of giving quantitative results, but at present this work is somewhat qualitative. Previously Wall (1938), by approximating to an actual liquid by a simple model, had derived a theoretical form for a single peak of a distribution function, and experimentally determined distribution functions can be reproduced fairly well (Wall, 1938; Gingrich and Wall, 1939; Coulson and Rushbrooke, 1939) by superimposing suitable Wall peaks similar to each other and at appropriate, empirically determined distances from the origin. But little other theoretical work has been done.

A limitation of Wall's theory is that his equation for the peaks of a distribution curve involves a parameter, σ, which bears little relation to the physical or atomic properties of the liquid concerned. Thus, even in the simplest cases, it is not possible to predict what the distribution curve for a liquid will be by using Wall's equation: we can only reproduce known distribution curves by means of Wall peaks in the way described above. Such a procedure is certainly not without value in the interpretation of the experimentally determined curves, but its semi-empirical nature shows that much remains to be done towards developing a satisfactory theory of atomic distribution curves for liquids in terms of the properties of the atoms themselves.

This limitation of Wall's theory is due to the very simple atomic model for a liquid which he employs. In this the atoms are treated as hard spheres. This has the advantage of generality but the disadvantage already mentioned, since it takes but a crude account of the interatomic forces existing within the liquid. Now Lennard-Jones and Devonshire have developed a theory of the liquid state of the inert gases, which may be supposed to form close-packed liquids, and though it is based on an Einstein (Mott and Gurney, 1938) model for the liquids, it takes sufficient account of the interatomic forces within them to predict fairly accurately such of their physical properties as their critical points and boiling temperatures (Lennard-Jones and Devonshire, 1937 and 1938 respectively). It is possible to use the same model to calculate atomic distribution curves, and this without introducing unknown parameters. As from Wall's theory we can find directly only a theoretical shape for the separate peaks of a distribution curve, and not the whole curve; but from Lennard-Jones and Devonshire's model we can predict not only the shape of the peaks of the atomic distribution curve for the liquid concerned but also the position and height of the first peak; and the nearness with which these predictions agree with experiment will be a good guide to the serviceability of such a model.

A general formula for deriving $\rho(r)$ from any Einstein model for a liquid has been given by Coulson and Rushbrooke (1939). An Einstein model is characterised by the assumption that each atom moves, independently of its neighbours, in the same average field; this field restricts the movements of the atom to vibratory motion about a fixed centre. If this average field be assumed to be spherically symmetrical and be denoted by $\phi(s)$, where s measures the displacement of the atom from its mean position, then the formula required is

$$4\pi r\rho_t(r)=\frac{4\pi^2 N_t}{R_t v^2}\int_0^\infty r_1 e^{-\phi(r_1)/kT}\Big[2r_1\psi(\infty)-\int_{|R_1-r|}^{|R_1+r|}\psi(\,|\,r-x\,|\,)dx\Big]dr_1, \quad . \qquad (1)$$

where

$$v=4\pi\int_0^\infty s^2 e^{-\phi(s)/kT}ds, \quad . \quad . \quad . \quad . \qquad (2)$$

$$\psi(r)=\int_0^r s e^{-\phi(s)/kT}ds, \quad . \quad . \quad . \quad . \qquad (3)$$

and N_i and R_i denote respectively the number of atoms in and the radius of the ith * co-ordination shell about a central atom. $\rho_i(r)$ is the contribution to $\rho(r)$ from the atoms of this ith co-ordination shell.

$r\rho_i(r)$ is then a function of $| R_i - r |$, which is symmetrical about $r = R_i$, and corresponds to the ith peak in the distribution curve. On this theory the various peaks of the function

$$r\rho(r) = \sum_i r\rho_i(r) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

differ only in the values to be ascribed to N_i and R_i. It is due to ignorance of these quantities that we cannot calculate theoretically the complete function $\rho(r)$ from any assumed form for the potential field $\phi(s)$. Lennard-Jones and Devonshire's theory, however, of the liquid state of the inert gases is based on the assumption that $N_1 = 12$ and relates R_1 to the density of the liquid: and so from it we can calculate precisely, with respect to both shape and position, the first peak of the distribution curve.

The purpose of the present paper is to give the result of a calculation, based on Lennard-Jones and Devonshire's model, of the first peak of the atomic distribution curve for liquid argon at 90° K. That temperature was chosen as one at which the density of liquid argon was known (*International Critical Tables*, vol. iii, p. 20), and also as being sufficiently near the freezing-point for the concept of co-ordination shells within the liquid to have meaning.

The peak calculated is compared with a Wall peak, which is found to reproduce it closely. The results of attempts to construct a complete distribution curve on two different assumptions about the magnitudes of N_2, N_3, . . ., R_2, R_3, . . ., are also given.

2. *The Average Field*, $\bar{\phi}(s)$.—The characteristic feature of Lennard-Jones and Devonshire's model for liquids, which distinguishes it from any other Einstein model, is the way in which the average potential energy, $\bar{\phi}(s)$, of an atom distant s from its mean position in the liquid is obtained from the potential function $\phi(r)$, which measures the mutual potential energy of two atoms of the substance concerned (inert gas) when distant r apart in free space. The potential $\bar{\phi}(s)$ at the point s is taken to be due only to the nearest neighbours (*i.e.* to the first co-ordination shell) of the atom whose mean position is at the origin. The field at s due to these atoms is averaged with respect to all positions of these atoms on the surface of a sphere with its centre at the origin and of radius equal to the mean distance between an atom and its nearest neighbours, *i.e.* of

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* In this paper the notation is a little different from that of Coulson and Rushbrooke (1939), where the successive peaks were numbered 0, 1, 2, . . . instead of 1, 2, 3, . . . The latter notation will be used below.

radius equal to R_1. For the liquid state of the inert gases the number of these neighbours is assumed to be 12, and R_1, which in accordance with the notation of Lennard-Jones and Devonshire we shall denote by a, is assumed to be related to the atomic volume v^* by the equation

$$v^* = \frac{1}{\sqrt{2}}a^3. \qquad (5)$$

This equation is that which relates the lattice parameter and the specific volume of a face-centred cubic lattice. Though the formula (1) is not affected thereby, it is convenient to take the zero of potential energy such that $\bar{\phi}(0)=0$.

If

$$\phi(r) = \frac{\lambda}{r^n} - \frac{\mu}{r^m}, \qquad (6)$$

then

$$\frac{1}{N_1}\bar{\phi}(s) = \frac{1}{2}\left(\frac{a}{s}\right)\left[\frac{\lambda}{(n-2)a^n}\left\{\left(1-\frac{s}{a}\right)^{-n+2} - \left(1+\frac{s}{a}\right)^{-n+2}\right\}\right.$$
$$\left. - \frac{\mu}{(m-2)a^m}\left\{\left(1-\frac{s}{a}\right)^{-m+2} - \left(1+\frac{s}{a}\right)^{-m+2}\right\}\right] - C, \qquad (7)$$

where C is chosen to make $\bar{\phi}(0)=0$.

Lennard-Jones and Devonshire assumed that for argon, and the other inert gases, $n=12$ and $m=6$. λ and μ were then chosen to give agreement with experimental values for the second virial coefficient of the gas (see Fowler, 1936, p. 306). Recent work of Corner (1939), however, suggests that when account is taken of zero-point energy better accord with the lattice constant, etc., of solid argon can be obtained with a somewhat smaller value of n. In the light of Corner's work n has been taken to be 11·4 in the present calculations. To find appropriate values of λ and μ, values of ϕ_0 and r_0 (see Corner's paper) corresponding to $n=11{\cdot}4$ were found by interpolation from Corner's Table I, and the related values of λ and μ calculated therefrom. The field $\phi(r)$ estimated in this way is

$$\phi(r) = \frac{5{\cdot}43 \,.\, 10^{-99}}{r^{11\cdot4}} - \frac{11{\cdot}1 \,.\, 10^{-59}}{r^6} \text{ ergs},$$

with r in cm. It is perhaps worth remarking that the coefficient of the Van der Waal's part of this potential, $11{\cdot}1 \,.\, 10^{-59}$, is just that ascribed to it by Herzfeld (1937), who gives as a satisfactory interatomic potential field for two argon atoms,

$$\phi(r) = -\frac{11{\cdot}1 \,.\, 10^{-59}}{r^6} + 1{\cdot}34 \,.\, 10^{-9} \exp\left(\frac{r}{0{\cdot}345 \,.\, 10^{-8}}\right) \text{ ergs},$$

with r in cm.

With these values of λ and μ (7) becomes

$$\frac{1}{12}\phi(s) = 10^{-14}\left(\frac{a}{s}\right)\left[0{\cdot}6065\left\{\left(1-\frac{s}{a}\right)^{-9\cdot4}-\left(1+\frac{s}{a}\right)^{-9\cdot4}\right\}\right.$$
$$\left.-3{\cdot}621\left\{\left(1-\frac{s}{a}\right)^{-4}-\left(1+\frac{s}{a}\right)^{-4}\right\}\right]+17{\cdot}566\,.\,10^{-14}, \qquad (8)$$

and its value was tabulated for 25 values of s/a between 0 and 0·25. The potential field closely resembles that of fig. 1*b* in Lennard-Jones and Devonshire (1937), which pertains to approximately the same density of liquid argon.

From the density of liquid argon at 90° K. ($\rho_0 = 0{\cdot}0209$ atom per Å^3) a is calculated to be 4·077 Å.

3. *Calculation of a Peak of $r\rho(r)$.*—There is no need to detail the calculation of $r\rho_1(r)$ from $\phi(s)$ by means of formula (1). It is enough to note that it was sufficient to tabulate $\psi(r)$ for 6 values of r/a between 0 and 0·25 (excluding the origin). For greater values of r/a, $\psi(r)$ is effectively constant. A "four strip" formula for numerical integration (Bickley, 1939) was used in obtaining $\psi(r)$. The next numerical integration, that of integrating $\psi(r)$, was performed graphically, and the final integration by the double application of a "six strip" formula. The result of the calculations is given in the table below. $4\pi r\rho_1(r)$ is tabulated for various values of $|\,R_1 - r\,|/R_1$, *i.e.* of $\left|\,1-\frac{r}{a}\,\right|$. The peak is shown graphically in fig. 1.

A check on the accuracy of the numerical integrations performed in calculating $r\rho_1(r)$ is given by $4\pi\int_0^\infty r^2\rho_1(r)dr$. By numerical integration, incorporating the data of the table and the double application of an "eight strip" formula, this is found to be $12{\cdot}0_4$, in good agreement with the value, 12, assumed for N_1.

$\left\|\,1-\frac{r}{a}\,\right\|$	$4\pi r\rho_1(r)$	"Wall's" $4\pi r\rho_1(r)$
0	$3{\cdot}08_{-3} \times 10^{16}$	$3{\cdot}08_{-3} \times 10^{16}$
0·02	$3{\cdot}01_{3}$,,	$3{\cdot}00_{3}$,,
0·04	$2{\cdot}82_{8}$,,	$2{\cdot}81_{1}$,,
0·08	$2{\cdot}17_{6}$,,	$2{\cdot}18_{4}$,,
0·12	$1{\cdot}38_{4}$,,	$1{\cdot}43_{4}$,,
0·16	$0{\cdot}72_{-1}$,,	$0{\cdot}75_{4}$,,
0·20	$0{\cdot}30_{0}$,,	$0{\cdot}28_{-4}$,,
0·24	$0{\cdot}09_{1}$,,	$0{\cdot}04_{3}$,,
0·28	$0{\cdot}02_{-1}$,,	0·00

On the basis of the present theory the complete curve $r\rho(r)$ is formed by superimposing peaks similar to $r\rho_1(r)$ and at various distances from the origin. The number of atoms in the shell corresponding to the ith peak,

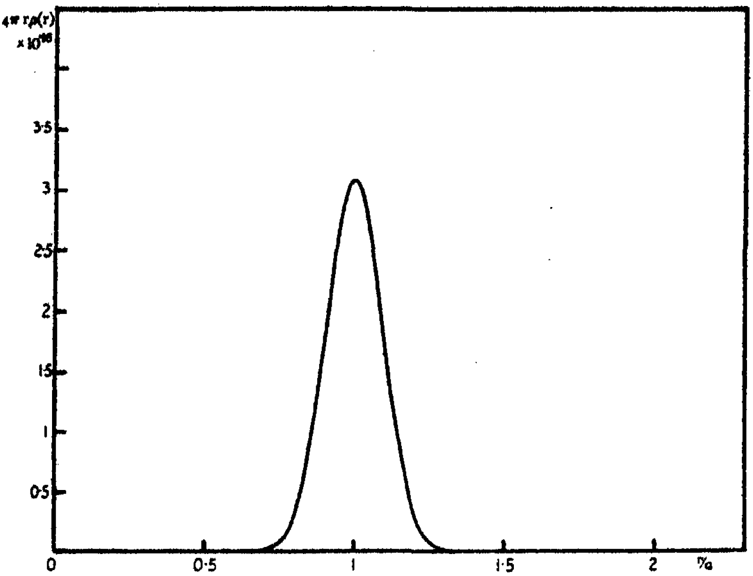

FIG. 1. ($a=4{\cdot}077$ Å)

the height of this peak, Y_i, and its mean ordinate, R_i, are connected (see equation (1)) by the formula

$$N_i/Y_iR_i = \text{constant}. \qquad (9)$$

4. *Comparison with a Wall Peak.*—The peak shown in fig. 1 is the contribution to $4\pi r\rho(r)$ made by the nearest neighbours of any "central" atom of liquid argon at 90° K., as calculated on the basis of Lennard-Jones and Devonshire's theory of the liquid state of the inert gases.

Its general shape is very much that of a Wall peak. In Wall's theory $\phi(s)$ is taken to be simply

$$\left.\begin{aligned} \phi(s) &= 0, \qquad s < \sigma, \\ \phi(s) &= \infty, \qquad s > \sigma, \end{aligned}\right\} \qquad (10)$$

where σ is an unknown parameter. Equation (1) then becomes

$$y_i \equiv 4\pi r\rho_i(r) = \frac{3N_i}{5\sigma R_i}\left\{1 - \frac{5(R_i-r)^2}{4\sigma^2}\left[1 - \frac{|R_i-r|}{2\sigma} + \frac{|R_i-r|^3}{40\sigma^3}\right]\right\} \qquad (11)$$

(which is equation (10) of Wall's paper), so that Y_i, the height of the ith peak, is given by

$$Y_i = \frac{3N_i}{5\sigma R_i}. \qquad (12)$$

Equation (12) enables us to compare the peak of fig. 1, calculated on the basis of (8) as an approximation to $\phi(s)$, with a Wall peak based on the cruder approximation to $\phi(s)$ given by equation (10). For from it we can find an appropriate value of σ for use in equation (11). We must assume that N_1 and R_1 have the values already ascribed to them, 12 and 4·077 Å respectively, and we shall further require that the Wall peak, with which we are going to contrast the curve of fig. 1, and this latter curve have the same height, Y_1, *i.e.* that $Y_1 = 3{\cdot}08_{-8}\,.\,10^{16}$. This is a suitable way of choosing a Wall curve for comparison with our newly calculated peak since when fitting Wall curves to experimentally determined distribution curves it is desirable to ensure that they pass through the maxima of the experimental peaks. In this way we find as an appropriate value for σ the value $0{\cdot}14_1$.

This is in quite remarkably close agreement with the conclusions which we might have drawn from fig. 1 itself. For in Wall's theory $\rho(r) = 0$ if $|R_i - r| > 2\sigma$, and for $\sigma = 0{\cdot}14_1$ the width of the base of the peak is almost precisely that of the curve shown in fig. 1. For this value of σ in fact, and the values of N_1 and R_1 already decided upon, formula (11) gives a very close approximation to that curve. In the third column of the table are the actual values of $4\pi r\rho_1(r)$ according to Wall's formula, and it will be seen that they follow those calculated on the basis of Lennard-Jones and Devonshire's model very closely. Near the base of the peak the values given by Wall's formula are less than those which we have estimated from equations (1) and (8), and then become greater than these for smaller values of $\left|1 - \frac{r}{a}\right|$. This behaviour is conveniently described by saying that the (presumably) more accurate calculation in which the atoms are not treated as hard spheres, but instead more explicit account is taken of the forces of attraction and repulsion between them, results in a distribution curve which is rather "fatter" towards the base than a Wall curve. This supports the previous contention of Coulson and Rushbrooke (1939) that, being based on a model which treats the atoms as hard spheres, Wall peaks are not quite sufficiently "fat" near the base to give an accurate reproduction of experimentally determined distribution curves.

5. *The Construction of Complete Distribution Curves.*—So far we have calculated only the first peak of $r\rho(r)$, the peak which corresponds to "first co-ordination shells" in the liquid. We cannot proceed further and obtain a more complete distribution curve without making explicit assumptions about R_2, N_2, R_3, N_3, . . . Knowing these parameters, we can of course construct $r\rho(r)$ for all values of r by superimposing peaks

similar to the first one, and to scales determined by equation (9). But the values of these parameters are not obtainable from Lennard-Jones and Devonshire's theory (in which they are not required). They can be introduced only by further elaborating our model for the liquid state of argon at 90° K.

In fig. 2, $\rho(r)$ is shown as a function of r on the basis of two alternative assumptions about R_2, N_2, R_3, N_3, . . . The full curve is based on the assumption that

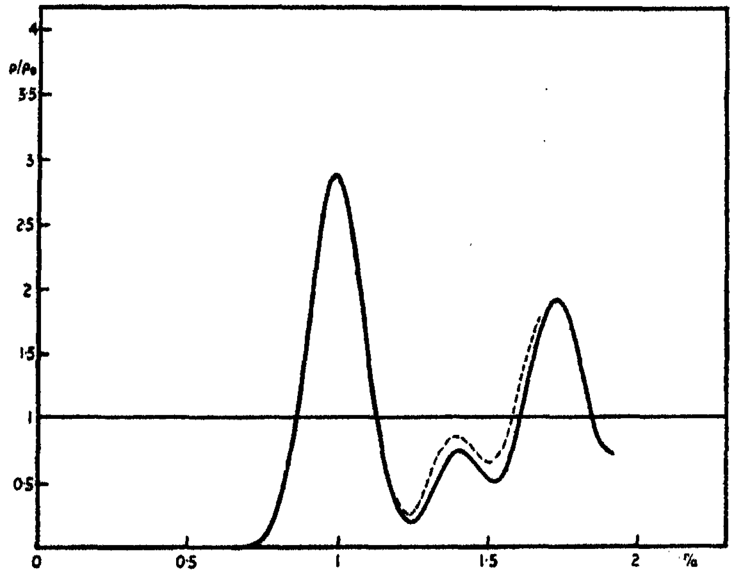

FIG. 2.

——— Face-centred-cubic model.
- - - - - - *Bernal's molecularly homogeneous model.*

$\rho_0 = 0{\cdot}0209$ atoms/Å^3
$a = 4{\cdot}077$ Å

$$R_2 = \sqrt{2}a \qquad N_2 = 6$$
$$R_3 = \sqrt{3}a \qquad N_3 = 24$$
$$R_4 = 2a \qquad N_4 = 12\ .\ .\ .$$

i.e. that these parameters have the values pertaining to a face-centred cubic lattice. The broken curve is based on values for R_2, N_2, R_3, and N_3 taken from Bernal's geometrical theory of molecular homogeneity (Bernal, 1936; the values are taken from fig. 3 of this paper), *i.e.*

$$R_2 = 1{\cdot}4a \qquad N_2 = 7$$
$$R_3 = 1{\cdot}7a \qquad N_3 = 22$$

The curves do not differ very greatly. They both show a small second peak with its maximum below the line of average density ρ_0. Since liquid argon at low temperatures is likely to be a close-packed liquid, either curve may reasonably be expected to approximate to the results of experiment.

In conclusion I have to thank Dr C. A. Coulson for helpful interest in these calculations, and the Council of University College for a research grant enabling me to work at Dundee.

SUMMARY.

The first peak of an atomic distribution curve for liquid argon at 90° K. is calculated theoretically, on the basis of Lennard-Jones and Devonshire's model for the liquid state of the inert gases. The result is compared with a Wall peak, which is found to reproduce it closely.

More complete distribution curves, showing further peaks, are also given; they are constructed on the basis of two alternative further assumptions about the structure of the liquid.

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(*Issued separately July* 18, 1940.)

XVI.—The Loss of Heat by Conduction from the Earth's Crust in Britain. By **E. M. Anderson,** M.A., D.Sc., F.G.S. (With Four Text-figures.)

(MS. received May 6, 1940. Read July 1, 1940.)

THE object of this paper is to record certain temperatures which were measured in the deep Boreland Bore, near Dysart, in Fife, and to discuss the bearing of these and some other observations on the amount of heat which escapes from the crust, by conduction, in Britain.

The flow of heat is determined from two factors: the gradient of temperature near the surface, and the conductivity of the rock or rocks in which the gradient has established itself. Until recently the two quantities had usually been estimated independently. In general the gradient in a bore or shaft was combined with observations of conductivity in specimens of rocks which were supposed to be similar to those which had been pierced in the sinking, but which might have been obtained from a distance. Unknown considerations might thus be involved, and for this reason a Committee of the British Association was formed, in 1934, "to direct determination of thermal conductivities of rocks in mines and borings where the temperature has been, or is likely to be, measured." This Committee should not be confused with the earlier Committee on thermal conductivities, which published reports from 1875 to 1882.

Acting in conjunction with the Committee, Mr A. E. Benfield made determinations on a number of specimens from the Boreland Bore; with these the Fife Coal Company had kindly allowed me to supply him. His results are included in a paper (1939) dealing with the terrestrial flow of heat in this country. In view of these investigations, it was particularly desirable that the gradient of temperature should be ascertained in the same bore-hole, and cordial thanks are due to the Fife Coal Company for granting permission and time for the necessary observations.

The bore is registered by the Geological Survey as Fife 36 NW, No. 52. It is situated less than a mile north of Dysart, and within the same distance from the shore of the Firth of Forth. The latitude is 56° 8′ 8″ N. and the longitude 3° 7′ 12″ W., while the ground level is approximately 200 ft. over O.D. After piercing about 16 ft. of boulder-clay, sinking began through Coal Measures. It continued through

Millstone Grit and the Upper Limestone Group, and was brought to a close at a depth of 3303 ft., or 1007 metres, after penetrating nearly the whole thickness of the Limestone Coal Group. Operations began in March 1937, and had reached the depth mentioned at 12 noon on 9th August 1939.

The great majority of bores in Scotland are found, after completion, to remain filled with water to within a short distance of the surface. It must be inferred that the strata are water-logged up to a comparatively shallow "water-level." This is in agreement with mining experience, as Scottish coal-workings are seldom free from water, contrasting in this respect, for instance, with those in the concealed coalfield of Yorkshire. The rule is not invariable, as occasionally the drilling water in a bore is lost; that is, it fails to rise again to the surface. In estimating the conductivity of the strata, however, it should probably always be assumed, in Scotland, that the pores of the rock present are full of water.

In a water-filled bore there is, in certain cases, a possibility that the temperatures at the base are lowered by convection currents, and are thus not representative of those in the surrounding rock. This possibility depends on the temperature gradient, and on the relation of the diameter of the bore-hole to the viscosity and conductivity of the water at the particular temperature. This has been shown by L. J. Krige (1939), who applies a deduction of A. L. Hales (1937). The diameter of the Boreland Bore was 5 in. at the depths at which the determinations were made, and a 5-in. hole would be "unstable" at any temperature, according to Hales's result. Van Orstrand has, however, failed to find any evidence of convection in holes which are less than 8 in. in diameter (1924).

It is probable that this effect is of less importance than that which is caused by the circulation of water in the bore-hole during the process of drilling. The so-called rods of the borers are, of course, hollow tubes, which serve a double purpose. They transmit the rotary motion, and they enable the necessary water to be pumped from the surface, down to the cutting "crown." The water then finds its way up between the tubes and the sides of the hole, so that, in general, there is circulation through the whole depth of the boring.

It is recognized that, in order to eliminate the effect, the temperature at any depth should not be taken until several times as long a period has elapsed as that through which circulation has continued at the particular level. At a point half-way down the Boreland Bore it might therefore have been necessary to wait for several years. This point will also have to be considered in discussing the temperatures found in the neighbouring Balfour Bore.

To overcome the difficulty it seems almost necessary to restrict observations to depths very near the bottom of a bore-hole, where the circulation has only been a short time in action. As already stated, boring had ceased at noon on 9th August at Boreland. It had continued that day for four hours, during which 5 ft. of strata had been penetrated. On the 8th, 2 ft. 4 in. had been pierced, and on the 7th, 6 ft. The progress is illustrated in fig. 1.

Apart from one reading which was rejected, the first observation was made with a thermometer (5782) which had remained at the bottom of the hole from 2.30 p.m. on the 10th till 2 p.m. on the 11th. The instrument was in a tube which had been specially constructed by Mr W. Boag, the foreman borer, and had perforations to ensure the free access of water, while it could be screwed on to the lowest of the boring rods. It was found necessary to lower the rods, as it was impossible to reach the bottom when the thermometer was let down in a container, by means of a winch, owing to irregularities of the hole. The attempt, indeed, led in one case to accident, and a thermometer fell about 2000 ft., but was afterwards recovered, and has not apparently been injured.

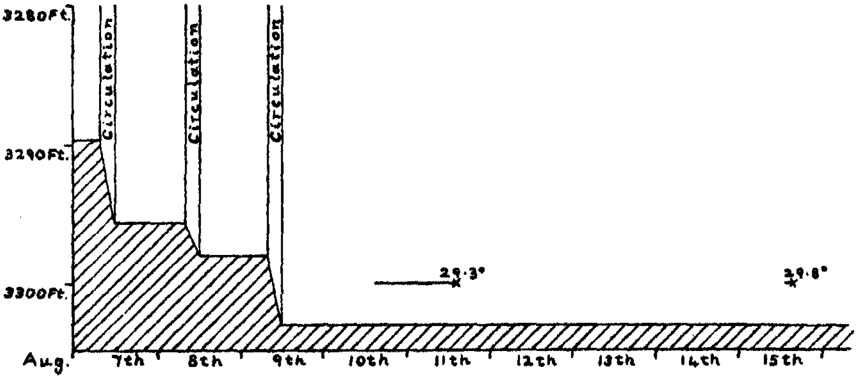

FIG. 1.—Progress of Boreland Bore, and times of observation.

The level of the thermometer on the 10th and 11th was about 3300 ft., or only 3 ft. above the bottom of the hole, unless there had been some slight accumulation of sediment. It can hardly, in any case, have been in a part of the hole where drilling had begun before the 8th. It seems probable, however, that the temperature was still somewhat influenced by the drilling water, as that registered was 29·3° C. A later observation was made at the same level, the thermometer 5782 being left at the bottom from 1 to 4 p.m. on the 15th. The reading was then 29·8°, and this may be regarded as the nearest to the undisturbed temperature of the strata, and the most reliable of the results (fig. 1).

It was thought that the two bottom readings might be somewhat influenced by the proximity of a coal seam, with possible cooling, owing to the evolution of gas. For this reason two thermometers were left at 3240 ft. from 9.10 a.m. to 11.40 a.m. on the 16th. The readings were 29·2° (5782) and 29·2° (5783), and so on the whole confirmatory of those at the lower level.

An observation was also made with two thermometers which were left at 2484 ft. from 1.40 p.m. to 4 p.m. on the 16th. The temperatures were 22·4° (5782) and 22·3° (5783), but there is slight uncertainty, as owing to the warmth at the surface it had not been possible to shake the mercury down to below 22°.

The main result of the observation was thus the determination of a temperature of 29·8° at 3300 ft., or 1006 metres. To find the flow of heat I have used this information along with the most probable value of the surface temperature. The Meteorological Office gives the mean temperature of the air, at Kirkcaldy, from 1914 to 1935, as 47·9° F. (1936). The bore is only two miles from Kirkcaldy, but at a height of 200 ft., while that of the recording station is 63 ft. Using a lapse rate of 1° F. for 300 ft., the average air temperature at Boreland must thus be about 47·4° F. To ascertain the ground temperature I have used the following data, which have been kindly supplied by the Meteorological Office. They refer to an observing station in Dunfermline at a height of 237 ft. above sea-level.

Year.	Average Air Temperature.*	Average at 1 ft. Depth.	Average at 4 ft. Depth.
	° F.	° F.	° F.
1933	48·3	49·7	49·7
1934	48·3	49·2	49·3
1935	47·1	48·7	49·2
1936	46·7	48·4	48·4
1937	47·0	48·9	49·1
1938	48·3	49·9	49·9
Six years	47·83	49·11	49·23

* The air temperatures quoted are strictly the averages of the means between the maximum and minimum recorded on each day of the year.

It may be inferred from these figures that the difference between the average temperature of the atmosphere and of the ground surface, in Fife, is about 1·6° F. Adding this to the deduced air temperature at Boreland, one obtains 49·0° F. as the most probable approximation to the mean temperature of the highest stratum in the bore.

From these data and the thermal conductivities the flow of heat may be calculated, provided the beds in the bore are inclined at only a gentle angle. The following dips were encountered:—

1 in 7 at 38 fms.; 1 in 12 at 42 fms.; 1 in 11 at 46 fms.; 1 in 6 at 228 fms.; 1 in 4 at 263 fms.; 1 in $4\frac{1}{2}$ at 265 fms.; 1 in $5\frac{1}{2}$ at 271 fms.; 1 in 8 at 313 fms.; 1 in 6 at 392 fms.; 1 in 7 at 509 fms.; 1 in 6 at 545 fms.

The highest inclination met with was thus 14°, and the average about 9°. With dips no greater than this one may apply the expression

$$\theta_2 - \theta_1 = \int_{z_1}^{z_2} \mathrm{FR}\,dz,$$

where θ_1 and θ_2 are the temperatures at depths z_1 and z_2, F is the flow of heat, regarded as variable, and R is the thermal resistance, or the reciprocal of the thermal conductivity k. If F is constant we have

$$\mathrm{F} = (\theta_2 - \theta_1) \div \int_{z_1}^{z_2} \mathrm{R}\,dz_1 = (\theta_2 - \theta_1)/\mathrm{I}, \text{ say,}$$

where I may be termed the Resistance Integral.

Benfield has determined the conductivities of a blaes and fireclay from the Boreland Bore, and of two samples of sandstone, and two of fakes, occurring at different levels. In view of what has been said about the condition of Scottish Carboniferous rocks, he used previously soaked material. The sandstone and fakes specimens gave consistent results; that for blaes had unfortunately to be rejected. It should be explained that blaes is the Scottish miners' term for shale, and fakes for a type which is intermediate in character between shale and sandstone.

In computing the Resistance Integral I have used resistances for sandstone, fakes, and fireclay which are the reciprocals of Benfield's figures, or, in the case of fakes, of the average. The three types constitute between them 77 per cent. of the whole column. For the remainder I have applied the tables published by the British Association Committee on Thermal Conductivities in the Fifth Report, taking into account a correction made in the Sixth Report (see references to literature). Only two resistances for Carboniferous shale are given by the Committee, one referring to "grey shale" and the other to "black shale." The density of the former type is 2·65, and of the latter 2·18. It is therefore clear that the figure given for black shale should only be applied to rock which is definitely bituminous. Of the 11 per cent. of blaes in the Boreland Bore, 9·7 per cent. is non-bituminous, and only 1·3 per cent. is bituminous. As there is a wide difference between the conductivities, the point is of some importance. The resistances of the other types have mainly been obtained by averaging. Only one resistance—1754—is given for ordinary or house coal. The figure for cannel coal is, however, 885, and although most of the coal met with in the bore is of the ordinary variety, I have hesitated to use the larger figure, and assigned a resistance of 1000 to both types. For miscellaneous material about which there is no information I have used the approximate figure of 250. These data may be tabulated as follows:—

	Percentage.	Thermal Resistance.
Sandstone	50·7	100
Fakes	13·7	308
Blaes, non-bituminous	9·7	333
Blaes, bituminous	1·3	602
Fireclay	12·7	228
Coal	3·2	1000
Limestone	0·3	214
Whinstone (basic igneous rock)	7·5	263
Miscellaneous	0·9	250

It is possible that the older British Association figures are not so accurate as those derived with a more modern technique by Benfield, and there are variations in type in the beds encountered in any bore, of which it is impossible to take account. But the journal of the Boreland Bore has been checked with great care by Mr W. Manson of the Geological Survey, and the integrated resistance to 3300 ft. is found to be $21{\cdot}50 \times 10^6$ c.g.s., using the above results. Taking the surface temperature to be 49·0° F., or 9·44° C., the increase of temperature in the interval is 20·36° C. This leads to a flow of heat of $0{\cdot}947 \times 10^{-6}$ calorie per square centimetre per second.

This is not, however, necessarily the surface flow, but strictly the flow averaged with regard to equal units of the resistance integral. If there is a glacial effect, the amount of heat crossing a horizontal plane will not be constant with depth, and the actual surface flow will be less than the average. The subject will be discussed more fully in a later paragraph.

Comparison may naturally be made with the results obtained at an earlier date from the deep Balfour Bore, four miles to the north of Boreland. The temperatures are fairly concordant; thus 29·8° was registered at 3300 ft. at Boreland, while by interpolation it may be gathered that about 28·5° would have been found at the same depth at Balfour. In discussing the Balfour results I at one time inferred that there was a surface flow of $1{\cdot}029 \times 10^{-6}$ cal./cm.^{2} sec. (1909). In a later publication I reduced the estimate to $0{\cdot}977 \times 10^{-6}$ (1934), and the question has now been reinvestigated by Benfield, using his own determinations of conductivities of similar strata from Boreland. These should be all the more applicable, because the two bores passed through much the same succession.

Benfield's results may be summarized as follows: He does not assume any surface temperature, but calculates certain quantities expressed as $\Sigma^D/_k$, which are the same as resistance integrals. By plotting the temperatures at different levels in the Balfour Bore against the values of the integrals, he obtains the least squares straight line represented by

$$T = (8{\cdot}87 \pm 0{\cdot}135) + (0{\cdot}867 \pm 0{\cdot}022)10^{-6}\Sigma^D/_k.$$

This implies that the average heat flow between the positions at which the temperatures were recorded is about ·867 × 10^{-6} cal./cm.^{2} sec., but it is not assumed that this is the surface flow. It is noted that the curve of temperature has an upward concavity. The reading taken at 2959 ft. is, for instance, about ·58° lower than would be obtained by drawing a straight line between those at 1783 ft. and 3955 ft. on the temperature-resistance diagram. To explain this Benfield supposes that there is a glacial after-effect, expressed by the formula

$$\beta\left(1 - \operatorname{erf}\frac{E}{2h\sqrt{t}}\right).$$

Here β is the present mean surface temperature, that during the Ice Age being supposed to have been constantly zero. E is a "fictitious depth," equal to the real depth at 3955 ft., the point of the deepest observation, and elsewhere proportional to the integrated resistance from the surface. h^2 is the diffusivity, and t the time which has elapsed since the end of the Ice Age. The total temperature T is then given by

$$T = \alpha E + \beta\left(1 - \operatorname{erf}\frac{E}{2h\sqrt{t}}\right),$$

where α is the equilibrium gradient, which would gradually establish itself throughout the crust if the surface temperature remained constant.

By "fitting" various curves of this type to the diagram, Benfield infers that $\alpha = 26{\cdot}7°$ C./km., $\beta = 10{\cdot}83°$ C., and $t = 1{\cdot}10 \times 10^4$ years. From these data it follows that the surface flow of heat is only ·68 × 10^{-6} cal./cm.^{2} sec., while the equilibrium flow is 1·16 in the same units.

It will be recalled, however, that the mean temperature of the ground at Boreland was estimated, from Meteorological Office data, to be 9·44° C. To find that at Balfour it is advisable to use the average yearly temperature at Leuchars, as well as at Kirkcaldy. The mean air temperature at Leuchars is 46·9° F. at 35 ft. over sea-level, or say 47·0° at sea-level, while the Kirkcaldy figure, reduced to sea-level, is probably 48·1°. The Balfour Bore is 6 miles from Kirkcaldy and 15 miles from Leuchars, in a nearly direct line between the localities. Allowing for the fact that the bore was sunk from 75 ft. above sea-level, the most probable value for the atmospheric temperature is 47·5° F. Adding 1·6° F. as before, the mean ground temperature should be 49·1° F., or 9·5° C.

The surface temperature inferred by Benfield is therefore almost certainly too large, and the method by which it is deduced seems open to the following objection. The measure of the curvature which has been mentioned is given by the deficit of ·58° C., at the depth of 2959 ft. An error of 0·1° in the reading at this level would make a large difference in the values of β and t. It is, however, extremely doubtful whether most of the readings can be considered accurate within 1°. The observations were made in August and September 1907, about three months after drilling had stopped. At the higher levels in the bore-hole, circulation had then continued for one or two years. The deepest measurement, at 3955 ft., might for two reasons be considered the most accurate. That point had been passed the most recently, and the diameter of the hole was there less than 1¾ in. At 2959 ft. it was 3 in., and these considerations may account for a part of the observed curvature.

The safest method to evaluate the surface flow at Balfour is, in my opinion, as follows. The surface temperature of 9·5° C., and that of 33·44° C. (92·2° F.), which was recorded at 3955 ft., will be assumed to be nearly accurate. The resistance between the two points will be integrated, and the difference of temperature will be divided by it. The quotient will be the average present flow of heat through the depth mentioned, but not the surface flow. To estimate the latter the glacial effect will be calculated, as nearly as possible, from the available chronological and physical evidence.

In integrating the resistance I have used the table given on p. 197. In drawing up this table I applied a correction mentioned in the sixth report of the Committee on Thermal Conductivities, to Benfield's figure for the conductivity of whinstone, which is taken from the fifth report. I have also distinguished between shale which is bituminous and non-bituminous, whereas Benfield's conductivity for shale is an average, taken from the final report, of the "grey shale" and "black shale" figures of the fifth report, with the required correction (p. 196). The result of the two changes has been to reduce the resistance calculated from $2\cdot774 \times 10^7$ c.g.s. to $2\cdot684 \times 10^7$.

In accordance with this I find an average flow of heat—$0\cdot892 \times 10^{-6}$ cal./cm.^{2} sec.—that is slightly larger than the $0\cdot867 \times 10^{-6}$ which is the gradient of the least squares line given by Benfield. The figures are not, however, strictly comparable, as my average applies to a greater range of depth.

To estimate the after-effect of the glacial period I shall use, in the first place, a datum provided by J. B. Simpson's work on the varve clays of the Hundred-foot Beach, near Dunning, in Perthshire (De Geer, 1935). From a correlation of varves found there, with the sequence determined by De Geer in Scandinavia, it is inferred that the ice had already retreated from the neighbourhood in 11,100 B.C. The correlation is not regarded as absolutely certain, but it agrees well enough with other estimates to make it reasonable to suppose that the ground became uncovered about 11,500 B.C. at low levels in Perthshire, and therefore probably in Fifeshire.

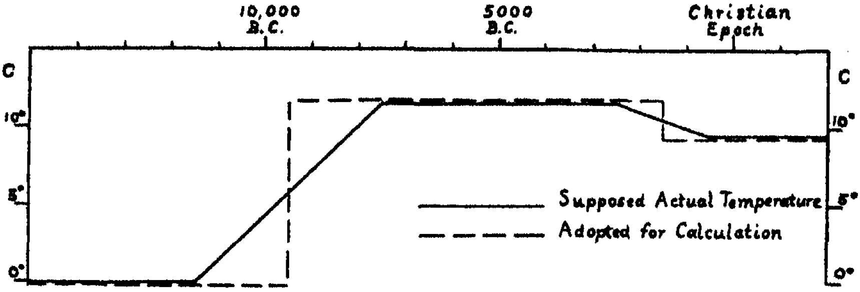

FIG. 2.—Temperatures assumed for post-glacial period.

It is probable that the temperature beneath the ice-sheet was approximately zero, and for the purpose of calculation it is assumed that the glaciation had lasted indefinitely.

The after-effect of the Ice Age must certainly have been reduced by the "climatic optimum" which followed it—that is, by the warm Boreal and Atlantic periods. It is stated by W. B. Wright that the average temperature in Scandinavia was at one time 2½° C. higher than at present (1937). The assumptions which are made for the present purpose are shown in fig. 2, but the climatic history may, of course, have been much more complicated. The further dates which are used are founded on De Geer's geochronology, as summarized by T. W. Woodhead (1928).

During the Pre-Boreal period the temperature was rising, and it is assumed that the rate of rise was constant, until the climate was about 2° warmer than at present, at the beginning of the Boreal period, about 7500 B.C. During the Boreal and Atlantic periods the temperature is taken to have been constant, but a fall set in about 2500 B.C., at the beginning of the Sub-Boreal. This continued during the whole of that period, until about 500 B.C. climatic conditions became those which have persisted until the present. As 9·5° is adopted for the present ground temperature at Balfour, that during the Boreal and Atlantic periods is taken to have been 11·5°.

Assuming a constant diffusivity, the climatic effect might be calculated so as to correspond with the temperatures shown by the firm line in the diagram. These are themselves, however, only intended to be a very rough approximation, and, to save labour, it has been supposed that the changes of temperature, instead of being gradual, were abrupt, at times half-way between the beginnings and ends of the periods in which they were effected. It is thus assumed that the ground remained at freezing-point for an indefinite time prior to 9500 B.C., or say 11,500 years ago. The surface temperature then rose to 11·5°, and it fell in 1500 B.C., or say 3500 years ago, to the 9·5° which has been maintained until the present. This supposition is shown by the dotted line, and it is probable that the results are very little modified by the simplification.

The total climatic effect is then given by

$$T = -11{\cdot}5^\circ \operatorname{erf} \frac{z}{2h\sqrt{t_1}} + 2^\circ \operatorname{erf} \frac{z}{2h\sqrt{t_2}},$$

where z is depth, h^2 the diffusivity, and t_1 and t_2 are 11,500 and 3500 years. The quantity h^2 may be written $\frac{1}{Rc\rho}$, where R is the thermal resistance, ρ the density, and c the specific heat per unit mass. The average value of R in the Balfour Bore is 219, while the table given by the Committee on Thermal Conductivities (in the 5th Report) shows that $c\rho$ for wet Carboniferous sediments has an average of about 0·55. h^2 is therefore about 0·0085, and by reducing t_1 and t_2 to seconds one obtains the expression

$$T = -11{\cdot}5^\circ \operatorname{erf} \frac{z}{1{\cdot}089} + 2^\circ \operatorname{erf} \frac{z}{0{\cdot}601},$$

where z is the depth in kilometres.

This curve is shown in fig. 3, and it is clear that the effect of the warm Boreal and Atlantic periods is to counteract the concavity which would otherwise occur in its shallower part, so that down to about 700 metres the result is nearly a straight line. The gradient at the surface is −8·16° per kilometre. The average slope of the curve between the surface and 3955 ft., or 1·206 km., is −6·76° per kilometre. It therefore follows that, in rock of the average diffusivity, the surface gradient must be about 1·40° per km. less than the average which has been found to a depth of 3955 ft. This must be allowed for in estimating the surface flow, and it reduces the $0{\cdot}892 \times 10^{-6}$ cal./cm.^{2} sec. to $0{\cdot}829 \times 10^{-6}$. This figure is, however, considerably larger than that found by Benfield, which was 0·68 in the same units.

The equilibrium flow must, however, be larger than the average flow by an amount corresponding to a gradient of 6·76° per kilometre, and it is thus found to be $1 \cdot 196 \times 10^{-6}$ cal./cm.^{2} sec. This is slightly larger than Benfield's figure of 1·16 in the same units, but the results are, of course, regarded in both cases as only approximate, as it has been necessary to assume that the rock sequence has uniform diffusivity, and this is contrary to fact. Apart from this the long continuation of drilling makes even the deepest temperature a little doubtful. The values deduced from the Boreland Bore may on this account be expected to be more accurate.

I have applied the same process to the Boreland Bore, allowing for

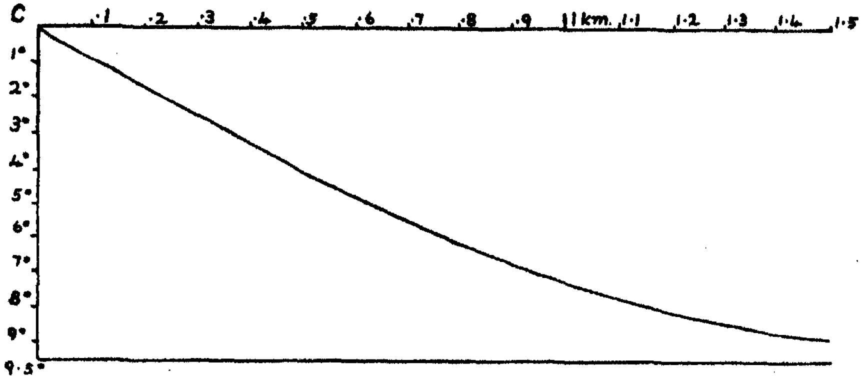

Fig. 3.—Additive effect of recent climatic changes on temperature curve. (Diffusivity = ·0082.)

the inferred slight differences of thermal resistance and of surface temperature. I find a surface flow of $0 \cdot 910 \times 10^{-6}$ cal./cm.^{2} sec., and an equilibrium flow of $1 \cdot 282 \times 10^{-6}$.

I have also investigated the two bores near Glasgow, which have been discussed by Benfield, and particulars about which are given by the British Association Committee on Underground Temperatures (2nd Report). The Blythswood Bore No. 1 is registered as Renfrew 8 NE No. 7 in the Geological Survey collection. Its position is 55° 53′ 2″ N., 4° 23′ 52″ W., and it begins less than 50 ft. above sea-level. I have been unable to identify the South Balgray Bore, but have used the particulars given by the Committee. Using the table on p. 197, I have integrated the resistance between the points of all the observations in the two cases, and applied the method of the least squares straight line adopted by Benfield. As, however, I have assumed the blaes to be non-bituminous, and blaes is present in great amount, I get smaller values for the integrals. As a consequence I find larger average and surface flows, and on applying the climatic correction as above indicated I also get larger equilibrium flows. The Blythswood Bore has not, however, been checked by the Geological Survey, and in the other case the data are somewhat meagre. If even 20 per cent. of the blaes is bituminous, Benfield's assumption is the more nearly correct. Both sets of figures are therefore given, with initials, in the following table, which summarizes the results inferred from temperature observations in Scotland:—

	Average Flow in Range Investigated.	Surface Flow.	Equilibrium Flow
	(In Units of 10^{-6} cal./cm.^{2} sec.)		
Boreland Bore	0·947	0·910	1·282
Balfour Bore	0·892	0·829	1·196
Blythswood Bore (A. E. B.) .	1·24	Same as averages	1·56
(E. M. A.) .	1·408		1·746
South Balgray Bore (A. E. B.) .	1·53	Same as averages	1·85
(E. M. A.) .	1·730		2·066

If allowance is made for the probable reduction of temperature, by circulation of water, in the Balfour Bore, the two results from Fifeshire more or less confirm one another. They show at least that there is not a wide variation in the flow of heat within a distance of four miles. The Glasgow results are less consistent, but they seem to indicate that there is a real difference in thermal conditions between the two parts of Scotland.

The variability which has been inferred for Scotland appears to extend into England. A high value of the heat flow is shown by the observations which were made in an underground bore at South Hetton, near Durham, and were recorded by the Committee on Underground Temperatures (5th Report). The bore was put down from the bottom of a shaft which began in Magnesian Limestone, and it reached the Hutton Seam of the Durham Coal Measures. The journals of the shaft and bore have been preserved by the North of England Institute of Mining and Mechanical Engineers (1894). For an interpretation of the mining terms which are used I am indebted to Mr R. G. Carruthers. In calculating the resistance integral I have used the table on p. 197, where it was applicable, except in two cases. There is reason to suppose that the Carboniferous part of the succession is here in a dry condition, although this does not apply to the overlying Permian. The data given by the Committee on Thermal Conductivities (5th Report) show that the resistance of dry sandstone is about 8 per cent. greater than that of the rock in a state of saturation. I have therefore increased the figure of 100 to 108, and I have increased the resistance of fakes by 4 per cent. to 320. For Magnesian Limestone I have adopted a resistance of 171, which is the mean of two (5th Report, with correction in 6th Report). Limestone marl, occurring in the Permian, has been assumed to have a resistance of 250.

The Meteorological Office give the mean air temperature at Durham as 46·7° F. at 336 ft. above sea-level. The shaft was sunk from a height of 420 ft. Allowing for this, and adding 1·6° F. to the air temperature, as in Scotland, the mean ground temperature is 48·0° F., or 8·89° C.

Most of the temperatures observed in the bore were probably fairly accurate, as they were taken long after the completion of the bore. The deepest reliable reading was at 1736 ft., where the thermometer case was pushed into an accumulation of sediment, which would entirely hinder any convection. The difference between this reading and the surface temperature is 16·18° C., while the integrated resistance to the depth in question is found to be $11{\cdot}02 \times 10^6$ c.g.s. The average flow throughout the depth investigated is found from these data to be $1{\cdot}468 \times 10^{-6}$ cal./cm.^{2} sec. If allowance is made for the glacial effect in the same way as before, the surface flow which results is $1{\cdot}466 \times 10^{-6}$ and the equilibrium flow is $1{\cdot}816 \times 10^{-6}$. The thermal condition of the crust in Durham is therefore more nearly similar to that in the neighbourhood of Glasgow than to that in Fifeshire.

The flow of heat in Lancashire may be inferred from the temperatures which were recorded during the sinking of the shaft of Rose Bridge Colliery, Wigan (Committee on Underground Temperatures, 3rd Report). The section of the shaft was in this case given by the Committee, and I am indebted to Mr F. W. Cope, of the Geological Survey, for the interpretation of the mining terms, and other particulars. The mean temperature of the air at Southport is 48·7° F. at a height of 35 ft. above sea-level. That at Bolton is 48·3° at 342 ft., while the Rose Bridge shaft was sunk from about 200 ft. Allowing for the differences of height, and for the fact that Wigan is about one-third of the distance from Bolton, in the Southport direction, the average air temperature near the shaft should be 48·5°. Adding 1·6° as before, the ground temperature may be taken to be 50·1° F., or 10·06° C. The deepest temperature taken was 94° F., or 34·44° C., at 2445 ft., while the resistance integral to this depth is found to be $23{\cdot}48 \times 10^6$. The average flow of heat inferred is therefore $1{\cdot}038 \times 10^{-6}$ cal./cm.^{2} sec., while the surface flow may be evaluated as $1{\cdot}011 \times 10^{-6}$ and the equilibrium flow as $1{\cdot}341 \times 10^{-6}$.

The section dealt with lies almost entirely in Middle Coal Measures, and it is interesting to contrast it with the Carboniferous part of the section near Durham, which constitutes 79 per cent. of the whole there present. At Wigan only 20 per cent. of the column consists of sandstone, 55 per cent. is shale, and 11 per cent. "linn and wool," possibly the local equivalent of what is "fakes" in Scotland. At Durham the whole of the Coal Measures and a part of the Millstone Grit may be represented in the range investigated. Of the Carboniferous sequence 60 per cent. is sandstone, while apparently about 14 per cent. is fakes, and only about 20 per cent. can be interpreted as shale. The difference in the apparent flow of heat cannot, however, be attributed to this dissimilarity, as, if the resistance used for shale needs revision, it is more likely that it should be increased than diminished, and this would lead to a greater disparity of the results. The temperatures measured near Wigan may be too low, as they were obtained by boring only 3 ft. from the base of the shaft during the process of sinking, and it is recorded that water was used in the drilling. It is, however, improbable that so large a difference as that between 1·038 and 1·468 units can be due to this cause. This would imply that the temperature found at 2445 ft. at Wigan was about 9° C. too small. The difference must be real, and the results obtained at Wigan may be compared with those deduced by Benfield from the boring at Holford, near Northwich (1939). The localities are about 18 miles apart, and for comparison it may be noted that Durham is about 95 miles from Wigan. The surface flow inferred at Holford is $0{\cdot}74 \times 10^{-6}$ cal./cm.^{2} sec., but as a large part of the thickness penetrated consists of rock salt, which has very high conductivity, there must be a large glacial effect. This has been taken into account by Benfield, who finds the equilibrium flow to be $1{\cdot}43 \times 10^{-6}$, not very different from the $1{\cdot}341 \times 10^{-6}$ which has been inferred for Wigan.

The two sets of observations are thus more or less confirmatory of one another, and they appear to show that there is a smaller flow of heat in Lancashire and Cheshire than in the county of Durham. It is probable that the smaller flow is the more usual, as will appear when attention has been paid to results obtained in other parts of England.

One may refer, in the first place, to the experimental bore-hole which was sunk to a depth of 100 ft. by the *present* Committee on Thermal Conductivities in a field near the Observatory at Cambridge (1938). There were irregularities in the gradient, believed to be due to the circulation of water in parts of the Gault clay penetrated, and on this account the Committee has hitherto refrained from publishing any estimate of an indicated amount of heat flow. There can be little doubt, however, that the loss of heat is similar in amount to the low values obtained in Lancashire and Cheshire rather than to that in Durham.

The Committee on Underground Temperatures published observations of temperature in a bore sunk from an old well in Kentish Town, in London (2nd Report). The deepest determination was at 1100 ft., and I have previously published a figure of $1 \cdot 28 \times 10^{-6}$ cal./cm.^{2} sec., deduced from this reading. This was intended for a surface and not for an equilibrium flow, but there is even greater uncertainty about the resistances which should be employed in this instance than there is in the Carboniferous. A much larger resistance than that of 269, which was used, might be applied to chalk, and, partly for this reason, I now think that the surface flow arrived at was too large, and that the equilibrium flow, at this locality, may perhaps lie between $1 \cdot 1 \times 10^{-6}$ and $1 \cdot 3 \times 10^{-6}$.

The only other particular estimate which has been made in Britain is that derived by Benfield from the temperatures and conductivities which he measured in a bore at Hankham, near Eastbourne (1939). The surface flow is found to be $0 \cdot 71 \times 10^{-6}$ cal./cm.^{2} sec. and the equilibrium flow is $1 \cdot 12 \times 10^{-6}$.

It may thus be inferred that the low loss of heat, by conduction, found in Lancashire and Cheshire is the result of a condition which extends from these counties to the south coast of England. There is fair reason to believe, however, that the flow of heat is not laterally constant, even if allowance is made for the after-effects of climatic changes, which themselves vary, more or less, as the square root of the diffusivity. It is possible that there is an area of enhanced flow extending north-westwards from Durham to Renfrewshire and the neighbourhood of Glasgow. Both to the north-east and to the south of this area the loss by conduction is smaller. Further observations in bore-holes may confirm the difference, or possibly disprove it, but they should be made under critical conditions. Further determinations of conductivity are also very necessary, particularly as regards the different varieties of shale and coal. At present it can hardly be said with confidence that most of the results arrived at are accurate within about 10 per cent.

It may be noted that an area extending from Renfrewshire to Durham is very nearly coincident with the south-eastern extension of the "Mull Swarm." This complex of Tertiary dykes is among the latest products of igneous activity in Britain. Some members of the swarm are shown in fig. 4, which indicates the localities which have been referred to in the text. The connection may be pure coincidence, but need not be rejected in the present state of investigation.

A possible value of the equilibrium heat flow was recently taken by Dr W. Q. Kennedy and myself to be 2×10^{-6} cal./cm.^{2} sec. (1934, 1938). This estimate was only intended to be provisional, and it did not refer to any special part of the earth's surface. We found, however, that the assumption made it possible to explain the facts of vulcanicity in a way that followed, in one respect, the lines of Professor J. Joly. Other attempts at explanation appear to be somewhat unsatisfactory. A general reduction of the value would involve vulcanologists in the previously existing difficulties, and the subject is therefore one of considerable importance.

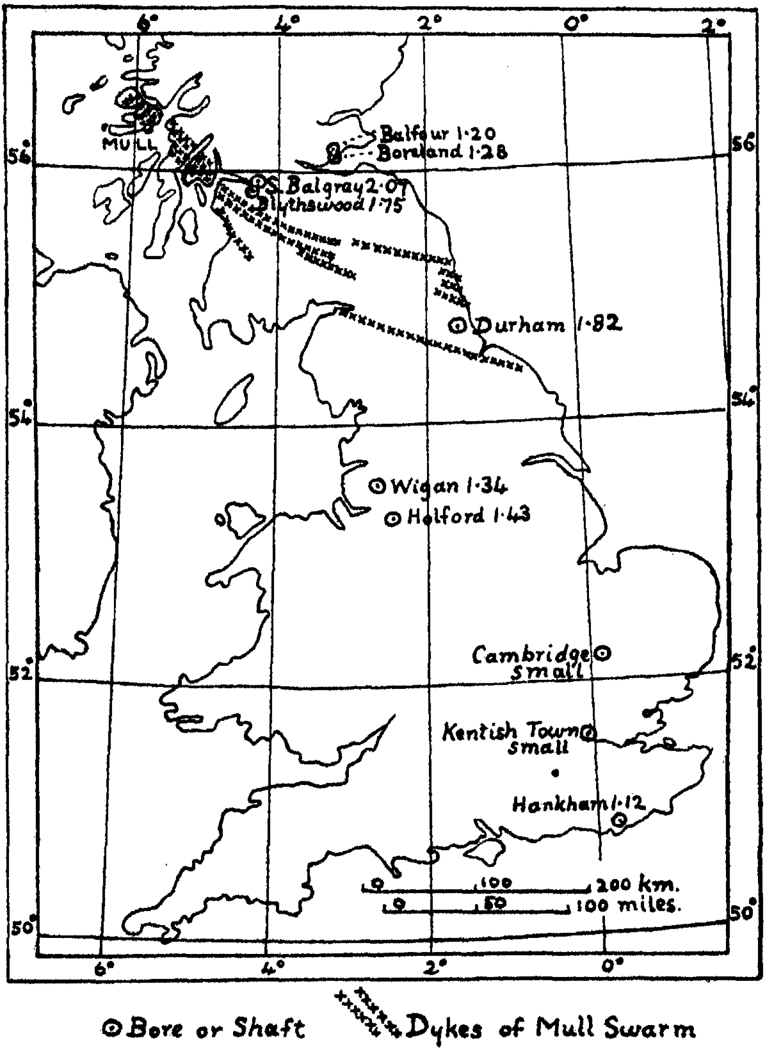

FIG. 4.—Equilibrium heat flow in Britain, in 10^{-6} cal./cm.^{2} sec. units.

The heat flow in South Africa has recently been very thoroughly investigated by E. C. Bullard (1939). The average value found is

$1{\cdot}16 \times 10^{-6}$ cal./cm.^{2} sec., and this is based on numerous measured conductivities from some of the bores considered. The data are not believed to point to the existence of any glacial effect. The effect might have been expected to be a large one owing to the high conductivities, but it seems quite impossible that the equilibrium heat flow, in most of South Africa, can be as great as 2×10^{-6}.

It is admitted that this statement must also apply to a large part of Britain, as has been shown by Benfield (1939). The facts, however, seem at present to point to a certain degree of lateral variability, and until this is further investigated the theory of igneous action, put forward by Kennedy and myself, may still be held.

The equilibrium flows at the localities which have been mentioned are shown in the following table. The average may be compared with Benfield's mean value of $(1{\cdot}424 \pm 0{\cdot}091)10^{-6}$ cal./cm.^{2} sec. If, however, the flow is variable, a question may arise as to the meaning of the average. The description "small" implies only that the value is small compared to those occurring at Durham and near Glasgow.

	Equilibrium Flow. (In Units of 10^{-6} cal./cm.^{2} sec.)
Balfour Bore	1·20
Boreland Bore	1·28
Blythswood Bore	1·75
South Balgray Bore	2·07
South Hetton Shaft and Bore, Durham	1·82
Rose Bridge Colliery Shaft, Wigan	1·34
Holford Bore	1·43
Cambridge Experimental Bore	Small
Kentish Town Well and Bore	Small
Hankham Bore	1·12
Average of 8	$(1{\cdot}500 \pm 0{\cdot}083)10^{-6}$ cal./cm.^{2} sec.

The thermometers which were used in the Boreland Bore (5782, 5783) had kindly been lent by the Department of Geodesy and Geophysics of Cambridge University. The facilities for the investigation which were given by the Fife Coal Company have already been mentioned, and I wish also to express my indebtedness to the boring contractor, Mr Andrew Kyle. Thanks are also due to Mr W. Boag, the foreman borer, and to Mr W. Manson of the Geological Survey, who assisted in the selection of specimens for thermal determination, and whose accurate checking of the bore journal has probably added to the reliability of the result.

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(*Issued separately July* 18, 1940.)

XVII.—The Energy Levels of a Rotating Vibrator. By **Ian Sandeman,** D.Sc., late Carnegie Research Scholar in the University of St Andrews.

(MS. received May 22, 1940. Read July 1, 1940.)

THE energy levels of a rotating vibrator have been calculated by the late J. L. Dunham (1932). The theory of the rotating vibrator is important, because it provides a method for direct quantitative study of molecular structure from the band spectra of diatomic molecules. The purpose of this paper is to extend Dunham's results and to express them in a form more convenient for numerical calculation.

The Schrödinger equation for the rotating-vibrator molecular model is

$$\frac{d^2\psi}{d\rho^2}+\frac{1}{B_e}\left\{F-U(\rho)-\frac{B_eK(K+1)}{\rho^2}\right\}\psi=0, \quad . \quad . \quad . \quad (1)$$

where ρ is the ratio of the internuclear distance to its equilibrium value, i.e. r/r_e, and B_e is a contraction for $h/8\pi^2\mu r_e^2c$ with the usual meaning of these symbols. $U(\rho)$ is the variable part of the oscillation energy expressed in wave-number units.

Dunham puts $\rho=1+\xi$, and expands U as a power series about $\xi=0$:

$$U=a_0\xi^2(1+a_1\xi+a_2\xi^2+a_3\xi^3+\ . \ . \ .). \quad . \quad . \quad . \quad (2)$$

He then proceeds to solve equation (1) by the Wentzel-Brillouin-Kramers method, and obtains an expression for the energy as a doubly infinite power series in the quantum numbers v and K of the form

$$F=\sum_{ij}Y_{ij}(v+\tfrac{1}{2})^iK^j(K+1)^j. \quad . \quad . \quad . \quad . \quad (3)$$

The coefficients Y_{ij} become rapidly smaller, as i and j rise in value, and therefore only the Y_{ij} with low values of i and j can be determined experimentally with accuracy. The theoretical values of the Y_{ij} are expressions involving B_e, a_0, and the configuration constants a_1, a_2, a_3, . . . of the potential expansion (2). As the highest suffix for the a_n occurring in the theoretical expression for any Y_{ij} is given by $n=2i+j-2$, it is possible to extract the values of these constants seriatim from the experimental Y_{ij} with low values of i and j. Hence, in certain circumstances to be explained below, the a_n with the lowest values of n can be determined with accuracy. This fact explains why the theory is of practical importance: it gives a method of working back from the spectrum to the molecular constants. The light which it throws on molecular potentials is of special interest, because we have no direct way of studying these functions mathematically except in the simplest molecular case, viz. that of the diatomic molecular ion.

As the whole usefulness of Dunham's theoretical work depends on the validity of his form of the potential expression (2), it has to be considered how far a molecular model with this form of potential expression can be expected to represent molecular behaviour. Two criticisms of this type of potential expression inevitably arise:

(1) Even if U is expressible near $\xi=0$ by such an expression, it does not necessarily follow that the series will converge over the whole range covered by the vibrational motion.

(2) Since U = const. for $r \to \infty$, a model in which U is represented by a power series is not necessarily the most suitable approximation to use.

As Dunham's analysis is based on the assumption that the potential expression has a single minimum, consideration can be confined to such expressions. We can rule out the use of any potential expression, for example, containing a term $C/(c \pm \xi)$, where c is of the order of magnitude which ξ assumes, as the expression would have a singularity at the point $\xi = \mp c$. Confining consideration to potential expressions with single minima is equivalent to selecting the stable vibrational states, but this limitation does not remove the objections raised above.

In order to justify the employment of Dunham's method in face of these objections, it is convenient to consider some known approximate potential functions which have been used to represent molecular behaviour. The potential function of Morse (1929), which may be written

$$U = (a_0/a^2)(1 - e^{-a\xi})^2, \qquad . \quad . \quad . \quad . \quad (4)$$

is one of great simplicity which follows the broad lines of expected molecular behaviour. The value of U which is positive for all values of ξ drops to zero at $\xi=0$ $(r=r_e)$, and, when ξ is positive, rises tangentially to its dissociation value a_0/a^2, as $\xi \to \infty$ $(r \to \infty)$. This expression is known to represent the behaviour of certain molecules extraordinarily well in spite of its lack of flexibility in containing only two arbitrary constants.

By expanding equation (4) we obtain

$$U = a_0\xi^2(1 - a\xi + 0{\cdot}583a^2\xi^2 - 0{\cdot}250a^3\xi^3 + 0{\cdot}086a^4\xi^4 - \;.\;.\;.). \qquad . \quad (5)$$

This expansion is convergent for all finite values of a and ξ, as it has been deliberately so constructed.

Another very simple form of potential function which has been employed by spectroscopic workers is that of Kratzer (1922) which approximates to the form

$$U = a_0\xi^2/(1+\xi)^2. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

The solution of equation (1) for this function was first given by Fues (1926). In this case U is infinite for $\xi = -1$ ($r=0$), *i.e.* when the vibrating nuclei approach infinitely near one another. U falls to zero at the point $\xi=0$, and tends tangentially to the value a_0, when $\xi \rightarrow \infty$. The function (6), while fitting a few molecular states, is inferior to that of Morse, as might be anticipated from the fact that it contains only one arbitrary constant.

By expanding equation (6) we obtain

$$U = a_0\xi^2(1 - 2\xi + 3\xi^2 - 4\xi^3 + 5\xi^4 - \ . \ . \ .). \quad . \quad . \quad . \qquad (7)$$

Expansion (7) is again convergent for all the values which ξ assumes, because it has been deliberately so constructed.

The potential expression of Dunham, equation (2), viz.

$$U = a_0\xi^2(1 - a_1\xi + a_2\xi^2 - a_3\xi^3 + a_4\xi^4 - \ . \ . \ .),$$

is only "general" in the sense that it may stand for any of the expressions such as (5) or (7) by suitable choice of the constants a_1, a_2, a_3, . . . Expansion in power series provides a convenient way of comparing known functions. The only advantage that can be claimed for Dunham's expression is flexibility.

The possibility which has to be faced is that, when the actual coefficients of Dunham's expansion (2) are calculated, the expreſsion obtained may not be convergent. This is essentially a matter which can be put to experimental test. Applying Gauss's test for convergency to expansion (2) we must have $\bar{\xi}(a_{n+1}/a_n) < 1$, where $\bar{\xi}$ represents the maximum value which ξ can assume during the motion. When the structure constants of the potential expression for any band state are calculated, it is generally found that these increase slightly in value with n, thus a_{n+1}/a_n generally exceeds unity, as is the case in equation (7). The value of $\bar{\xi}$ is, however, generally small.

Some idea of the magnitude of $\bar{\xi}$ may be obtained from the approximate relation:

$$\bar{\xi}^2 \doteqdot (v+\tfrac{1}{2})x_e, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

where v is the vibration quantum number and x_e is a contraction for

$\sqrt{B_e/a_0}$. (In terms of the usual constants of band theory $u_e = 2B_e/\omega_e$, where ω_e is the frequency in wave-numbers of a vibration of infinitesimal amplitude. Equation (8) can be obtained without much difficulty from any convenient form of the potential function, U. As U is by supposition measured in wave-numbers, the potential in ergs is hcU. We therefore have, by considering the force acting on either nucleus, the relation $\mu\ddot{r} = -hc(dU/dr)$. By integrating this equation twice we obtain an expression for the period, and hence for the frequency, of an anharmonic oscillation, which, when compared with the vibrational part of equation (3), gives the result of equation (8).)

For most of the hydrogen states u_e is about 0·027; this value is higher than for the band states of any other substance. For some band states u_e is extremely small. For the states of the K_2 molecule, for example, according to the data given by Jevons (1932) it is about 0·0006.

To take an unfavourable hypothetical case, let us assume $u_e = 0{\cdot}03$, *i.e.* a greater value than for any known hydrogen state, and $v = 14$, the highest value of the vibrational quantum number which has been observed in hydrogen by Beutler (1934). We then have

$$\xi^2 \doteqdot 14{\cdot}5 \times 0{\cdot}03,$$
$$\doteqdot 0{\cdot}435.$$

Whence

$$\xi \doteqdot 0{\cdot}66.$$

The writer considers that, in the band states of hydrogen to which he has applied Dunham's analysis, expansion (2) is valid, although in some it is not as rapidly convergent as it should be to give very good results.

Besides the simple potential functions (4) and (6) there are other simple functions which have been employed by spectroscopic workers. Among these is Davidson's (1932) function:

$$U = (a_0/4)\left(1 + \xi - \frac{1}{1+\xi}\right)^2. \quad . \quad . \quad . \quad . \quad (9)$$

This, like expression (6), contains only one arbitrary constant and lends itself to direct solution of equation (1). Morse's function (4) contains two arbitrary constants, and lends itself to a simple solution of equation (1) only when $K = 0$.

By the addition of a third constant, as for example in the functions of Rosen and Morse and of Pöschl and Teller, the suitability of which have been tested by Lotmar (1935), we introduce greater elasticity, and so can obtain a closer approximation to molecular behaviour. It is evident that trial functions can be evolved based on any mathematical functions of ξ which, when expanded as power series in ξ, do not contain the first power.

For the purpose of representing molecular behaviour all the other functions which have been mentioned possess the advantage over Dunham's expression (2) that they do not demand a condition of convergency. On the other hand, all these simple functions are insufficiently flexible to represent molecular behaviour in any more than its broad outlines. Given satisfactory convergency, Dunham's expression becomes the most suitable, because it possesses the advantage of flexibility to any desired extent. In discarding the other expressions in favour of Dunham's, we put aside expressions which cannot from their very nature represent molecular behaviour with precision, and adopt one which may or may not be capable of precise representation according to whether it turns out to be sufficiently convergent or not for the molecular states to which it is applied.

If we wish to study the form of the molecular potential function from spectral data for diatomic molecules, Dunham's expression (2) gives the only analytic method conveniently available for carrying such theoretical study beyond the stage of a first rough outline. This conclusion justifies employment of the method as an instrument of exploratory research, but does not justify the assumption that it will work in every case.

A limitation to the utility of Dunham's method, and indeed of any other similar method, is that it implies the assumption that it is possible to represent the electronic structure by a function $U(\xi)$ independent of the rotational-vibrational motion of the molecule. Although the assumption is known to be a good first approximation, it is not self-evident that it is worth carrying Dunham's analysis to any high stage of accuracy. The effect of interaction between the electronic and vibrational-rotational motions may well be that, while the potential has the form of equation (2) at the point $\xi = 0$, it may change to a different form, say,

$$U = \epsilon_0 + \epsilon_1\xi + a_0'\xi^2(1 + a_1'\xi + a_2'\xi^2 + \ldots), \quad . \quad . \quad . \quad (10)$$

at some other point, say $\xi = \xi'$. The whole validity of the assumption depends on the supposition that at any point within the range of values which ξ assumes the potential differs very slightly from equation (2).

This limitation lies at the root of all attempts so far made at a quantitative analytical study of band spectra. The question may be asked: In view of the doubtful validity of the assumption that the vibration-rotation potential is independent of electronic structure, is it worth while extending Dunham's results to the accuracy here contemplated? In face of this question it is necessary to state very clearly that the writer has in fact not *extended* Dunham's analysis, if "extended" means carried to terms of lower magnitude. He has merely rounded off Dunham's analysis by adding further vibrational and vibrational-rotational coefficients in order to give all the coefficients of the spectral-term expansion (3) which are of equal magnitude to that of the small rotational coefficient Y_{03} already given by Dunham. Dunham gives a still smaller rotational coefficient, Y_{04}.

The position is that, while the theoretical work published by Dunham carries the rotational analysis a little further than is necessary for the study of most band states, the vibrational analysis is not carried nearly far enough. This is because the vibrational coefficients contain the higher configuration constants. For example, the pure-rotational coefficient Y_{03}, calculated to a first approximation, contains only the structure constant a_1, whereas the pure-vibrational coefficient of the same magnitude, Y_{60}, contains structure constants up to a_{10}. It may be said that it is *unsafe* to assume that interaction between the electronic and vibrational-rotational motions will be so small as to justify taking so many of the structure constants into account, but this is scarcely a fair way of stating the matter, since it suggests that the precautions essential to a mathematical investigation have been so far relaxed as to admit the possibility of a misleading result. This is not the case.

We have to admit two possibilities: (1) that the interaction between electronic and vibrational-rotational motions is so small as scarcely to affect the values of the structure constants, and (2) that the interaction is large enough to affect all these constants, the effect being progressively greater as we pass from a_1 towards a_{10}. It must be remembered that the manifold of coefficients Y_{lj} has a self-testing property. This is because there are more coefficients than constants. If the constants are calculated from one group of coefficients, their validity can be tested by substitution in the theoretical expressions for the remaining group. If the results of this substitution are inconsistent with observation, the whole method must be rejected. Thus, if for any band state Dunham's method fails for reason (2) above, or for any other reason, we reject it as unsuitable for the band state in question. The value of the method must rest on experiment, and no absolute *a priori* validity is, or can be, claimed for it.

Another reason for caution in the use of Dunham's analysis arises from the fact that the form of the constant of integration corresponding to the rotation quantum number of older theory has been taken as $K(K+1)$. This expression, while perhaps adequate for Σ states, is probably not so for $\Pi, \Delta, \ldots$ states. Davidson (1932) has remarked that the effect of changing the form of this constant of integration would be to alter all the structure constants of the potential expansion by factors of the form $1+cu_e^2$. (Davidson uses the Greek letter κ instead of u_e.) As modifications can be introduced into the theory to allow of deviations from Dunham's model, no attempt has been made to generalise equations (1) and (2), which are in the form in which they were employed by him.

A practical objection to Dunham's theoretical work is that it involves awkward expressions for the Y_{ij} which do not lend themselves to rapid calculation. This difficulty is inherent in the use of a potential function with a large number of arbitrary constants, but a certain simplification is possible, as will be shown, by the use of another set of constants instead of $a_1, a_2, a_3, \ldots$ of expression (2).

It is convenient to split the Y_{ij} of equation (3) into components:

$$Y_{ij}=x_{ij}+y_{ij}+z_{ij}+\ldots \qquad (11)$$

Here x_{ij} represents the major part of the coefficient Y_{ij}. The various x_{ij} in fact correspond to the band constants which have been known for a considerable period from the quantum-mechanical treatment of the anharmonic-oscillator molecular model with a potential expression analogous to (2), first carried out by Kratzer (1922). For example, $x_{10}=\omega_e$, $x_{01}=B_e$, $x_{11}=(B_e^2/\omega_e)6(1+a_1)$. The y_{ij} are correction terms introduced by Dunham's analysis. Each y_{ij} is smaller than the corresponding x_{ij} in the ratio $u_e^2 : 1$. Perhaps the most important of the y_{ij} for general theory is $y_{00}=(B_e/8)(3a_2-7a_1^2/4)$, which has thrown some light on the spacing between isotopic states. The z_{ij} are second correction terms, the ratio in magnitude of the three components being $x_{ij} : y_{ij} : z_{ij} :: 1 : u_e^2 : u_e^4$. Dunham does not give the second correction terms, although he gives a method by which they may be calculated. Their inclusion is necessary to make clear the implications of his theory and to ensure that all coefficients are calculated to the same order of magnitude.

The addition of yet a third correction term, say t_{ij}, is unnecessary, as all the t_{ij}, except t_{00} which is not required for ordinary band theory, would be of smaller magnitude than the small rotational coefficient Y_{03} (or x_{03}) which has been taken as typical of the order of magnitude to which it is assumed to be necessary to work.

It may be remarked that, if we confine ourselves to the notation of equations (1) and (2), we should express our results in terms of B_e, a_0, and the configuration constants $a_1, a_2, a_3, \ldots$ of the potential function. a_0 has, however, never been popular with spectroscopic workers, who use $\omega_e=2(a_0B_e)^{\frac{1}{2}}$ instead, because ω_e has a definite physical meaning, viz. the classical frequency in wave-numbers of an infinitely small vibration. The writer prefers to make a further change in notation, and to use ω_e, u_e, and the configuration constants. The advantage of this is that, as u_e is a very small quantity, the power of u_e entering into any term gives a measure of the magnitude of the term.

If, for example, we assume that the highest power of u_e which need be taken into account is the fifth power, and use the simple potential function of Kratzer (6) as the U in equation (1), we find without much difficulty that the x_{lj} come out as in the following table:—

	$j=0.$	1.	2.	3.
$l=0$	..	$\frac{1}{2}\omega_e u_e$	$-\frac{1}{2}\omega_e u_e^3$	$\frac{1}{2}\omega_e u_e^5$
1	ω_e	$-\frac{3}{2}\omega_e u_e^2$	$\frac{15}{8}\omega_e u_e^4$	
2	$-\frac{3}{2}\omega_e u_e$	$3\omega_e u_e^3$	$-\frac{9}{2}\omega_e u_e^5$	
3	$2\omega_e u_e^2$	$-5\omega_e u_e^4$		
4	$-\frac{5}{2}\omega_e u_e^3$	$\frac{15}{2}\omega_e u_e^5$		
5	$3\omega_e u_e^4$			
6	$-\frac{7}{2}\omega_e u_e^5.$			

The y_{lj} come out as follows:—

	$j=0.$	1.	2.
$l=0$	$\frac{1}{8}\omega_e u_e$	$-\frac{1}{4}\omega_e u_e^3$	$\frac{3}{8}\omega_e u_e^5$
1	$-\frac{3}{8}\omega_e u_e^2$	$\frac{15}{16}\omega_e u_e^4$	
2	$\frac{9}{4}\omega_e u_e^3$	$-\frac{9}{4}\omega_e u_e^5$	
3	$-\frac{5}{4}\omega_e u_e^4$		
4	$\frac{15}{8}\omega_e u_e^5.$		

And the z_{lj} come out as

	$j=0.$	1.
$l=0$	$-\frac{1}{32}\omega_e u_e^3$	$\frac{3}{32}\omega_e u_e^5$
1	$\frac{15}{128}\omega_e u_e^4$	
2	$-\frac{9}{32}\omega_e u_e^5.$	

It can be seen from inspection of these tables that in this simple case we can make the y_{lj} and z_{lj} vanish by substituting $(K+\frac{1}{2})^2$ in place of $K(K+1)$ in our term-formula (3).

If we solve equation (1) using the general potential function (2) as U, we find that the magnitude of each term is the same as in the above tables, but instead of simple fractional numerical coefficients we have expressions containing the structure constants $a_1, a_2, a_3, \ldots$

We can invert the potential equation (2), and write it in the form

$$\xi = \left(\frac{U}{a_0}\right)^{\frac{1}{2}}\left\{1 + c_1\left(\frac{U}{a_0}\right)^{\frac{1}{2}} + c_2\frac{U}{a_0} + c_3\left(\frac{U}{a_0}\right)^{\frac{3}{2}} + \ldots\right\}. \quad . \quad . \quad (12)$$

We can then achieve a slight simplification by expressing our solution of equation (1) in terms of $c_1, c_2, c_3, \ldots$ instead of $a_1, a_2, a_3, \ldots$ After the values of $c_1, c_2, c_3, \ldots$ have been found from spectral data, we can then evaluate $a_1, a_2, a_3, \ldots$ from the conversion formulæ

$$a_1 = -2c_1,$$

$$a_2 = -2c_2 + 5c_1^2,$$

$$a_3 = -2c_3 + 12c_1c_2 - 14c_1^3,$$

$$a_4 = -2c_4 + 7(2c_1c_3 + c_2^2) - 56c_1^2c_2 + 42c_1^4,$$

$$a_5 = -2c_5 + 16(c_1c_4 + c_2c_3) - 72(c_1^2c_3 + c_1c_2^2) + 240c_1^3c_2 - 132c_1^5,$$

$$\begin{aligned} a_6 = {} & -2c_6 + 9(2c_1c_5 + 2c_2c_4 + c_3^2) - 30(6c_1c_2c_3 + 3c_1^2c_4 + c_2^3) + 165(2c_1^3c_3 + 3c_1^2c_2^2) \\ & - 990c_1^4c_2 + 429c_1^6, \end{aligned}$$

$$\begin{aligned} a_7 = {} & -2c_7 + 20(c_1c_6 + c_2c_5 + c_3c_4) - 110(2c_1c_2c_4 + c_1^2c_5 + c_2^2c_3 + c_1c_3^2) + 440(3c_1^2c_2c_3 \\ & + c_1^3c_4 + c_1c_2^3) - 1430(c_1^4c_3 + 2c_1^3c_2^2) + 4004c_1^5c_2 - 1430c_1^7, \end{aligned}$$

$$\begin{aligned} a_8 = {} & -2c_8 + 11(2c_1c_7 + 2c_2c_6 + 2c_3c_5 + c_4^2) - 132(2c_1c_2c_5 + 2c_1c_3c_4 + c_1^2c_6 + c_2^2c_4 \\ & + c_2c_3^2) + 143(12c_1^2c_2c_4 + 12c_1c_2^2c_3 + 4c_1^3c_5 + 6c_1^2c_3^2 + c_2^4) - 2002(4c_1^3c_2c_3 \\ & + c_1^4c_4 + 2c_1^2c_2^3) + 3003(2c_1^5c_3 + 5c_1^4c_2^2) - 16016c_1^6c_2 + 4862c_1^8. \quad . \quad . \quad (13) \end{aligned}$$

It may be mentioned that the simple function of Kratzer (6) corresponds to the case where $c_1, c_2, c_3, \ldots$ are all unity*—a fact which helps to check the accuracy of any expression for the x_{ij}, y_{ij}, or z_{ij}, because by equating all the c_n to unity we should find that our expressions reduce to those given in the preceding tables. Davidson's potential function (9) corresponds to the case where all the c_n for which n is even vanish and those for which n is odd correspond to the numerical coefficients of the expansion of $(1+x)^{\frac{1}{2}}$, viz. $c_1 = \frac{1}{2}$, $c_3 = -\frac{1}{8}$, $c_5 = \frac{1}{16}$, . . . Morse's potential function (4) corresponds to the case, $c_1 = a/2$, $c_2 = a^2/3$, . . ., $c_n = a^n/(n+1)$.

Expressed in terms of the c_n, the x_{ij} come out as follows:—

$$x_{10} = \omega_e,$$

$$x_{20} = -\omega_e x_e \tfrac{3}{2} c_2,$$

$$x_{30} = -\omega_e x_e^2 \tfrac{1}{2}(5c_4 - 9c_2^2),$$

$$x_{40} = -\omega_e u_e^3 \tfrac{5}{8}(7c_6 - 30c_2c_4 + 27c_2^3),$$

$$x_{50} = -\omega_e u_e^4 \tfrac{3}{8}(21c_8 - 105c_2c_6 - 50c_4^2 + 315c_2^2c_4 - 189c_2^4),$$

$$x_{60} = -\omega_e u_e^5 \tfrac{7}{16}(33c_{10} - 189c_2c_8 - 175c_4c_6 + 630c_2^2c_6 + 600c_2c_4^2 - 1620c_2^3c_4 + 729c_2^5),$$

$$x_{11} = \omega_e u_e^2 \tfrac{3}{2}(1 - 2c_1),$$

$$x_{21} = \omega_e u_e^3 \tfrac{3}{4}(5 - 20c_1 + 6c_2 - 10c_3 + 15c_1^2 + 8c_1c_2),$$

$$\begin{aligned} x_{31} = \omega_e u_e^4 \tfrac{5}{4}(&7 - 42c_1 + 17c_2 - 28c_3 + 9c_4 - 14c_5 + 70c_1^2 - 12c_1c_2 + 42c_1c_3 + 10c_1c_4 \\ &- 6c_2^2 + 22c_2c_3 - 28c_1^3 - 33c_1^2c_2 - 18c_1c_2^2), \end{aligned}$$

$$\begin{aligned} x_{41} = \omega_e u_e^5 \tfrac{15}{16}(&21 - 168c_1 + 77c_2 - 126c_3 + 55c_4 - 84c_5 + 28c_6 - 42c_7 + 441c_1^2 \\ &- 210c_1c_2 + 420c_1c_3 - 52c_1c_4 + 126c_1c_5 + 28c_1c_6 - 5c_2^2 + 28c_2c_3 - 54c_2c_4 \\ &+ 98c_2c_5 + 63c_3^2 + 58c_3c_4 - 420c_1^3 - 70c_1^2c_2 - 252c_1^2c_3 - 87c_1^2c_4 + 48c_1c_2^2 \\ &- 294c_1c_2c_3 - 120c_1c_2c_4 + 27c_2^3 - 130c_2^2c_3 + 105c_1^4 + 196c_1^3c_2 + 195c_1^2c_2^2 \\ &+ 108c_1c_2^3), \end{aligned}$$

$$x_{12} = -\omega_e u_e^4 \tfrac{3}{2}(\tfrac{19}{4} - 9c_1 + 4c_2 - c_1^2),$$

$$\begin{aligned} x_{22} = -\omega_e u_e^5 \tfrac{3}{4}(&65 - 250c_1 + 122c_2 - 60c_3 + 30c_4 + 190c_1^2 - 108c_1c_2 - 20c_1c_3 - c_2^2 \\ &+ 30c_1^3 + 8c_1^2c_2), \end{aligned}$$

$$x_{01} = \tfrac{1}{2}\omega_e u_e,$$

$$x_{02} = -\tfrac{1}{2}\omega_e u_e^3,$$

$$x_{03} = \tfrac{1}{2}\omega_e u_e^5(3 - 2c_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

As Dunham has shown, we can obtain a first approximation to the values of the constants by assuming each Y_{ij} to be sufficiently well represented by its major part x_{ij} and by using the experimental values of the Y_{ij} for which j is 0 and 1 to solve for the constants. ω_e and u_e are found from x_{10} and x_{01}, c_1 from x_{11}, c_2 from x_{20}, and so on. We can thus determine from the x_{i0} and x_{i1} approximate values of all the constants up to c_{10} except c_9, the value of which we can usually guess by graphing the c_n. We cannot in any case hope to discover the values of c_9 and c_{10} with any very great numerical accuracy from any normal spectral measurements.

The simplicity which the set of formulæ (14) possesses is that the numerical coefficients of the c_n do not contain fractions. By expressing results in this way we also somewhat reduce the number of terms: for example, the expression for x_{41} which contains 33 terms would, if expressed in Dunham's manner in terms of the a_n, contain 45 terms. This reduction in the number of terms affects only the x_{ij} and not the y_{ij} and z_{ij}, but, as it is sufficient for a rough analysis of most spectra to take only the x_{ij}

into account, this simplification occurs at the point where it is most useful.

If a better approximation to the values of the constants is required, the correction terms y_{ij} must be taken into account. We do this by accepting the approximate values of the constants obtained from the x_{ij} and calculating each y_{ij} from these. More correct values of the x_{ij} are then obtained from the experimental Y_{ij}, since we have

$$x_{ij} \doteq Y_{ij} - y_{ij}. \qquad (15)$$

We then re-calculate the constants from the new x_{ij}.

The theoretical values of the y_{ij} are as follows. As these coefficients may not often be required, to avoid the setting down of cumbersome numerical coefficients, y_{30} and y_{40} have been expressed in terms of $d_1, d_2, d_3, \ldots$ instead of $c_1, c_2, c_3, \ldots$, where we have put $c_1 = d_1/2$, $c_2 = d_2/3, \ldots, c_n = d_n/(n+1)$:

$$y_{00} = -\omega_e u_e \tfrac{1}{8}(3c_2 - 4c_1^2),$$

$$y_{10} = -\omega_e u_e^2 \tfrac{1}{8}(25c_4 - 80c_1c_3 - 54c_2^2 + 192c_1^2c_2 - 80c_1^4),$$

$$\begin{aligned} y_{20} = -\omega_e u_e^3 \tfrac{1}{16}(&245c_6 - 840c_1c_5 - 1320c_2c_4 - 560c_3^2 + 5760c_1c_2c_3 + 2160c_1^2c_4 \\ &+ 1431c_2^3 - 4480c_1^3c_3 - 9288c_1^2c_2^2 + 9120c_1^4c_2 - 2240c_1^6), \end{aligned}$$

$$\begin{aligned} y_{30} = -\omega_e u_e^4 \tfrac{5}{16}(&21d_8 - 42d_1d_7 - 57d_2d_6 - 42d_3d_5 - 25d_4^2 + 154d_1d_2d_5 + 134d_1d_3d_4 \\ &+ 64d_1^2d_6 + 102d_2^2d_4 + 77d_2d_3^2 - 308d_1^2d_2d_4 - 346d_1d_2^2d_3 - 84d_1^3d_5 \\ &- 126d_1^2d_3^2 - 41d_2^4 + 476d_1^3d_2d_3 + 109d_1^4d_4 + 310d_1^2d_2^3 - 126d_1^5d_3 \\ &- 390d_1^4d_2^2 + 161d_1^6d_2 - 21d_1^8), \end{aligned}$$

$$\begin{aligned} y_{40} = -\omega_e u_e^5 \tfrac{5}{64}(&231d_{10} - 462d_1d_9 - 679d_2d_8 - 462d_3d_7 - 557d_4d_6 - 231d_5^2 \\ &+ 1806d_1d_2d_7 + 1436d_1d_3d_6 + 1526d_1d_4d_5 + 1806d_2d_3d_5 + 700d_1^2d_8 \\ &+ 1258d_2^2d_6 + 763d_3^2d_4 + 1109d_2d_4^2 - 7344d_1d_2d_3d_4 - 3492d_1^2d_2d_6 \\ &- 2772d_1^2d_3d_5 - 4242d_1d_2^2d_5 - 924d_1^3d_7 - 1950d_2^3d_4 - 924d_1d_3^3 \\ &- 1602d_1^2d_4^2 - 2121d_2^2d_3^2 + 5460d_1^3d_2d_5 + 4900d_1^3d_3d_4 + 7840d_1d_2^3d_3 \\ &+ 10386d_1^2d_2^2d_4 + 8190d_1^2d_2d_3^2 + 1180d_1^4d_6 + 588d_2^5 - 8390d_1^4d_2d_4 \\ &- 18480d_1^3d_2^2d_3 - 1386d_1^5d_5 - 3465d_1^4d_3^2 - 6307d_1^2d_2^4 + 10962d_1^5d_2d_3 \\ &+ 1687d_1^6d_4 + 11795d_1^4d_2^3 - 1848d_1^7d_3 - 8043d_1^6d_2^2 + 2289d_1^8d_2 - 231d_1^{10}), \end{aligned}$$

$$y_{01} = \omega_e u_e^3 \tfrac{1}{16}(15 - 28c_1 + 18c_2 - 30c_3 - 3c_1^2 + 88c_1c_2 - 64c_1^3),$$

$$\begin{aligned} y_{11} = \omega_e u_e^4 \tfrac{1}{16}(&175 - 570c_1 + 335c_2 - 380c_3 + 225c_4 - 350c_5 + 310c_1^2 + 444c_1c_2 \\ &- 150c_1c_3 + 1210c_1c_4 - 285c_2^2 + 1370c_2c_3 - 380c_1^3 + 57c_1^2c_2 - 3040c_1^2c_3 \\ &- 3132c_1c_2^2 + 240c_1^4 + 6016c_1^3c_2 - 2080c_1^5), \end{aligned}$$

$$y_{21}=\omega_e u_e^5 \tfrac{3}{32}(735-3640c_1+2065c_2-2170c_3+1475c_4-1820c_5+980c_6-1470c_7$$
$$+4795c_1^2-1650c_1c_2+1900c_1c_3+3180c_1c_4-630c_1c_5+5460c_1c_6$$
$$-895c_2^2+4780c_2c_3-3510c_2c_4+6930c_2c_5-1155c_3^2+7410c_3c_4$$
$$-2380c_1^3-1710c_1^2c_2-9140c_1^2c_3-555c_1^2c_4-14560c_1^2c_5-8352c_1c_2^2$$
$$+5130c_1c_2c_3-35360c_1c_2c_4-19200c_1c_3^2+2943c_2^3-21270c_2^2c_3$$
$$+2075c_1^4+17700c_1^3c_2+5280c_1^3c_3+32320c_1^3c_4-3135c_1^2c_2^2+118400c_1^2c_2c_3$$
$$+43740c_1c_2^3-7040c_1^5-10560c_1^4c_2-66240c_1^4c_3-150240c_1^3c_2^2$$
$$+5280c_1^6+120960c_1^5c_2-26880c_1^7),$$

$$y_{02}=-\omega_e u_e^5 \tfrac{1}{16}(163-398c_1+238c_2-180c_3+90c_4+98c_1^2+252c_1c_2-220c_1c_3$$
$$-131c_2^2-198c_1^3+472c_1^2c_2-192c_1^4). \quad . \quad . \quad . \quad . \quad . \quad (16)$$

For very accurate work we may have to take the z_{ij} into account. The values of these are

$$z_{00}=-\omega_e u_e^3 \tfrac{1}{281}(315c_6-1680c_1c_5-1890c_2c_4-864c_3^2+11520c_1c_2c_3+5600c_1^2c_4$$
$$+2187c_2^3-14080c_1^3c_3-22032c_1^2c_2^2+30528c_1^4c_2-9600c_1^6),$$

$$z_{10}=-\omega_e u_e^4 \tfrac{1}{128}(623d_8-2100d_1d_7-1935d_2d_6-1652d_3d_5-848d_4^2+7980d_1d_2d_5$$
$$+7340d_1d_3d_4+4480d_1^2d_6+3819d_2^2d_4+3414d_2d_3^2-20696d_1^2d_2d_4$$
$$-19200d_1d_2^2d_3-7560d_1^3d_5-9720d_1^2d_3^2-1658d_2^4+41960d_1^3d_2d_3$$
$$+11338d_1^4d_4+20596d_1^2d_2^3-15768d_1^5d_3-38298d_1^4d_2^2+21470d_1^6d_2$$
$$-3585d_1^8),$$

$$z_{20}=-\omega_e u_e^5 \tfrac{3}{256}(3073d_{10}-10780d_1d_9-10437d_2d_8-8764d_3d_7-8655d_4d_6$$
$$-4046d_5^2+44660d_1d_2d_7+39480d_1d_3d_6+37940d_1d_4d_5+37492d_2d_3d_5$$
$$+23800d_1^2d_8+21750d_2^2d_6+17274d_3^2d_4+19254d_2d_4^2-205200d_1d_2d_3d_4$$
$$-117960d_1^2d_2d_6-105560d_1^2d_3d_5-111300d_1d_2^2d_5-41720d_1^3d_7$$
$$-37096d_2^3d_4-31920d_1d_3^3-52180d_1^2d_4^2-50034d_2^2d_3^2+242200d_1^3d_2d_5$$
$$+223800d_1^3d_3d_4+226080d_1d_2^3d_3+349500d_1^2d_2^2d_4+327300d_1^2d_2d_3^2$$
$$+64440d_1^4d_6+12111d_2^5-435140d_1^4d_2d_4-842000d_1^3d_2^2d_3-91308d_1^5d_5$$
$$-205310d_1^4d_3^2-216070d_1^2d_2^4+712412d_1^5d_2d_3+123170d_1^6d_4$$
$$+603130d_1^4d_2^3-160440d_1^7d_3-566670d_1^6d_2^2+210310d_1^8d_2-26586d_1^{10}),$$

$$z_{01}=\omega_e u_e^5 \tfrac{1}{256}(2835-9240c_1+6615c_2-6642c_3+4725c_4-4620c_5+3780c_6$$
$$-5670c_7+4655c_1^2+5850c_1c_2-4740c_1c_3+12740c_1c_4-13230c_1c_5$$
$$+30660c_1c_6-4995c_2^2+14292c_2c_3-17010c_2c_4+34230c_2c_5-7047c_3^2$$
$$+33966c_3c_4-5820c_1^3+5230c_1^2c_2-28772c_1^2c_3+33915c_1^2c_4$$
$$-105280c_1^2c_5-26064c_1c_2^2+71262c_1c_2c_3-223640c_1c_2c_4-112128c_1c_3^2$$
$$+15633c_2^3-120942c_2^2c_3+2015c_1^4+48492c_1^3c_2-65088c_1^3c_3$$
$$+280960c_1^3c_4-110235c_1^2c_2^2+884480c_1^2c_2c_3+309636c_1c_2^3-16640c_1^5$$
$$+117888c_1^4c_2-643200c_1^4c_3-1333056c_1^3c_2^2-29760c_1^6+1315584c_1^5c_2$$
$$-345600c_1^7). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Here again, to shorten the numerical coefficients, z_{10} and z_{20} have been expressed in terms of d_1, d_2, d_3, . . . where $c_n = d_n/(n+1)$. For the analysis of many spectra these small correction terms will scarcely affect results. We must, however, make sure that, with the constants determined, the z_{ij} are too small to have appreciable effect. As the numerical coefficients become larger, as we pass from the y's to the z's, in some particular case we may find the z's to be inconveniently large. In such a case we have to abandon the attempt to obtain very accurate values of the constants by the method.

It may be noted that all the x_{ij}, y_{ij}, and z_{ij} for which j is zero can be simplified numerically by replacing the c_n by d_n. This is not, however, the case for those coefficients for which j is not zero. As each of the x_{i0}, y_{i0}, and z_{i0} is made up of terms of equal weight (*i.e.* terms for which the sum of the suffixes is constant), if we express these coefficients in terms of the d_n, and set each d_n equal to unity, we obtain the numerical parts of the coefficients for the case when U is Morse's function, and all these coefficients should vanish except x_{10} and x_{20}. This provides another way in which the accuracy of the theoretical expressions for the components of the Y_{i0} can be checked.

It is sometimes thought that a simplification could be introduced by expressing the integration constant corresponding to the rotation quantum number in terms of $(K+\frac{1}{2})^2$. To do this we should have to replace $K(K+1)$ in equation (3) by $(K+\frac{1}{2})^2-\frac{1}{4}$. The effect would merely be to make the theoretical expressions more complicated. Further terms would be introduced into the y_{i0} and z_{i0} which are limited to terms of equal weight, and the consequent complication of these coefficients would not be outweighed by any corresponding advantage.

SUMMARY.

The theoretical work of the late J. L. Dunham based on the rotating vibrator for the diatomic molecule is amplified and simplified. While the simplifications introduced are necessarily not very far-reaching, they mean an appreciable gain in the rapidity with which the molecular constants can be calculated by Dunham's method from spectral data.

I am indebted to Professor H. Stanley Allen for encouragement and helpful criticisms.

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XVIII.—On the Histogenesis of some Teleost Pituitaries.

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INTRODUCTION.

THE importance of the pituitary in relation to many body processes has long been recognised, and may be due either to its direct influence on the body or to its control over other endocrine organs, in particular the thyroids and gonads. It is evident that various hormones are elaborated in the gland, and it appears likely that they originate in particular regions or even in particular cell types. It is of interest therefore to ascertain at what period in the development of the animal these hormones first appear. This problem can be approached from two directions, experimental and histological. In the former case glands may be removed or implanted at various stages in the course of development and the extent to which normal functioning is disturbed noted, whilst in the latter case development may be followed in the normal gland so that histological changes may be correlated with changes elsewhere. Both these methods have been applied in Mammals, Birds, and Amphibia, but little has yet been done in relation to Fishes. In the Anura, experiments have shown that the elaboration of its hormone by the thyroid is definitely dependent on the presence of the anterior lobe of the pituitary. A recent study of pituitary histogenesis in the Frog and Toad (Kerr, 1939) showed that the anterior lobe acidophiles appeared at approximately the same time as colloid in the vesicles of the thyroid, whilst the appearance of the anterior lobe basophiles was long delayed until the definitive gonads began to form. In the present work a similar study has been made of some Teleost pituitaries, and although no attempt has been made to describe in detail the parallel development of the thyroids and gonads their condition has been observed at the various stages to determine whether any correlation with pituitary differentiation might be suggested. The Teleost pituitary differs greatly from those of other Vertebrates in the arrangement and histology of its parts. The histological sequence during development will show the differentiation of these parts and in consequence may throw some light on their homologies with the components of the pituitaries of higher types.

The Trout was selected as the most easily obtainable of the relatively primitive Teleosts (Norman, 1931), and for comparison a series of developing Perch were examined together with a few stages of Minnow, Stickleback, Plaice, and Flounder. The young Trout emerges from its tough egg-membrane relatively large in size and well supplied with yolk, whereas the others are minute with little yolk. The differences between these two types of early development therefore should bring out any basic similarities there may be in the development of the pituitaries. Stendell's (1914) division of the glandular component of the pituitary into anterior, transitional, and intermediate lobes has been retained, since no more satisfactory system has so far been suggested.

The only previous work along the present lines is by Bock (1928), who very briefly described the histological conditions in the early pituitaries of the Stickleback (*Gasterosteus aculeatus*). He found no secretory differentiation in the glands of newly hatched (5 mm.) fish and the lobes could hardly be defined, in two-day-old (6·5–7 mm.) fish some differentiation is apparent, and by three to four days old (8 mm.) the first acidophile cells have appeared in the anterior and transitional lobes. At about the end of the second week (14 mm.) the pituitaries possess all the histological elements of the adult. The main accounts of pituitary development from a purely morphological point of view are given by Haller (1896) for the Trout (*Salmo fario* and *S. irideus*) and by Matthews (1937) for *Fundulus*. In the latter Matthews found that no differentiation of secretory cells had occurred some days after hatching.

MATERIALS AND METHODS.

Trout (*Salmo trutta* L.) eggs were kept in aërated tanks in the laboratory, and a series of stages at close intervals was fixed from about a fortnight before until about a month after hatching, and then at intervals up to six months old. Duplicate groups of all important stages were fixed at the hatchery, to make sure that development was not being affected by unnatural conditions, also yearling and fully mature fish. A series of Perch (*Perca fluviatilis* L.) stages were preserved, and a few stages of developing Minnow (*Phoxinus laevis* Ag.), Stickleback (*Gasterosteus aculeatus* L.), Plaice (*Pleuronectes platessa* L.), and Flounder (*Pleuronectes flesus* L.).

The routine technique was fixation in corrosive-formol (using neutralised formaldehyde), sectioning at 5 μ, and staining with Mallory's (1936) modified anilin-blue stain. Other series were fixed in other ways, particularly in Susa, to give effective comparison with the results of workers using acid fixatives on the glands of adult fish. A number of series were stained in Azan. It is noteworthy that strongly acid fixatives have a very deleterious effect on the staining of basophiles, especially in early stages; decalcification has a similar but more pronounced effect.

The designation of different stages according to length or age alone gives quite divergent results; together, however, these standards give a sufficiently accurate indication of the relationship between the stages.

OBSERVATIONS.

The first appearance of the pituitary in all the Teleosts so far described consists of a solid ectodermal inpushing of the buccal epithelium (Haller, 1896; Matthews, 1937), which extends below the brain and becomes stalked. In early stages this component shows no cellular differentiation and does not at any time possess a hypophyseal cavity. The process is similar in the early stages of all the species examined here.

DEVELOPMENT IN THE TROUT.

The series of stages in the Trout is more complete and hence treated in more detail. It soon became apparent that the development of the thyroid and gonads did not suggest the same degree of correlation with the differentiation of the pituitary in their early stages noticeable in the Frog and Toad. Consequently their development has been dealt with separately.

A. *The Pituitary.*

Stage 1.—The first stage to be described is about a week before hatching and 8 mm. in length (fig. 1). The hypophyseal stalk is normally present, though it may disappear at any time from now until just after hatching, but in any case its presence is associated with an embryonal condition of the gland. The pituitary has a lens-shaped ectodermal element whose lower surface is connected mid-ventrally by the forwardly inclined stalk to the ectoderm of the mouth, with its upper surface pressed against the thin floor of the infundibulum. There is a flat and continuous layer of connective tissue between this element and the infundibulum, and no nervous lobe has yet appeared. There is no differentiation into lobes of the ectodermal element and no special grouping of the cells. The peripheral cells, however, tend to pack closer and are frequently elongated at right angles to the periphery, whereas the medullary cells are grouped irregularly and often loosely. In some series there are already present rounded cavities in the region of the future anterior lobe. They can be distinguished by their thin lining of connective tissue, and are not connected with each other to form a system of tubes or with the vascular system. Their mode of origin is not clear. In later stages anterior lobe cells are arranged around them in a characteristic way giving the impression that the cavities have arisen after the orientation of the cells, but early stages show that they may exist with little or no orientation of the cells around them. Their number and size vary greatly; sometimes they are quite large and may then give a superficial impression of representing the remains of a hypophyseal cavity. Blood-vessels are established immediately anterior and posterior to the gland, but no vascularisation of the organ itself has so far commenced.

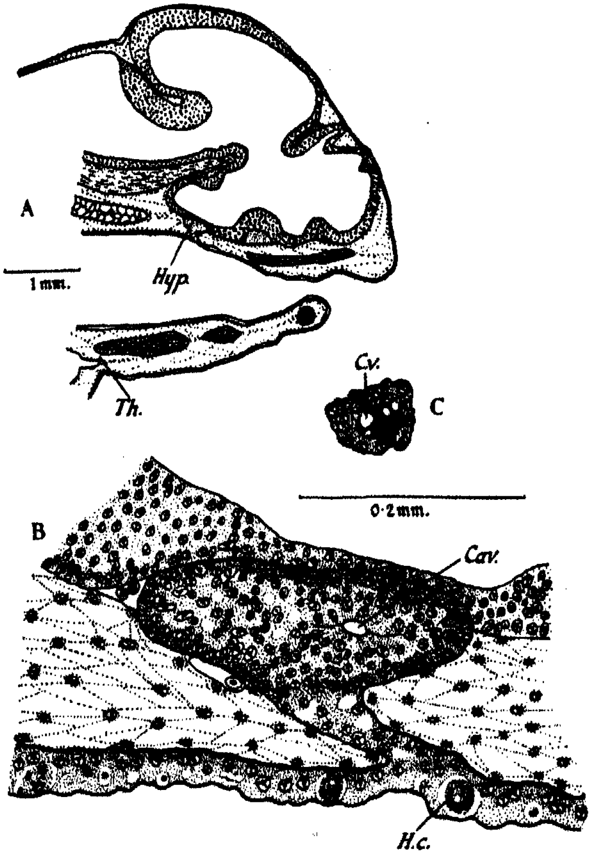

FIG. 1.—L.S. Trout head at hatching.

A, complete head; B, pituitary region enlarged; C, thyroid vesicle. *Cav.*, cavity in anterior lobe; *C.v.*, clear vacuole in colloid; *H.c.*, "hatching cell"; *Hyp.*, hypophysis; *Th.*, position of thyroid. Head end to the right in this and other text-figures.

In the skin of the mouth and head region in stages before hatching are curious large cells with basophilic contents (fig. 1). They disintegrate and disappear with hatching, and the mucous cells of the skin, present before hatching in an inactive state, gradually replace them. They may be the "hatching cells" reported by Armstrong (1936) in *Fundulus*. They are frequent in the stalk and ectodermal element.

Stage 2.—The pituitaries of fish fixed immediately upon hatching (12–14 mm.) are still of an embryonal order. The stalk and the cavities of the anterior region vary in their presence or otherwise, and any "hatching cells" in the ectodermal element are degenerating. No colour reactions in the cytoplasm are yet established nor any but the first traces of a nervous component. The oral membrane has ruptured at a much earlier stage, and despite the amount of yolk still carried at hatching the young fish begin to feed almost at once.

Stage 3.—Changes in the pituitary commence immediately after hatching and in three-day-old fish (16–18 mm.) have become marked. The nervous lobe becomes established as a gradually thickening layer of tissue of a fibrous appearance which invades the glandular component from its infundibular surface, particularly in the posterior region. It is the continuation of this downgrowth mainly which causes the change in shape of the flattened pituitary of early stages into the conical organ of the adult. At this time the process is in its first stages and is marked by the formation of small finger-shaped processes projecting into the middle of the glandular component (the future transitional lobe) and by a much larger regular-walled process into the posterior region (the future intermediate lobe).

The first sign of differentiation in the glandular component is in the future anterior lobe. Cavities in this region have appeared irregularly at earlier stages, and around some the cells may become orientated without any differential colour reaction. With hatching, the orientation of these cells becomes more clearly marked, they become elongated towards the cavities in a characteristic manner, retained in later stages (fig. 2), and their cytoplasm stains brown. The cavities vary from two or three quite large ones to a dozen small ones, but in no case with granular or other contents.

In the transitional and intermediate lobes no differentiation has appeared. The layer of connective tissue between the nervous and glandular components is still quite continuous and the relative sizes of the lobes remains embryonal.

Stage 4.—When five days old (18–20 mm.) the young Trout are feeding actively and the yolk is diminishing. A great divergence in size becomes obvious even in fish from the same batch of eggs. The picture presented by the pituitary differs mainly in the appearance of definite acidophiles in the transitional lobe. As these acidophiles appear there occur also the first ruptures in the connective-tissue lamina separating the nervous and glandular components, but although it is in the transitional lobe region that these ruptures occur there is no definite evidence for associating the two phenomena.

During ensuing development the differentiation of the pituitary continues along the lines already established. The nervous lobe enlarges rapidly, particularly in its posterior downgrowth into the intermediate lobe, and the latter remains relatively small and stretched out into an approximately homogeneous layer around the downgrowth. The acidophiles of the transitional lobe increase in number and in the brightness of their staining and extend in an ill-defined band from one side of the pituitary to the other, concentrating particularly towards the boundary with the nervous lobe. The acidophiles of the anterior lobe become a reddish brown and more precise in outline, the cavities round which they are grouped tend to become more numerous, and the lobe as a whole remains relatively bulky. No acidophiles of the anterior or transitional lobes have ever been noticed passing through the connective-tissue boundary into the nervous lobe.

The acidophiles of these two lobes can be distinguished at once, either by their orientation or by their coloration. Those of the transitional lobe are browner in early stages than later, but from the beginning are brighter and redder than those of the anterior lobe. They take mainly the acid fuchsin of the Mallory stain and are typically a bright red. The cells are oval or rounded in shape with the nucleus usually towards one side; they are frequently noticeably large and without any particular orientation. The anterior lobe acidophiles take mainly the orange G component of the stain in early stages and are brown; later they take both and are brown-red. They are typically very elongated cells with the nucleus towards the outer end and the cytoplasm directed towards a cavity. No discharge of their secretion into the cavities, however, is ever apparent.

Stage 5.—At two to three weeks after hatching (about 24 mm.) the yolk-sac has more than half disappeared. The nervous lobe (fig. 2) has increased in size; it possesses the beginnings of a good blood-supply, partly as capillaries on the connective-tissue boundary and partly as larger vessels in the substance of the lobe. The intermediate lobe consists of loose and homogeneous tissue in which darker staining spindle-shaped cells occur. These particular cells disappear later and are replaced by an adult type, so they are not considered to represent differentiation. The boundary with the nervous lobe is smooth and the connective-tissue layer unbroken, but anteriorly the tissue of the lobe passes insensibly into that of the transitional lobe. The two lobes contrast fairly clearly at low magnifications owing to a general difference of colour, but there is no connective-tissue boundary between them. It is normal to find what appear to be transitional lobe acidophiles isolated in the substance of the intermediate lobe (fig. 2). In the transitional lobe the acidophiles are numerous and bright, and occupy a relatively great longitudinal area compared with later stages; they show a slight tendency to concentrate dorsally. As before, the connective-tissue boundary is broken in an irregular manner by the finger-like projections of the nervous lobe.

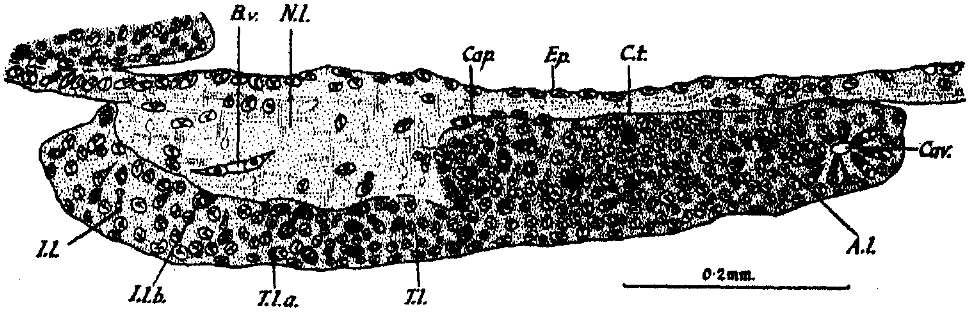

FIG. 2.—L.S. pituitary of Trout two to three weeks old and about 24 mm. in length. *A.l.*, anterior lobe; *B.v.*, blood-vessel; *Cap.*, capillary; *Cav.*, anterior lobe cavity with acidophiles orientated around it; *C.t.*, connective-tissue boundary layer; *Ep.*, ependyma layer; *I.l.*, intermediate lobe; *I.l.b.*, intermediate lobe basophile; *N.l.*, nervous lobe; *T.l.*, transitional lobe; *T.l.a.*, transitional lobe acidophile.

In the anterior lobe the brown acidophiles are orientated around a variable number of cavities, which lie in otherwise homogeneous tissue. Throughout the glandular component of the pituitary, and particularly in its posterior regions, there is a marked range of nuclear variations, though mitotic figures are rare. Typically the nuclei are round or slightly oval and vesicular, but pointed and kidney-shaped specimens are common. Pycnotic nuclei also exist, sometimes closely applied to vesicular nuclei, and such may resemble basophiles with dense blue cytoplasm. There identification is possible by Azan, however, and no true basophiles are present at this stage.

Stage 6.—At four to six weeks (about 28 mm.) differentiation has proceeded with little change. The downgrowth of the nervous lobe into the intermediate is still rounded (fig. 3), but may extend as two blunt horns directed posteriorly and laterally. The projections into the transitional lobe may pass almost through the gland to the ventral surface, but most are short and not arranged in any regular manner. Breaks in the connective-tissue layer between the two lobes do not become extensive. Blood-vessels are common in the substance of the nervous lobe and all along its boundary with the glandular lobes, but not in the latter themselves.

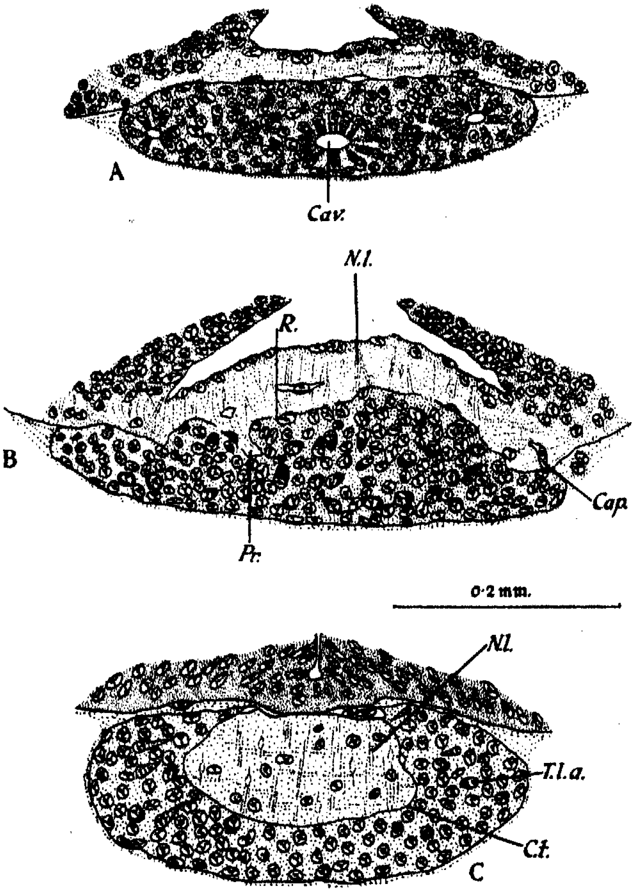

FIG. 3.—T.S. pituitary of Trout four to six weeks old and about 28 mm. in length.

A, anterior lobe level; B, transitional lobe level; C, intermediate lobe level; *Pr.*, nervous lobe inpushing into transitional lobe; *R.*, break in connective-tissue boundary layer. Other lettering as before.

The floor of the infundibular cavity is flat, with no hypophyseal recess as yet, and now, as at all other stages, the nervous processes are completely solid. The ependyma lining the infundibular surface is an ill-defined layer with oval vesicular nuclei not arranged in any precise way. Isolated nuclei with little or no visible cytoplasm are scattered commonly throughout the nervous lobe. Their origin is doubtful, but it would appear that they are derived from brain rather than glandular cells, since they are present before the rupture of the boundary connective tissue and never show any concentration towards the glandular side of the nervous lobe. These nuclei, moreover, show no general sign of degeneration. Distinct amounts of the secretory material of Herring (1908, 1913) are present from this stage.

In the intermediate lobe differentiation has begun with the cells showing a dull basophile reaction. A few cells, however, are amphiphilic; they stain purplish with Mallory, and with Azan show red granules in the blue cytoplasm. Occasional acidophiles identical with those of the transitional lobe are present. In the transitional lobe itself the acidophiles stretch right across the gland, becoming rarer towards the edges. The lobe is relatively more elongated than it will later become, but the distribution of its acidophiles is already foreshadowing the two broad horns which will pass upwards round the ends of the nervous lobe towards the dorsal surface. Occasionally there are present strongly acidophilic inclusions, perhaps representing the "secretory spheres" of Charipper (1937). No basophiles have appeared, but there is a fairly distinct zone of non-reacting cells between the acidophilic areas of the transitional and anterior lobes where the majority of the basophiles will appear later. There is no significant change in the anterior lobe.

Later Stages.—Apart from general growth, little change takes place in the pituitary for some time. At three months (about 27 mm.) it is still flattened, but enlarging posteriorly. The main downgrowth of the nervous lobe is developing many processes into the intermediate lobe, and in lesser degree into the transitional lobe. The intermediate lobe consists of loosely arranged amphiphiles and basophiles, with an occasional acidophile. The transitional and anterior lobes have altered little, though the latter is extending dorsoventrally. The zone of non-reacting cells between the areas of acidophiles is more distinct, but no basophiles have appeared.

At six months (about 4 cm.) the shape of the gland is changing towards that of the adult by the continued downgrowth of the nervous lobe and the consequent dorsoventral stretching of the anterior lobe. The floor of the hypophyseal cavity is still flat, but the nervous-lobe processes penetrating into the transitional and intermediate lobes have increased in number and complexity. Little histological change is apparent in the intermediate lobe. It has increased relatively in size, and its boundary with the nervous lobe, still marked by a continuous layer of connective tissue, is becoming very complex. The most important feature of this stage is the appearance of basophiles in the region between the two acidophile groups and in the lower levels of the transitional lobe.

At twelve to fourteen months (6–8 cm.) the pituitary resembles that of the adult in shape and general structure (fig. 4). It is now almost enclosed in connective tissue and held in a hollow in the base of the cranium. The nervous lobe is much enlarged, with its processes into the intermediate and transitional lobes becoming elaborately branched. No projections of the nervous lobe ever pass into the anterior lobe proper. The ependyma may be depressed into two lateral inpushings connected by a shallow gutter, comprising a hypophyseal recess.

The intermediate lobe is relatively enlarged, with loosely packed cells having scanty cytoplasm and often well-defined cell boundaries. They have typically a dull basophilic reaction, but red granules are sufficiently numerous in many to give an amphiphilic or dull acidophilic effect. The blood-vessels in the nervous lobe give rise to a network of capillaries against the connective-tissue boundary which supply the intermediate lobe, and this is supplemented by capillaries carried in its substance by connective-tissue laminæ. The transitional lobe type of acidophile is now rare in the intermediate lobe. There is no connective-tissue division between the two lobes, but their structure and staining reactions distinguish them.

The transitional lobe is now approaching closely to the adult form and condition. In sagittal section it appears much more restricted than it actually is, for the lobe broadens laterally and dorsally to pass forwards over the anterior lobe and backwards over the intermediate and nervous lobes, though these extensions do not actually meet on the dorsal surface. Numerous rounded or oval basophiles, whose cytoplasm appears homogeneous, are scattered throughout the lobe, especially anteriorly and ventrally. The anterior limit of the lobe, however, is not precise, making it difficult to assign the zone of mixed basophiles and neutrophiles definitely to this lobe or to the adjacent anterior lobe. The zone closely resembles the remainder of the transitional lobe, but there is a process of the nervous lobe partially separating the two, and also in *Gasterosteus* (Bock, 1928) the anterior lobe is differentiated into two parts. Sometimes quite large blood-vessels are now to be found in the substance of the transitional lobe, not associated with laminæ of connective tissue; no orientation of chromophiles towards these vessels occurs.

The anterior lobe has the deep dorsoventral shape of the adult lobe, but its structure is much as before. The acidophiles are brown-red, and

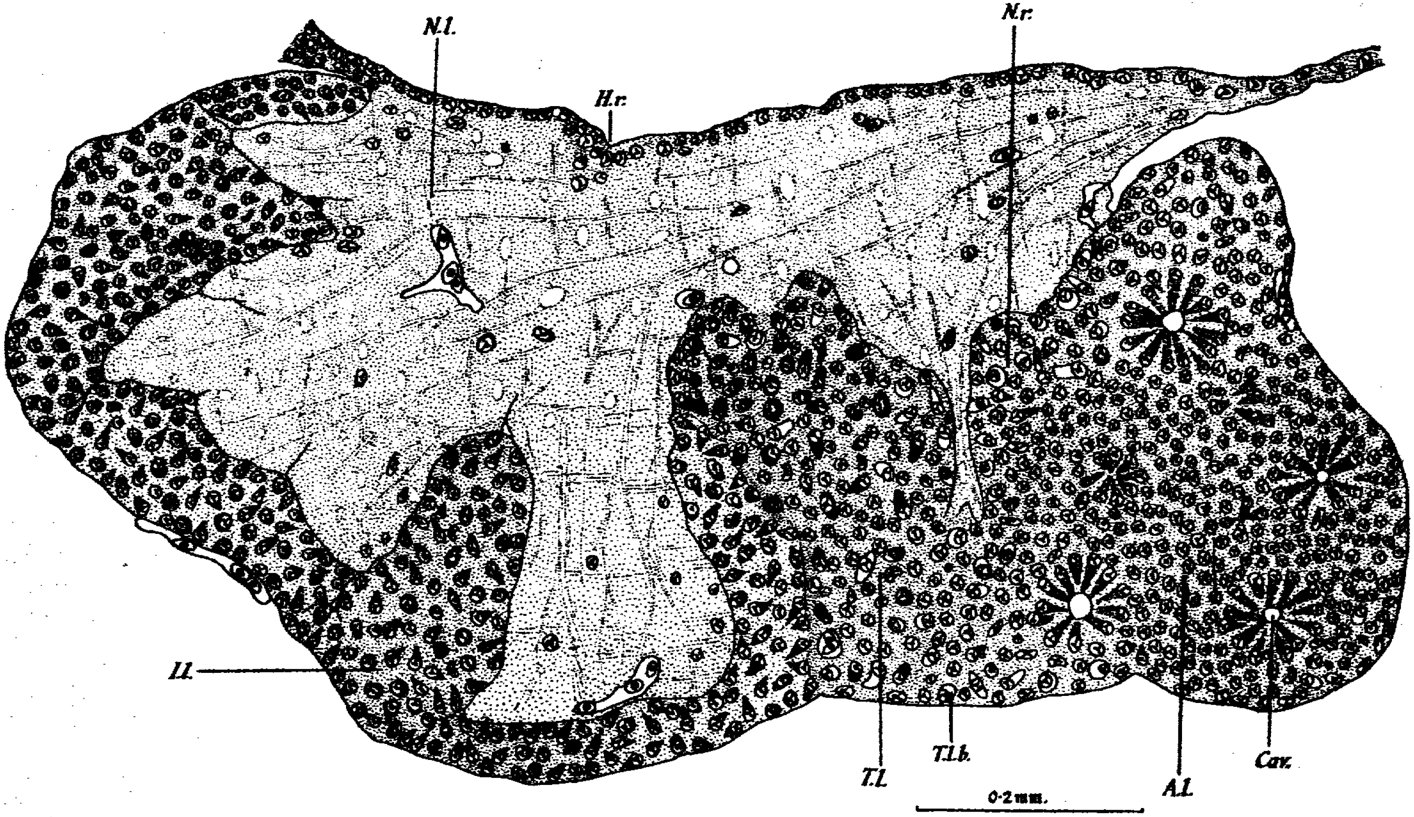

FIG. 4.—L.S. pituitary of Trout about one year old and 7 cm. in length.
H.r., hypophyseal recess; *N.r.*, region of mixed basophiles and chromophobes; *T.l.b.*, transitional lobe basophile. Other lettering as before.

the cavities are as variable as in earlier stages. Basophiles are to be found among the acidophiles, but they are not common. The blood-supply is the poorest of the three lobes, as there are no inpushings of the nervous lobe into its substance; vessels, however, are present in the dorsal angle between the gland and the brain floor, and some are present on the external surface which may act as a source of supply.

The glandular lobe of the pituitary therefore contains three types of acidophiles and two of basophiles, all of which have distinctive features, as well as amphiphiles and numerous neutrophiles.

B. *The Thyroid.*

Maurer (1886) described the development of the thyroid in the Trout. It appears as an unpaired invagination of the ventral throat epithelium at twenty-six to twenty-eight days after fertilisation, when the animals are about 6 mm. in length. The vesicle loses its connection with the wall, comes to lie in the fork of the ventral aorta, and by five weeks comes to show a distinct lumen. This primary vesicle elongates underneath the aorta and by about six weeks colloid has appeared in the lumen, new vesicles budding from the wall of the primary vesicle. This corresponds to the first stage described, about a week before hatching when the pituitary is still undifferentiated. One to four vesicles are present, lined with cubical epithelium and filled with deep blue-staining colloid (fig. 1). From the present point of view the appearance of colloid in the vesicles is taken as the first signs of thyroid activity. A number of clear vacuoles also may be present in the colloid; these are the so-called "absorption vacuoles" which are supposed to represent a stage either of secretion or of excretion of the colloid. The shape of the epithelium and the occurrence of these vacuoles indicates a glandular condition of continuous but not extreme activity, a condition which persists throughout early development. The well-developed state of the thyroid relative to that of the pituitary at this stage contrasts strongly with conditions in the Frog and Toad (Kerr, 1939) and renders any initial control of colloid production in this form by the pituitary very unlikely.

During the next stages up to four to six weeks after hatching the thyroid vesicles increase in number by budding, and move apart and anteriorly from their early positions around and lateral to the base of the aorta. The vesicles are still generally small and separate, though thyroid tissue may form an unbroken band round the aorta, their epithelium remains cubical, and clear vacuoles are present. Blood-vessels become frequent and take over the vascular supply, which was at first apparently by lymph channels. By three months after hatching the epithelial cells lining the vesicles are distinctly flatter and, although a certain number of clear vacuoles can still be found, the gland is approaching the less active adult state. None of these changes, however, can be correlated with changes in the pituitary. Whether therefore control of the thyroid is established during these later stages cannot be settled histologically.

C. *The Gonads.*

Moore (1937) has described the origin of the germ cells and gonads in the Rainbow Trout (*Salmo irideus*) in an account which largely covers the anatomical details in the present form. Sex cells in the first stage are present along the flat genital ridges, which become prominent on either side of the swim bladder during the first month after hatching. Further changes continue slowly and it is not until about six months after hatching that the sex cells are enclosed in definitive gonads. It is between the three- and six-month stages that basophiles appear in the lower portion of the transitional lobe of the pituitary and in the area between the two groups of acidophiles of the transitional and anterior lobes. Tentatively, therefore, an association between these two phenomena may be suggested, but the slow development of the gonads makes a correlation upon purely histological grounds difficult.

DEVELOPMENT IN THE PERCH.

A. *The Pituitary.*

In the stages up to about the sixth day after hatching (6–8 mm.) there is no histological differentiation in the pituitary. The ectodermal component is rounded dorsally and pressed closely into the infundibular floor, ventrally it is flat and applied to the dermis of the mouth. There is no connection with the buccal ectoderm after hatching.

Stage 1.—By about ten days after hatching (10–12 mm.) changes are beginning. The fibrous downgrowth from the infundibular floor to form the nervous lobe has begun, the posterior region of the glandular component has the looser texture characteristic of the intermediate lobe, and acidophiles have appeared in the centre of the gland. These are transitional lobe cells and the first to show differentiation.

Stage 2.—At three to four weeks after hatching (14–16 mm.) the nervous lobe has increased in size, with loose undifferentiated intermediate lobe tissue around it. The transitional lobe is U-shaped in transverse section, with the arms of the U embracing the nervous lobe on either side. Many bright red acidophiles are present, and the lobe is relatively solid. In the anterior lobe brown acidophiles are appearing and small cavities may appear, but without any orientation of the cells around them.

Stage 3.—At about eight weeks after hatching (about 4 cm.) the pituitary is approaching the adult condition (fig. 5), and differentiation, slower at first, is now ahead of that of the Trout. The flattened early pituitary has become more conical in shape. The nervous lobe has grown down into the intermediate, with smaller wedges into the transitional, whilst the differentiation of the lobes of the glandular component is far advanced. The intermediate lobe cells are loosely arranged and more or less amphiphilic, though basophiles and acidophiles are present. These cell types may all be related, but there tends to be a gradation in size from large amphiphiles through basophiles to small acidophiles. The amphiphiles are larger than the intermediate lobe cells of the Trout and are frequently orientated towards the connective-tissue membrane of the nervous lobe, otherwise the lobes of the two species are similar.

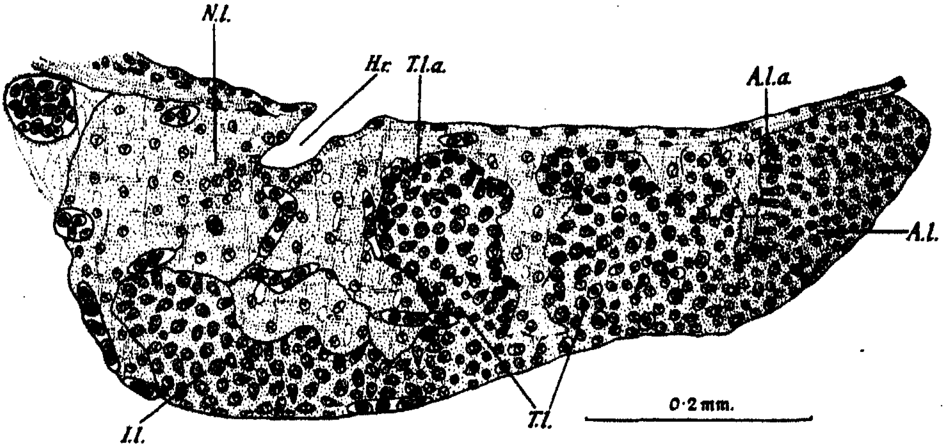

FIG. 5.—L.S. pituitary of Perch about two months old and 4 cm. in length. *A.l.a.*, anterior lobe acidophile. Other lettering as before.

The boundary between intermediate and transitional lobes is indefinite, although a distinct colour difference exists. As in the Trout, the anterior face of the main part of the nervous lobe bounds the transitional lobe behind, although there is often a posterior wedge of intermediate cells in this region (fig. 5). In front the transitional lobe is bounded by a much thinner process of the nervous lobe. Two thick lateral sheets of nervous-lobe tissue pass into the transitional lobe on either side of the mid-line, producing a main central mass and two lateral arms of transitional tissue. The arms pass half-way up the outer sides of the pituitary, whence smaller processes extend dorsally over the intermediate lobe. The acidophiles of the transitional lobe are more regularly arranged than in the Trout, lying along the boundary with the nervous lobe, though without any particular orientation. Basophiles are present, generally anteriorly and ventrally and often in groups. A zone of basophiles and neutrophiles lie just posterior to the anterior lobe, much as in the Trout, but may be separated from the anterior lobe by a process of the nervous lobe (fig. 5). The blood-supply of the lobe is from vessels along the nervous-lobe boundary, and to a less extent on the external surface of the gland.

The anterior lobe merges ventrally into the transitional, but the difference between the two types of acidophiles is again striking. Those of the anterior lobe are brown rather than red, lack compactness, and are small or indefinite in shape, or larger and elongated towards laminæ of connective tissue. There are no cavities at this stage, and acidophiles and neutrophiles are scattered irregularly throughout the lobe. Basophiles are rare and are situated mainly towards the transitional lobe. The blood-supply is poor and does not affect the orientation of the acidophiles.

A hypophyseal recess appears as a backwardly slanting funnel (fig. 5), with a short lateral forking at its inner end. There is a plentiful blood-supply to the nervous lobe, carried in large vessels in the substance of the lobe, with a restricted number of capillaries on the connective-tissue boundary. A substance representing the secretory material of Herring (1908, 1913) can be demonstrated throughout the nervous lobe shortly after differentiation has commenced and by the present stage is very plentiful. It takes the form of irregularly shaped granules and small masses, staining particularly well with Azan, and most numerous in the posterior and ventral portions of the nervous lobe. There is no tendency for it to concentrate near the boundary with the glandular lobes or near blood-vessels, nor does it appear to be associated with disintegrating cells of any kind as it does in *Gasterosteus* (Bock, 1928).

B. *The Thyroid.*

Colloid appears in the thyroid vesicles during the first week after hatching, relatively later than in the Trout. The pituitary, however, is also later and its differentiation has not yet begun. The gland increases rapidly in bulk, showing the same cubical condition of the epithelium and presence of clear "absorption vacuoles" as in the young Trout. This state still persists about eight weeks after hatching. Again, therefore, colloid production does not appear to be under the control of the pituitary at first, and there is no indication when such control is established.

C. *The Gonads.*

The development of the gonads, though similar in plan, is much more rapid than in the young Trout. By eight weeks after hatching the genital ridges have developed into quite distinct gonads, and at this much earlier stage basophiles have also appeared in the pituitary. The tentative suggestion that these two phenomena may be linked is therefore strengthened.

The differentiation of the pituitary and the indications of correlation between it and the thyroids and gonads are basically similar to the Trout and Perch in the other Teleosts examined (Minnow, Stickleback, Plaice, and Flounder). The pituitary at hatching is a flattened organ with no histological differentiation. The nervous lobe appears as a posterior downgrowth into the intermediate lobe and as smaller ingrowths into the transitional lobe within a week or so of hatching. At the same time or shortly afterwards red acidophiles appear in the transitional lobe and brown acidophiles in the anterior lobe, which latter contains no organised cavities. Later still, differentiation appears in the intermediate lobe, and finally basophiles appear in the transitional and anterior lobes. Variations exist in the onset of differentiation relative to hatching, *e.g.* in the Stickleback it commences within the first week, also there are differences foreshadowing the very considerable divergences in the detailed histology of the adult glands, but the general process is similar throughout. It is of interest to note that the amount of yolk present has no noticeable effect upon the differentiation of the pituitary.

DISCUSSION.

The general course of development in these forms is basically similar. Specific differences are slight in early stages, and result eventually in conditions which suggest comparable functional capacity in the various lobes. There is some variation in the onset of differentiation relative to hatching, but this does not appear to be significantly affected by the amount of yolk carried or by a freshwater or marine habitat, and the subsequent sequence of changes is much the same in each case.

In the organisation of the gland the importance of the nervous lobe is early emphasised, and a close time relationship exists between the rupture of its connective-tissue boundary and the appearance of acidophiles in the adjoining transitional lobe. Some question of food or oxygen supply may possibly be involved, but as there is a complete lack of orientation between the acidophiles and the breaks in the connective tissue the importance of the relationship is difficult to estimate. The appearance of blood-vessels in the nervous lobe and along its boundary with the glandular lobes also coincides roughly with the beginning of histological differentiation, but no correlation ever exists between blood-vessels and differentiated cells. Acidophiles in the Trout appear in the anterior lobe before they appear in the transitional lobe, while in the Perch, Minnow, Stickleback, Plaice, and Flounder the reverse is the case. The difference is not great, however, and may be partly due to their distinctive orientation in the Trout, which makes them conspicuous from the earliest possible stage. The relative time and course of differentiation in the intermediate lobe is much the same throughout. The presence in this lobe of isolated acidophiles of transitional lobe type in early stages may be the result of the initial continuity of the two lobes. The difference in time of appearance of the basophiles of the transitional and anterior lobes in different forms is perhaps significant.

The beginning of histological differentiation in the pituitary shortly after hatching in all species indicates that its functions have now commenced, and justifies the view that it has responsibilities in the adjustments between external and internal environments (Spaul, 1930). The final differentiation shows three quite distinct types of acidophiles and two of basophiles, as well as amphiphiles and neutrophiles, with which to correlate functions, if indeed it is possible to do so with any precision. In the Teleost, however, the functions of the pituitary are little known. Its pressor and melanophore functions have been investigated (Herring, 1908 and 1913; Hogben and de Beer, 1925; Zondek and Krohn, 1932; Kleinholz, 1935), but these investigations do not enable correlations between particular cell types and particular activities to be established. The acidophiles of the transitional lobe, perhaps along with those of the anterior lobe, might be associated with the general metabolism of the body, including growth, in view of their early appearance here as in other forms (Kerr, 1939). With regard to a relationship between the pituitary and the gonads in the adult fish, Matthews (1939) injected pituitary extracts with indefinite results, but, on the other hand, found great changes in the gonads following hypophysectomy. The appearance of basophiles during development in the transitional and anterior lobes at approximately the time of formation of the definitive gonads would suggest a possible association of these cells with the maturation of the gonads and, coupled with Matthews' observations, the control of sexual rhythms when these become established. The independence of the gonads in early development, here as in the Anura, would imply that such control was exercised during the later maturation rather than during the early developmental stages.

The lack of any probable pituitary control, as compared with the Anura, over the early development and functioning of the thyroid may be an indication that in more primitive forms the pituitary has not established the same ascendancy over the other endocrine organs that occurs later, or simply that the peculiar conditions in the Anura with their metamorphosis require a particularly early co-operation. It appears likely that pituitary control of the thyroid is established at some stage, although there is no histological evidence of this.

The cavities in the anterior lobe of the Trout may be compared only with transitory structures in the Perch. There seems to be no justification for regarding them as the remnants of a hypophyseal cavity such as arises secondarily in *Lepidosiren* (Kerr, 1933). They appear at different levels, always in the anterior region of the gland, showing no regularity in their number or time of appearance, and increasing in number for some time after the onset of differentiation without showing any tendency to fuse. They are round or oval spaces and never form any sort of complex, however simple, but they may represent a simplified form of the elaborate tube system found in Ganoids (Stendell, 1914). The orientation of the acidophiles around them is very consistent in the Trout; when orientation is present in the other forms it is towards the connective-tissue laminæ in the substance of the anterior lobe or the connective-tissue boundary with the nervous lobe.

The present nomenclature of the glandular region was put forward tentatively by Stendell (1914), but is less satisfactory in the present state of knowledge. The intermediate lobe was so named in higher types because its position is either between the nervous lobe and the hypophyseal cavity or between the nervous lobe and the anterior lobe. There is no hypophyseal cavity in the Teleost, and the topographical position of the so-called intermediate lobe is quite different. Histological appearances, moreover, are too varied to give any decisive evidence. The name given to the transitional lobe is also misleading in that it is a distinct region and does not share the characteristics of the neighbouring intermediate and anterior lobes or show any gradation from the one to the other. It is "transitional" between these two lobes only in the degree of its contact with the nervous lobe. Histologically it is closer to the anterior lobe of higher types than is the Teleost anterior lobe, but this may have more physiological than morphological significance. Anyhow its intimate connection with the nervous lobe seems to make any close comparison between the two unjustifiable. The remainder of the glandular region is the anterior lobe; it may receive processes from the nervous lobe (e.g. *Carassius*, Bell, 1938) or it may not, and it may consist of two histologically distinct regions, acidophile and neutrophile (e.g. *Gasterosteus*, Bock, 1928). The relatively large size of the lobe in early development, its initial condition as an essential part of the ectodermal ingrowth, and the lack of lateral lobes appear to make Bell's (1938) suggestion, that it is the homologue of the tuberal lobes, unlikely. It could be combined with the transitional lobe as the homologue of the later anterior lobe, but only on negative grounds. Indeed, the confluence of the glandular lobes in development, and often later, and the close association of the nervous lobe with two or all of these lobes suggests that only the ectodermal rudiment in the Teleosts and higher Vertebrates is truly homologous. In this case the subsequent development of the pituitaries of these groups illustrates their divergence during the course of evolution, and the anatomical subdivisions of the Teleost pituitary would be considered peculiar to the group and perhaps best referred to as the anterior, middle, and posterior regions of the glandular component. Scruggs' (1939) observations upon a series of American Teleosts could be interpreted in this way, although the author actually only emphasizes the essentially tripartite nature of the glandular element.

No definite support could be found for the view that the granules and small masses of material in the nervous lobe are products of glandular lobe secretion on its way to blood-vessels or to the ventricles of the brain. No secretion of the glandular lobes into the nervous lobe was detected, nor any migration of glandular-lobe cells, as, for example, in *Gasterosteus* (Bock, 1928).

SUMMARY.

1. The development of the Trout pituitary has been followed from before the time of hatching until histological differentiation of the cells is established. A briefer study has been made in the Perch, and also in the Minnow, Stickleback, Plaice, and Flounder. These pituitaries differ amongst themselves, but no satisfactory homologies appear to exist between the glandular subdivisions of the ectodermal element in the Teleosts and those in higher Vertebrates.

2. The invasion of the glandular by the nervous component is described. It consists of processes which are numerous and elaborate in

the intermediate lobe, fewer and simpler in the transitional lobe, and mere indentations in the anterior lobe. In the Trout acidophiles differentiate first round the cavities in the anterior lobe and are followed very shortly by those of the transitional lobe; the acidophiles appear in the reverse order in the other fish studied, and cavities are lacking in the anterior lobes. Later in all cases dull basophiles appear in the intermediate lobe, and finally brighter basophiles in the other two lobes.

3. The condition of the thyroid and gonads was noted at the various stages for purposes of correlation. Early differentiation in the pituitary is associated with free-living existence, and the basophiles of the anterior and transitional lobes may be related to the maturation of the gonads. The early development and functioning of the thyroid would appear to be independent of the pituitary.

4. From an early stage the nervous lobe in all cases has a good blood-supply which, by branches along the boundary, supplies the glandular lobes also. Granules and masses representing the Herring material are concentrated in the posterior processes of the nervous lobe. No migration of glandular-lobe cells into the nervous lobe has been observed.

I wish to express my appreciation to Professor E. A. Spaul, D.Sc., for his encouragement and advice throughout the course of the work. My thanks are also due to Mr David Smith of the West of Scotland Trout Farm for the generous gift of the bulk of the Trout material, and to Dr E. B. Worthington, Director of the Freshwater Biological Association's Station at Wray Castle, and Mr G. Thompson for their kind co-operation in supplying stages of Perch and other fish.

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XIX.—The Effect of the Inhibition of Respiration and Assimilation on the Diatom *Ditylum Brightwelli* (West).

By **D. Bhatia,** Ph.D. (From the Department of Zoology, University of Edinburgh.) *Communicated by* Professor JAMES RITCHIE, M.A., D.Sc. (With Eight Text-figures.)

(MS. received February 29, 1940. Revised MS. received May 29, 1940. Read June 3, 1940.)

INTRODUCTION.

DITYLUM BRIGHTWELLI, a marine plankton diatom, shows very peculiar osmotic relations which have recently been investigated by Gross (1939). The cells (fig. 1) consist of a central mass of protoplasm containing the nucleus and connected by a few protoplasmic strands with the peripheral plasma membrane in which are embedded numerous chromatophores. The remainder of the space inside the cell is filled with cell sap. As in other diatoms, the protoplast is enclosed by a siliceous cell wall. Gross found that *Ditylum* plasmolyses rapidly in isotonic and hypotonic NaCl and sugar solutions, being reduced in a few seconds to a small spherical body about one-third to one-twentieth of its original volume, similar in all respects to a resting spore. In a solution of $NaCl + CaCl_2$, and in unbuffered artificial sea-water, plasmolysis occurs in all cells, but the process is considerably delayed and completed only after several hours. In artificial sea-water, the pH of which is adjusted to about 8, the cells remain unchanged.

In the course of plasmolysis (figs. 1–3) the cell membrane shrinks from the cell wall, at first remaining connected to it by a few protoplasmic filaments; the chromatophores travel towards the centre, the filaments are gradually withdrawn, and the protoplast assumes the spherical shape of a resting spore. Removed to fresh culture medium, the "resting spore" begins to germinate. For the purpose of description the following three stages may be recognised in the recovery process: the "early amœboid" stage (fig. 4), when it becomes somewhat amœboid in shape and gives off delicate protoplasmic filaments similar to fine pseudopodia; the "tubular" stage (fig. 5), when the protoplast lengthens and the chromatophores move towards the poles of the cell; and the "late amœboid" stage (fig. 6), when the protoplast has greatly increased in volume and gives off branched pseudopodia which attach themselves to the cell wall. Finally, the protoplast fills out the shell and recovery is complete.

Vegetative cells, when placed in the dark, cannot maintain their turgor for longer than 15–22 hours even if kept in fresh culture medium. After that they plasmolyse slowly and in 5–10 days form resting spores. "Artificial" resting spores formed in NaCl solutions when placed in fresh culture medium may recover as far as the "late amœboid" stage and sometimes even further (fig. 9) before they begin to plasmolyse again. The resting spores formed in the dark, on the other hand, only recover if placed in light. They have thus lost something which the "artificial" resting spores still retained, and may, therefore, be termed "exhausted" resting spores.

From these observations Gross (1939) concluded that *Ditylum* does not behave as an osmotic system, and that the factors determining the exchange of water between these cells and their environment are (1) the presence of NaCl and $CaCl_2$, (2) a pH similar to that of sea-water, and (3) a vital activity of the cells, involving perhaps the secretion of water.

This work was undertaken to investigate the third factor mentioned above, *i.e.* the importance of metabolic activity for the maintenance of turgor in *Ditylum*. In particular, the effects of the inhibition of photosynthesis and respiration were studied by using certain anæsthetics and reduced sea-water.

I am greatly indebted to Professor J. Ritchie and Dr F. Gross for their interest and encouragement in carrying out this work, and to Dr C. F. A. Pantin for the technique of preparing reduced sea-water.

THE EFFECT OF ANÆSTHETICS.

It is well known that respiratory and assimilatory processes are respectively sensitive to different anæsthetics. Warburg (1921) showed that in weak concentrations cyanide inhibits respiration without much affecting other metabolic processes. Keilin (1925) found that the effect of cyanide and sulphide is due to their action in preventing the oxidation of cytochrome and thereby cutting off the supply of O_2 available to the protoplasm. The action of alcohols, on the other hand, is less specific (Warburg, 1921) and appears to be due to general adsorption and rendering the protoplasm unable to reduce cytochrome. Warburg (1919) and van der Paauw (1932) have demonstrated that, unless used in strong concentrations, urethane has an inhibitory effect on assimilation without greatly affecting the respiratory mechanism of the cell.

Suitable concentrations of cyanide, sulphide, alcohols, and urethane were employed alone or in conjunction with the absence of light to control the respiratory and assimilatory mechanism of *Ditylum* cells.

The narcotic solutions were prepared in the culture medium ("Erdschreiber"; see Gross, 1937) and their pH adjusted to correspond with that of normal sea-water. As some of the anæsthetics used were highly volatile, the watch-glasses used in the experiments were filled completely and covered with glass plates and made air-tight with vaseline.

Cyanides.—Various concentrations of sodium and potassium cyanide were employed, and their effects with regard to plasmolysis and recovery observed.

NaCN $3 \cdot 16 \times 10^{-3}$ to 10^{-3} M. concentration caused rapid plasmolysis and formation of small dark resting spores in 10–15 minutes, but many cells were damaged by the narcotic and speedily disintegrated. These resting spores were similar to the "artificial" resting spores and recovered completely in 20–24 hours when placed in the culture medium and light; and, like them, recovered partially up to the "late amœboid" stage before shrinking again when transferred into culture medium in the dark.

NaCN 10^{-4} to $3 \cdot 16 \times 10^{-5}$ M. concentrations caused slow plasmolysis (figs. 1–3). The protoplast began to shrink almost at once; the delay of 1–2 hours in some cases was probably due to the adherence of a thin layer of culture medium to the cell wall after the transference of the cells from a culture into the cyanide medium. Contracted resting spores were formed in 8–10 hours. Placed back in the culture medium and light, they recovered completely in about 20 hours. Left for a longer period in the narcotic, the subsequent recovery did not begin immediately. Removed from the narcotic after 15 hours, the resting spores began to germinate after remaining quiescent for 2–3 hours; and after 20 hours in the narcotic the quiescent period was about 3–5 hours. But after the recovery began the process was completed in about the same time, *i.e.* 20 hours. After 24 hours in the narcotic most cells disintegrated and no recovery was possible.

Vegetative cells kept in the dark in this narcotic plasmolysed in about the same time as in light.

The "cyanide" resting spores formed in this concentration in 8–10 hours were placed in the culture medium and dark. They recovered up to the "tubular" stage (fig. 7) in 15–20 hours, after which they again began shrinking to form resting spores in 5–6 days. It may be noted that rapidly formed "artificial" resting spores recovered in the culture medium and dark far beyond the "tubular stage."

"Artificial" resting spores (formed in 3·5 per cent. NaCl) began to recover immediately when placed in the narcotic solution, but the recovery did not go beyond the "early amœboid" stage, whether placed in light or dark (fig. 4). After 3–4 hours they began to shrink again and formed resting spores in 8–10 hours. Placed in the culture medium and light they recovered completely in about 20 hours. Kept for 15–20 hours in the narcotic, the commencement of the subsequent recovery was delayed for 3–4 hours. It may be noted that the "artificial" resting spores recovered in the culture medium and dark as far as to the "late amœboid" stage or nearly completely.

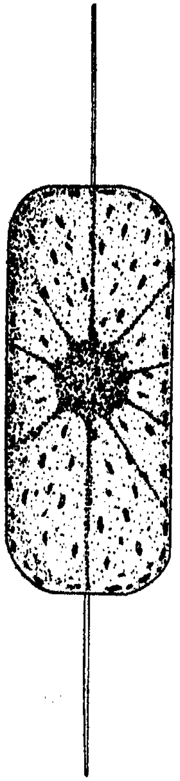

FIG. 1.—Normal cell of *Ditylum Brightwelli.*

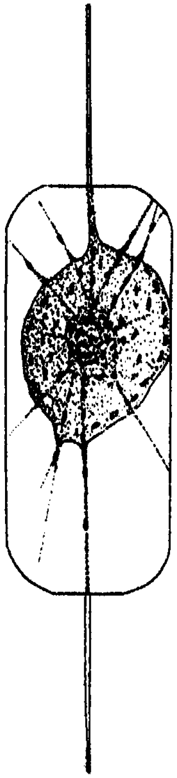

FIG. 2.—Amœboid stage of plasmolysis.

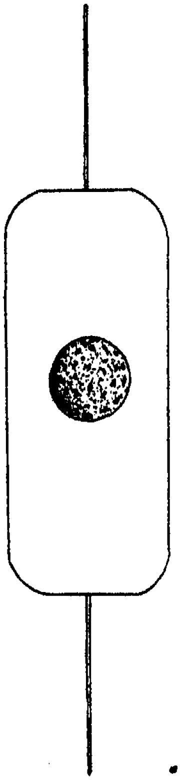

FIG. 3.—Resting spore stage of plasmolysis.

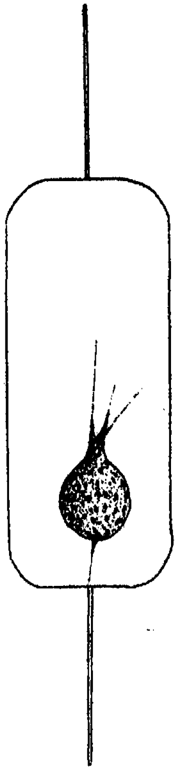

FIG. 4.—"Early amœboid" stage in the process of recovery.

The "exhausted" resting spores formed by slow and prolonged plasmolysis in the absence of light did not show any recovery in the narcotic solution.

NaCN 10^{-5} M. caused plasmolysis after 1–2 hours, but the shrinkage did not go beyond the formation of a somewhat contracted protoplast connected with the cell wall by fine protoplasmic filaments. After about 18 hours the protoplast began to recover in the narcotic medium and was completely normal in about 12 hours, when the cells began to divide.

"Artificial" resting spores recovered completely in this solution in about 24 hours.

NaCN 10^{-6} M. solution did not cause plasmolysis.

Experiments with various concentrations of KCN yielded similar results to those with the sodium salt. But slightly stronger concentrations were needed to produce these effects. 10^{-2} M. was destructive; 10^{-3} to 10^{-4} M. produced slow plasmolysis in about 10 hours, and 10^{-5} had no obvious effect.

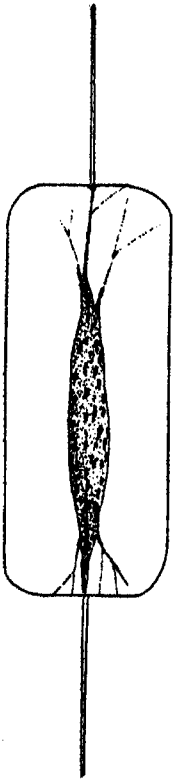
FIG. 5.—"Tubular" stage in the process of recovery.

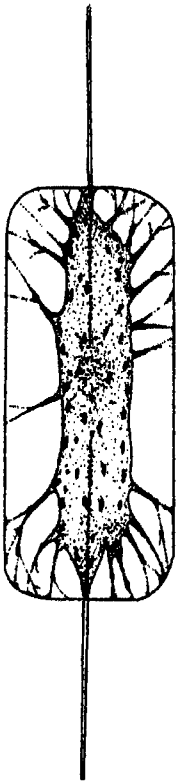
FIG. 6.—"Late amœboid" stage in the process of recovery.

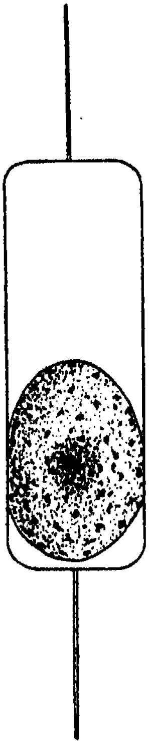
FIG. 7.—Incomplete plasmolysis, with weak concentration of alcohols: sphæroidal protoplast of same diameter as shell.

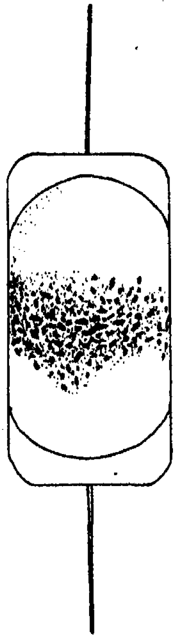
FIG. 8.—First stage of contraction in urethane: slight shrinkage of protoplast and median concentration of chromatophores.

Sulphide.—The effect of Na_2S as regards plasmolysis was similar to that of the cyanides, but much stronger concentrations were required to produce similar results.

At 10^{-2} M. it was destructive and at 10^{-4} M. it was found to be ineffective. In intermediate concentrations slow plasmolysis took place, which began almost immediately and resulted in the formation of small resting spores. In 2×10^{-3} M. resting spores were formed in 15–20 hours, and in 10^{-3} M. in about 30 hours. Complete recovery occurred in 20–30 hours on return to the culture medium and light. In $4{\cdot}0 \times 10^{-4}$ M. the cells recovered in the narcotic medium after a partial shrinkage for about 10 hours. Absence of light did not cause any appreciable difference in the effect of this narcotic.

The "sulphide," "artificial," and "exhausted" resting spores behaved in exactly the same way as those formed in the cyanides.

Alcohols.—Solutions of methyl, ethyl, butyl, amyl, and octyl alcohols were made in the culture medium, and watch-glasses sealed to prevent evaporation.

It was noted that though their effective range in causing plasmolysis followed the series methyl < ethyl < butyl < amyl < octyl the general effects produced were very similar. Table I indicates the comparative effects of various concentrations of different alcohols.

TABLE I.—THE EFFECT OF VARIOUS CONCENTRATIONS OF ALCOHOLS.

+ + = Complete plasmolysis to form "artificial" resting spores.
+ – = Partial plasmolysis to form contracted protoplast.
– – = No plasmolysis.
⊕⊕ = Complete recovery.
⊕⊖ = Partial recovery.
⊖⊖ = No recovery.

Alcohol.	Molar Concentration.	$3{\cdot}16 \times 10^{-1}$	10^{-1}	$3{\cdot}16 \times 10^{-2}$	10^{-2}	$3{\cdot}16 \times 10^{-3}$	10^{-3}	$3{\cdot}16 \times 10^{-4}$	10^{-4}	$3{\cdot}16 \times 10^{-5}$	10^{-5}	$3{\cdot}16 \times 10^{-6}$	10^{-6}
Methyl	Veg. cells	+ +	+ –		– –								
	Art. r. sp.	⊖⊖	⊕⊖		⊕⊕								
Ethyl	Veg. cells		+ +	+ –		– –							
	Art. r. sp.		⊖⊖	⊕⊖		⊕⊕							
Butyl	Veg. cells				+ +	+ –		– –					
	Art. r. sp.				⊖⊖	⊕⊖		⊕⊕					
Amyl	Veg. cells					+ +	+ –		– –				
	Art. r. sp.					⊖⊖	⊕⊖		⊕⊕				
Octyl	Veg. cells									+ +	+ –		– –
	Art. r. sp.									⊖⊖	⊕⊖		⊕⊕

Comparatively strong concentrations were either destructive or produced an immediate plasmolysis forming "artificial" resting spores not unlike those formed by 3·5 per cent. NaCl. These recovered completely in 15–20 hours if removed immediately to the culture medium and light. Suitable weaker concentrations caused a slow and uniform contraction, which began after 1–2 hours in the narcotic. The plasmolysis was, however, incomplete, producing a spheroidal protoplast of the same diameter as the shell (fig. 7), and took 15–18 hours in all alcohols except octyl in which 6–8 hours sufficed to produce the same result. In the culture medium and light recovery began almost immediately and was completed in 20–24 hours. A prolonged action of the narcotic did not cause any further shrinkage. The cells either disintegrated in the narcotic solution or did so when removed to the fresh culture medium. Still weaker solutions had no effect, but after a few days the cells looked unhealthy and began to form auxospores.

"Artificial" resting spores recovered only to the "early amœboid" or "tubular" stages in the effective concentrations of alcohols in 6–10 hours before shrinking again. They were, however, able to recover completely if removed to the culture medium within 6–8 hours in octyl and 15–20 hours in the other alcohols.

There appeared to be no difference in the effect of alcohols in light or in the dark.

The action of alcohols was thus somewhat different from that of other narcotics, for they did not cause complete plasmolysis of the cell unless used in strong concentrations, when they were mostly destructive. Besides, the range between the concentration producing an effect and the one which destroys the cell was very small. No effect was produced up to a certain liminal concentration, and then only ten times the liminal concentration produced an immediate effect. All concentrations more than ten times the liminal tended to have a cytolytic effect, and unless the cell was immediately removed from the narcotic recovery was not possible. Such cytolytic effect was particularly noticeable in the case of amyl and octyl alcohols. Methyl and ethyl alcohols produced irreversible effects due probably to dehydration, as high concentrations were required to produce plasmolysis.

As pointed out by Warburg (1921), the sequence observed in the narcotic effect of the homologous series of alcohols is probably related to their ability to become adsorbed at the surface, and their liminal concentration falls with their increasing molecular weight.

Urethane.—In suitable concentrations urethane is known to inhibit, though not completely, the assimilation of algal cells (Warburg, 1919;

van der Paauw, 1932). It was observed that much stronger concentrations of urethane than those of cyanide or sulphide were needed to produce plasmolysis. 2×10^{-1} to 10^{-1} M. solutions were destructive, but 2×10^{-2} M. solution was altogether ineffective. Intermediate concentrations produced plasmolysis which differed in many respects from that caused by cyanide or sulphide.

Though plasmolysis was rapid in stronger concentrations, due probably to their inhibitory effect on oxidation, its beginning was, on the whole, delayed for 10–15 hours in comparatively weaker concentrations. The process was much slower and contraction not quite uniform. After a slight shrinkage of the protoplast from the cell wall, the chromatophores rapidly concentrated into a median mass leaving the rest of the protoplast almost empty (fig. 8). Further contraction resulted in the formation of a large resting spore almost twice as large as that produced by the cyanide solutions. The time taken for plasmolysis varied with different concentrations.

The "urethane" resting spore recovered completely in 24–30 hours in the culture medium and light, the process of recovery beginning without any delay or quiescent period. A prolonged exposure to urethane after the complete withdrawal of pseudopodia and the formation of the resting spore seemed to damage the cell, for then there was either no recovery or only a partial one before the protoplast became yellow and strongly refractive.

Unlike the plasmolysis effected by cyanide, sulphide, or alcohols, the process was somewhat quicker with urethane in dark than in light and the resting spore produced was more contracted, the absence of light supplementing the partial inhibition of assimilation caused by urethane. The period before plasmolysis began, however, remained the same in light as in dark.

The following table indicates the effect of various concentrations of urethane on plasmolysis in light and in dark:—

Concentrations.	Period before Plasmolysis began.	Plasmolysis in Light.	Plasmolysis in Dark.
$6{\cdot}5 \times 10^{-2}$ M.	10–15 minutes.	10–15 hours.	10–15 hours.
$5{\cdot}0 \times 10^{-2}$ M.	2–5 hours.	15–20 ,,	15–20 ,,
$4{\cdot}0 \times 10^{-2}$ M.	10–15 ,,	36 ,,	30–36 ,,
$3{\cdot}3 \times 10^{-2}$ M.	12–15 ,,	50 ,,	40 ,,
$2{\cdot}5 \times 10^{-2}$ M.	12–15 ,,	90–100 ,,	60–70 ,,
$2{\cdot}0 \times 10^{-2}$ M.	..	No effect.	5–7 days as in dark alone.

In urethane 4×10^{-2} M. the "artificial" resting spores recovered in 20–24 hours up to the "late amœboid" stage; but the "exhausted" resting spores recovered only to the "early amœboid" stage in this medium before shrinking again.

THE EFFECT OF REDUCED SEA-WATER.

Ditylum "vegetative" cells and "artificial" resting spores were placed in sealed petri dishes containing sea-water from which dissolved gases had been removed. Its pH was adjusted to that of normal sea-water.

The following method was employed for obtaining the reduced sea-water for the experiments: To a litre of sea-water a little brom-thymol-blue and enough 10 per cent. HCl was added to bring the pH to about 6. Exactly 500 c.c. of this was measured out into a boiling flask of 700 c.c. capacity. It was then boiled vigorously for 10 minutes to drive off all dissolved gases. While still boiling it was poured into a 500 c.c. flask from which air had been swept off with nitrogen. When cold, the volume was made up to 500 c.c. with freshly boiled distilled water. Before using, it was tested for its pH, which, with the escape of CO_2, was now 8·2. With a little practice it was possible to exactly judge the amount of HCl needed to bring the pH down before boiling the water. The reduced sea-water was siphoned out to supply the petri dishes containing the cells so that only the water at the bottom of the flask, which had not been in contact with the air, was employed.

The cells were placed in watch-glasses with as little water around them as possible, and then gently and rapidly flooded with a considerable excess of reduced sea-water. They were removed with fine pipettes to new watch-glasses and the flooding process repeated twice or three times till all traces of the culture medium had been removed. The cells were finally placed in a petri dish filled with reduced sea-water with just enough methylene blue to give a clear tint. The dish was then almost covered with a glass plate. A merest trace of 10 per cent. sodium hydrosulphite solution was rapidly added till the methylene dye just bleached on gentle mixing. The glass plate was then pushed completely over the dish excluding all air bubbles and sealed with warm vaseline. The sodium hydrosulphite solution was prepared by dissolving 0·5 g. in 8 c.c. of cooled boiled distilled water together with the same amount of sodium carbonate. Of this 0·05–0·1 c.c. was mixed with 20 c.c. of cooled boiled distilled water. The amount of carbonate added to the first solution was determined by trial to be that which would prevent any change in the final pH of the sea-water. When added to the sea-water a merest trace of this solution discharged the colour of the methylene blue; and as long as the blue tint was absent no O_2 was present in the water.

After completing the experiments the reduced sea-water in the petri dish was again tested for its pH, which an accidental presence of CO_2 would alter.

In order to obviate any possibility of toxic effects of methylene blue or hydrosulphite solution on the cells parallel experiments were performed with the same stock of sea-water without the addition of these indicators.

"Vegetative" cells placed in the reduced sea-water began to plasmolyse immediately; the process was slow and regular, and contracted resting spores (fig. 3) were formed in 8–9 hours. These, when placed in the culture medium, recovered completely in light in about 20 hours.

Exposed to the reduced sea-water for longer periods, *i.e.* 12–20 hours, before removal to the culture medium, the recovery did not begin immediately but after a quiescent period of 1–4 hours. No recovery took place if cells were kept longer than 24 hours in the reduced sea-water.

The resting spores, produced with reduced sea-water and removed after 8 hours into the culture medium but kept in dark, recovered only up to the "tubular" stage in about 20 hours and then began to shrink again.

"Artificial" resting spores produced rapidly with NaCl began to recover immediately in the reduced sea-water; but their recovery did not go beyond "early amœboid" stage (fig. 4). After 3–4 hours they began to shrink again and formed resting spores in 7–8 hours. Placed in the culture medium and light, they began to recover immediately and were normal in 20 hours; but in the dark the recovery proceeded only to the "tubular" stage (fig. 5).

"Exhausted" resting spores formed by prolonged plasmolysis did not show any recovery in the reduced sea-water.

"Vegetative" cells and resting spores first washed with NaCN 10^{-4} M. solution before being placed in the reduced sea-water showed the same results. Similar results were obtained with the reduced sea-water without any indicators, so that the results obtained could be safely assumed to be due to the absence of gases rather than to any toxic effects of the indicators introduced.

The behaviour of *Ditylum* cells and the "artificial" resting spores in reduced sea-water bore a marked resemblance in almost every detail to that in the cyanide or sulphide solutions which are known to inhibit respiration. Since the effect of the cyanide or sulphide on plasmolysis is the same in light and in dark, it may be assumed that the effects produced by the reduced sea-water from which both O_2 and CO_2 had been removed were due mainly to the inhibition of respiration rather than to the inhibition of assimilation. It was also made plain that rapidly plasmolysed "artificial" resting spores were able to show an amœboid recovery under completely anaerobic conditions not very unlike that observed by Pantin (1930) in marine amœbæ.

DISCUSSION.

Suitable concentrations of urethane, which did not affect respiration but inhibited assimilation, produced effects similar to those due to the absence of light (see p. 246). Plasmolysis began after 10–15 hours in weak concentrations, *i.e.* after about the same interval as in darkness alone. This was, however, not the direct effect of the inhibition of assimilation, for "artificial" resting spores began to recover almost at once when placed in urethane solutions in which vegetative cells would slowly plasmolyse after some time. In many instances they recovered almost completely before shrinking again.

On the other hand, in suitable concentrations of cyanide and sulphide the cells began to plasmolyse almost immediately. Cyanides are known to inhibit respiration though not to affect assimilation at all or beyond a certain limit which, according to Warburg (1919), is the compensation point of carbon assimilation and respiration. The complete plasmolysis of *Ditylum* in cyanide must, therefore, have been due mainly to the inhibition of oxidation. This is confirmed by the fact that plasmolysis similar in almost every detail was obtained with sea-water from which all oxygen had been removed.

Kitching (1936, 1938) has observed the inhibitory effects of cyanide and sulphide on the fluid output of the contractile vacuoles of *Peritricha*. Plasmolysis observed in *Ditylum* is also probably caused by a similar inhibition of a secretory mechanism which normally makes up for the loss of water by exosmosis (Gross, 1939).

The partial plasmolysis observed in NaCN 10^{-5} M. solution and the subsequent complete recovery in the narcotic medium can perhaps be correlated with changes in the oxidation process due to gradual reduction of the strength of the solution in the presence of aerated sea-water. As described by Commoner (1939) in yeast, and Emerson (1927), Genevois (1927), and van der Paauw (1932) in algal cells, very weak concentrations of cyanide actually accelerate oxidation.

Resting spores formed in cyanide in 6–8 hours recovered completely when placed in culture medium and light; but they also recovered in the dark, though only to the "tubular" stage, before they began to contract again. It may be inferred that the cells which plasmolysed in the cyanide owing to the inhibition of oxidation, retained enough energy precursor to set free energy necessary for partial recovery as soon as they were placed in the culture medium, although in the absence of light no further energy-precursor was produced.

The relation of the recovery time to the period during which oxidation was inhibited in cyanides and sulphides and in reduced sea-water suggests that *Ditylum* cells were passing through a condition analogous to the "oxygen debt" observed by Pantin (1930) during anaerobiosis of marine amœbæ. Pantin found that substances which require the presence of oxygen for their removal accumulate in the cells during the inhibition of oxidation, and, although oxidation is subsequently restored, the amœboid activity by which the recovery is initiated does not recommence until this debt has been paid.

Comparison of the effects of cyanide, sulphide, and reduced sea-water on the one hand, and urethane on the other, shows that assimilation is necessary only for producing and replenishing the energy precursor, while respiration seems to be essential for the production of the energy required for the secretion of cell sap. The inhibition of oxidation would thus cause an immediate cessation of the secretory process and resulting plasmolysis, but if oxidation is allowed, the cell is able to maintain its turgor for some time by acting on the stored energy-precursor even in the absence of assimilation. After 15–20 hours, however, this store is depleted to a level at which it does not yield enough energy to maintain the cell in an expanded state, and its further exhaustion causes slow plasmolysis. The resting spore thus formed is almost completely exhausted of its store of energy precursor and does not show any recovery in light under anaerobic conditions in reduced sea-water or in NaCN 10^{-4} M.

On the other hand, rapidly formed "artificial" resting spores began to recover in the reduced sea-water as well as in NaCN 10^{-4} M. solution both in light and in the dark. In both conditions, however, their recovery proceeded only to the "early amœboid" stage. They obviously had retained the energy precursor which the "exhausted" spores had lost. The amœboid activity shown by the unexhausted spores is an anaerobic reaction similar perhaps to one described by Pantin (1930) in marine amœbæ.

The recovery of the resting spore appears thus to be a double process. In its initial stage it can take place even under anaerobic conditions as long as the cells have enough energy precursor. But this recovery does not go beyond an amœboid activity of the protoplast, and there is no increase in its volume by any secretion of cell sap. Further recovery of the resting spore and increase in volume until it fills the cell depends first upon the removal by oxidation of the reaction products of anaerobiosis, and then upon a release of energy by the oxidation of energy precursors which may be already stored in the cell or are freshly produced by photosynthesis.

SUMMARY.

1. The process of plasmolysis of *Ditylum Brightwelli* caused by anæsthetics like cyanide, sulphide, alcohols, and urethane, and by reduced sea-water has been studied.

2. In weak concentrations urethane produces results similar to those obtained in the absence of light, *i.e.* delayed plasmolysis and immediate recovery on return to normal conditions. The effect of urethane is increased in the absence of light.

3. Plasmolysis produced by cyanide and sulphide and the subsequent recovery resemble the plasmolysis in reduced sea-water due to the inhibition of oxidation. The process begins immediately, and the subsequent recovery is somewhat delayed if oxygen inhibition is prolonged beyond 8–10 hours; the delay may be related to the accumulation of an "oxygen debt." Absence of light makes no difference in the effect of oxygen deficiency with regard to plasmolysis.

4. The effects of alcohols do not resemble those of light or oxygen deficiency, for plasmolysis produced is incomplete or else the effects are destructive. Their effectiveness follows the series methyl < ethyl < butyl < amyl < octyl.

5. The rapidly formed "artificial" resting spores recover partially in the absence of light or in urethane before they begin to shrink again.

6. Under anaerobic conditions the "artificial" resting spores recover to the "early amœboid" stage, but the "exhausted" resting spores do not show any recovery.

7. Resting spores formed by oxygen deficiency recover to the "tubular" stage if oxidation is restored but assimilation inhibited.

8. It is suggested that the cell produces energy-precursor by assimilation and stores it, and that respiration is required to release energy by oxidizing it. The energy is expended in secreting cell sap necessary for the maintenance of the turgor. With the inhibition of assimilation the energy-precursor is reduced, and after using up the store in 15–20 hours the cell is unable to maintain its turgor and begins to plasmolyse. The inhibition of oxidation, on the other hand, causes immediate plasmolysis as there is a stoppage of the release of energy necessary to maintain the secretory processes.

9. The recovery process consists of two stages: an initial anaerobic process which does not cause any increase in volume and depends on the presence of an energy-precursor, and the subsequent complete recovery with secretion of cell sap which depends on the presence of oxygen.

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XX.—The Structure and Behaviour of the Chromosomes of the Sheep during Mitosis and Meiosis.

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Communicated by Dr P. C. KOLLER. (With Twelve Text-figures.)

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INTRODUCTION.

MANY studies on the chromosomes of the sheep have been published in the last decade, but their results are unsatisfactory and contradictory, and hence it was found necessary and justifiable to undertake a further analysis, and the present paper reports the data obtained.

MATERIAL AND METHODS.

For the present study, testicular material from rams of the Leicester breed was obtained from the Edinburgh City Abattoir. The following fixatives were found to give good results:—

(1) Minouchi A—1 part 2 per cent. Osmic acid.
2 parts 1 per cent. Chromic acid.
2 parts 3 per cent. Potassium bichromate.
(2) Minouchi B—1 part 2 per cent. Osmic acid.
4 parts 1 per cent. Chromic acid.

Each of these two fixatives was used (*a*) undiluted, or (*b*) diluted with distilled water in a 1 : 1 dilution or (*c*) a 4 : 1 dilution.

(3) Medium Flemming.

This was used (*a*) undiluted, or (*b*) with distilled water in a 1 : 1 dilution.

The best results were given by Minouchi A (*b*) and Minouchi B (*c*).

The period of fixation was between 15 and 18 hours.

Sections were cut 16 μ thick and were stained with gentian violet. The drawings were made with the aid of an Abbé camera lucida, Zeiss 1·3 apochr. oil immersion objective and 30x comp. eyepiece.

CHROMOSOME BEHAVIOUR DURING MITOSIS.

Several spermatogonial cells were encountered showing various stages of mitotic prophase. During this stage the nucleus is large in comparison with the total volume of the cell. In late prophase the chromosomes are identified as thick threads, which are either scattered within the nucleus or gathered near the periphery of the nuclear membrane (fig. 1). Some of them show twisting, which may be described as the "relic coiling" (Darlington, 1937). The daughter chromatids of prophase chromosomes are tightly pressed together and their double structure cannot usually be seen. No nucleolus nor precociously condensed bodies were found in the nucleus of the spermatogonium during the mitotic prophase.

At mitotic metaphase the chromosomes are aggregated on the equatorial plate, which is small, indicating a narrow spindle. The crowding of the chromosomes makes it difficult to identify them and to analyse the structure of each individual chromosome. Several spermatogonial cells were encountered, however, in which the chromosomes were evenly distributed on the equatorial plate. The diploid number of chromosomes in these cells was invariably found to be 54. This number was furthermore verified by analysis of the primary spermatocytes, in which 27 bivalents were counted. This observation confirms the findings of Shivago (1930, 1931) and of Berry (1938), both of whom found that the diploid chromosome number in the sheep is 54. The table on p. 262 summarizes the data obtained by various investigators.

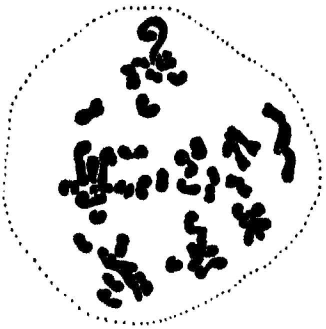


FIG. 1.—Mitotic prophase in the spermatogonium. The double structure of some chromosomes can be seen. ×3700.

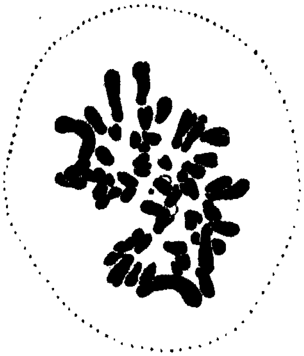


FIG. 2.—Mitotic metaphase. The chromosomes differ in length and size. ×3700.

It is more than probable that the discrepancies in the findings of the different authors are due to inferior technique, and to the fact that chromosome counts in the spermatogonia were not compared with the number of bivalents in the first spermatocytes. Butarin (1933) suggested that different breeds of sheep may have a different chromosome number, but this suggestion cannot be accepted without further proof.

The chromosomes at mitotic metaphase are of different shapes (fig. 2); there are V-shaped chromosomes, with equal and unequal arms, and

TABLE I.—CHROMOSOME NUMBER IN THE SHEEP.

Name of Investigator.	Year.	Breed of Sheep.	Diploid Chrom. No.	Haploid Chrom. No.	Sex Chrom.	Material Studied.
Wodsedalek, J. E.	1929	Not given	33 ♂ 34 ♀	.. ..	XO ♂ XX ♀	Spermatogonial cells
Shivago, P. I. .	1930	Not given	54	..	XY ♂	Amnion cells
Krallinger . .	1931	Not given	50–60	..	XY ♂	Spermatogonial cells
Shivago, P. I. .	1931	Not given	54 (48–56 also reported)	..	XY ♂	Spermatogonial cells
Bruce, H. A. .	1934	Not given	Ca. 60	30	XY ♂	Spermatogonial cells
Butarin, N. S. .	1935 *a*	Arkhar fat-rumped	54–56 52–54 (cells with 60 also found)	..	XY ♂	
Butarin, N. S. .	1935 *b*	Arkhar	60 (52–59 also found)	30	XY ♂	Spermatogonial cells
Novikov, I. I. .	1935	Merinox mouflon	60 (56, 57, 58, 61 also reported)	30	XY ♂	Spermatogonial cells.
Pchakadze, G. M.	1936	Georgian fat-tailed	..	30	..	Primary spermatocytes
Berry, R. O. .	1938	Not given	54	..	..	Mitosis in the amnion cells

others which are more or less rod-shaped. The chromosomes of the sheep can be classified into four groups in respect of the position of the centromere. One chromosome pair has a median or nearly median centromere, 4 have submedian, 15 have subterminal, and 6 have almost terminal centromeres (fig. 3). While 52 chromosomes of the diploid complement can be arranged in pairs according to size and shape, members of the remaining pair differ in shape and length. By analogy with other organisms, these two unequal chromosomes were assumed to represent the sex chromosomes, X and Y; this view was further supported by a study of the meiotic division which also showed that the sex-determining mechanism in the sheep

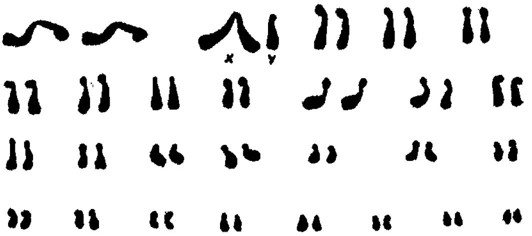


FIG. 3.—The chromosomes of the sheep arranged in pairs according to size. The sex chromosomes are indicated. × 3700.

consists of a pair of unequal chromosomes. The large one, which has a median or nearly median centromere, is designated as the X, while the smaller, which has a subterminal centromere, is described as the Y chromosome.

It is interesting to note that chromosomes of the same shape and length were very often seen lying adjacent to each other on the equatorial plate during mitosis. Such an arrangement resembles the somatic pairing of chromosomes found in Diptera (Metz, 1916), and also reported in mammals (Koller, 1936), and assumed to be due to homology. The chromosomes of the sheep fall into groups according to size, as well as to the position of the centromere. There are one large chromosome pair and eleven very small pairs; the largest chromosome is about $2 \cdot 7\ \mu$ long and is four times the size of the smallest one. The remaining chromosomes are intermediate and show a slight but continuous gradation in size, several of them being very similar in size and position of the centromere. Such similarity of non-homologous chromosomes makes it difficult, if not impossible, to follow the behaviour of individual chromosomes during mitosis and meiosis.

Chromosomes in different spermatogonia were found to exhibit different degrees of contraction during metaphase. In some spermatogonia, the chromosomes are short and thick, indicating strong contraction, while in others they are thin and long, suggesting a lesser degree of spiralization. Similar behaviour of chromosomes was described in insects (White, 1935) and in mammals (Koller, 1937).

CHROMOSOME BEHAVIOUR DURING MEIOSIS.

Butarin (1933–1934) described spermatogenesis in a six-months-old ram. In the present material, however, rams of that age did not show spermatogenesis at all, or only very few primary spermatocytes were encountered. On the other hand, in a five-years-old Leicester ram spermatogenesis was very active, and a great number of cells showing meiotic divisions were seen.

The prophase stages of meiosis are very similar to those found in various other mammals. Chiasmata, representing the loci of genetical crossing over, were observed at diplotene. Their number and position, however, could be determined accurately only at late diplotene and diakinesis (figs. 4, 5). The chiasma frequencies were counted in the late diplotene stage and compared with those found in diakinesis and during the first meiotic metaphase.

The number of chiasmata in ten late diplotene nuclei was found to vary from 60 to 73 per nucleus. The high number of chiasmata in some of the larger bivalents is probably due to overlapping loops being mistaken for chiasmata. Only five diakinesis nuclei were analysed and the number of chiasmata was found to range between 58 and 64 per nucleus. At metaphase the number of chiasmata in ten nuclei varied from 56 to 62. The data indicate a reduction in the number of chiasmata at diakinesis and metaphase, as compared with diplotene (Table II).

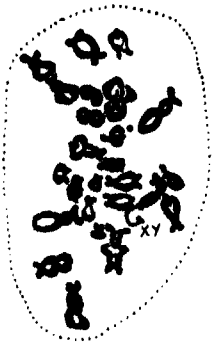


FIG. 4.—Late diplotene stage showing the symmetrical XY bivalent. × 3700.

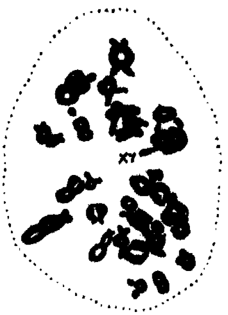


FIG. 5.—Diakinesis stage showing the asymmetrical XY bivalent. × 3700.

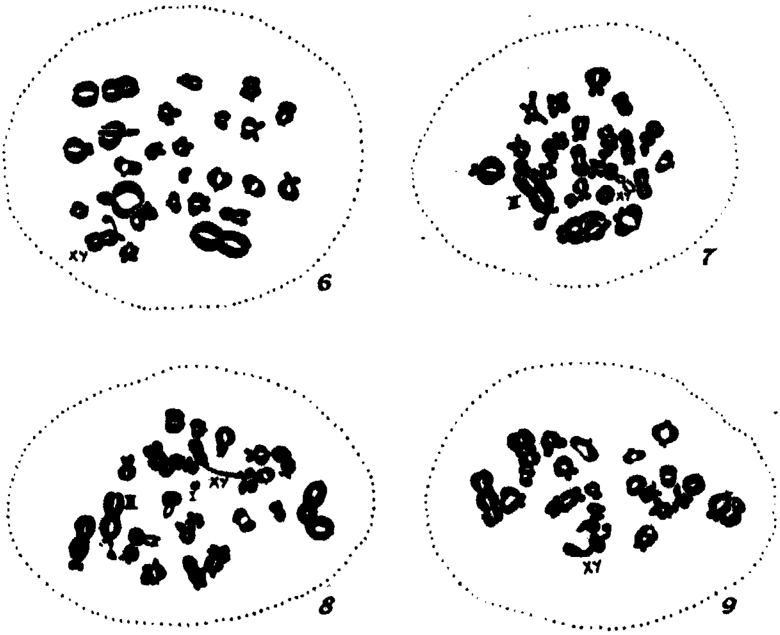
TABLE II.—CHIASMA FREQUENCIES AT LATE DIPLOTENE, DIAKINESIS, AND METAPHASE.

	Total No. of Biv.*	No. of Bivalents with							Total Xta. *	Total Term.	Mean No. of Xta per Nucleus.
		1 Xa.†	2 Xta.‡	3 Xta.	4 Xta.	5 Xta.	6 Xta.	7 Xta.			
Diplotene .	260	3	137	98	14	6	1	± 1	670	186	67·0 ± ·922
Diakinesis	130	..	80	34	6	..	..	..	306	139	61·2 ± ·837
Metaphase	260	3	207	42	8	..	..	..	575	330	57·5 ± ·427

* The XY bivalent is not counted.
† Xa = chiasma. ‡ Xta = chiasmata.

The largest bivalents are at least four to five times larger than the smallest ones, and the average number of chiasmata in the largest bivalents during metaphase is between 3 and 4, while that of the smallest bivalent is 2 (figs. 6, 7, 8, 9). It can be seen that there is no direct proportionality between chromosome length and number of chiasmata. In this respect the chromosomes of the sheep exhibit the same behaviour as various other organisms (Darlington, 1937). Table II shows that there is an increase in the number of bivalents in diakinesis and metaphase, with a decrease in the number of chiasmata at the expense of bivalents with a large number of chiasmata; the number of terminal chiasmata is also higher at metaphase than it was at diplotene and diakinesis.

At diakinesis and at metaphase, some bivalents were found showing peculiar structure; one part of the bivalent is thick and strongly contracted, while the other part remains very thin (figs. 5, 8). The configuration of the larger bivalents, as determined by the number and position of chiasmata, was found to be very similar in different nuclei. Similar behaviour in the bivalents of the pig was described by Crew and Koller (1939), who also discussed the genetical implication of restricted chiasma formation.



FIGS. 6–9.—Polar view of first meiotic metaphase. The sex bivalent is indicated, and two metaphases are shown, with trivalent and univalent. × 3700.

The type of chiasmata could not be identified owing to the strong contraction of bivalents during diakinesis and metaphase. In polar view at meiotic metaphase it can be seen that the bivalents are distributed on the equatorial plate without undue crowding. This is due to the formation of a broad spindle.

In a few primary spermatocytes during meiotic metaphase a univalent and a trivalent were found (figs. 7, 8). It is assumed that two small chromosomes have failed to pair, and while one is left unpaired as a univalent, the other is associated with one of the large bivalents; the association is apparently terminal. This behaviour is explained by assuming that two pairs of chromosomes, consisting of two large and two small chromosomes, have homologous regions which are terminally located. The very low incidence of trivalent formation and the absence of quadrivalents indicate that this homologous region is very short. It is not improbable that only the telomeres are homologous, and therefore the association may be described as "terminal junction," without chiasma formation, similar to that described by Slack (1938) in Corixidæ.

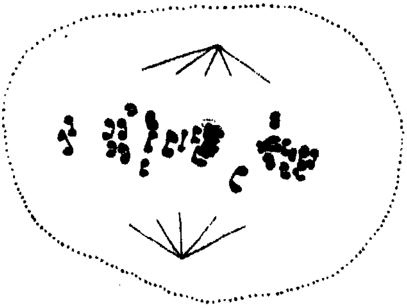


FIG. 10.—Side view of second meiotic metaphase. × 3700.

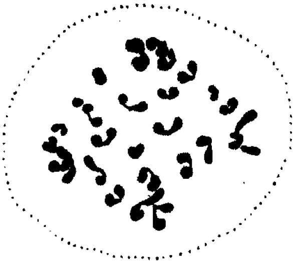


FIG. 11.—Polar view of second meiotic metaphase. × 3700.

The separation of the members of bivalents at the first meiotic anaphase is more or less simultaneous. In some instances the smaller bivalents separated before the larger ones. Lagging of the larger bivalents is probably due to the higher number of chiasmata and also to the type of chiasma, it being known that the non-compensating chiasmata lead to lagging of bivalents at first anaphase (Darlington, 1937).

After the first anaphase the chromosomes arrange themselves into an equatorial plate on the pole, an interphase being absent. During the second meiotic metaphase the sister chromatids of the chromosomes separate and are held together only in the region of the centromere (figs. 10, 11). The haploid chromosome number in the secondary spermatocytes is 27.

THE STRUCTURE OF THE SEX CHROMOSOMES.

All the authors who have investigated the chromosomes of the sheep agree that the male is the heterogametic sex. Wodsedalek (1929) alone holds the view that the male has only one sex chromosome. Novikov

(1935), Butarin (1935 *a*), and Shivago (1930) consider that the Y is the smallest chromosome in the diploid chromosome complement, while the X belongs to the group of the largest chromosomes.

At the first meiotic division the sex chromosomes are associated and form the XY bivalent. During early prophase this bivalent is strongly heteropycnotic, precociously condensed, and can be easily identified. The method of pairing of the X and Y could not be determined. The heteropycnotic XY consists of two more or less well-defined regions, which differ in staining capacity: one is heavily stained, while the other is made up of slightly stained diffuse material. Its shape resembles a "frying-pan" or a ring. During the growth period, when the autosomal bivalents are more or less indiscernible, the heteropycnotic sex bivalent exhibits the same staining reaction as before. The XY complex shows great variation in shape, because of the varying degree of contraction and condensation. The structure of the sex bivalent can be analysed at late diplotene and diakinesis.

Two different configurations of the XY were identified at diakinesis and metaphase; one type resembles an autosomal bivalent with three, or two, interstitial chiasmata, except that it has a thin, double thread terminally attached (figs. 4, 6, 7, 8); this is designated the symmetrical sex bivalent. The other type is an unequal bivalent, and is described as the asymmetrical sex bivalent (figs. 5, 9); the members of this bivalent are always attached terminally or subterminally. It was found that the asymmetrical XY was smaller than the symmetrical. These two types of sex bivalent as seen during meiosis enable the determination of the inner structure of the sex chromosomes of the sheep. It is known through the work of Koller and Darlington (1934) and of Koller (1936) that the configurations of the XY bivalent are determined by the position of the pairing or homologous segment of the X and Y with respect to the centromere, and the localisation of chiasmata in the pairing segment. It is assumed that the large X chromosome is made up of three regions: (*a*) the differential segment, which is not represented in the Y, (*b*) the inner pairing segment which includes the region lying between the centromere and the differential segment, and (*c*) the outer pairing segment (fig. 12). In the Y chromosome we can distinguish only the two pairing segments; the differential segment is either entirely absent or very small. The frequency of the two types more or less indicates the relative length of the two pairing segments. In 33 primary spermatocytes of the ram in which the XY bivalent could clearly be seen, 23 symmetrical and 10 asymmetrical sex bivalents were identified. This suggests that in the sheep the outer pairing segment of the sex chromosomes is at least twice as large as the inner. The minimum number of chiasmata in the inner and in the outer pairing segments is one, though more frequently two chiasmata are formed in the outer. The symmetrical sex bivalent of the sheep differs from that of the rat, hamster (Koller, 1938), or man (Koller, 1937) in that the outer pairing segments of the X and Y have more than one chiasma.

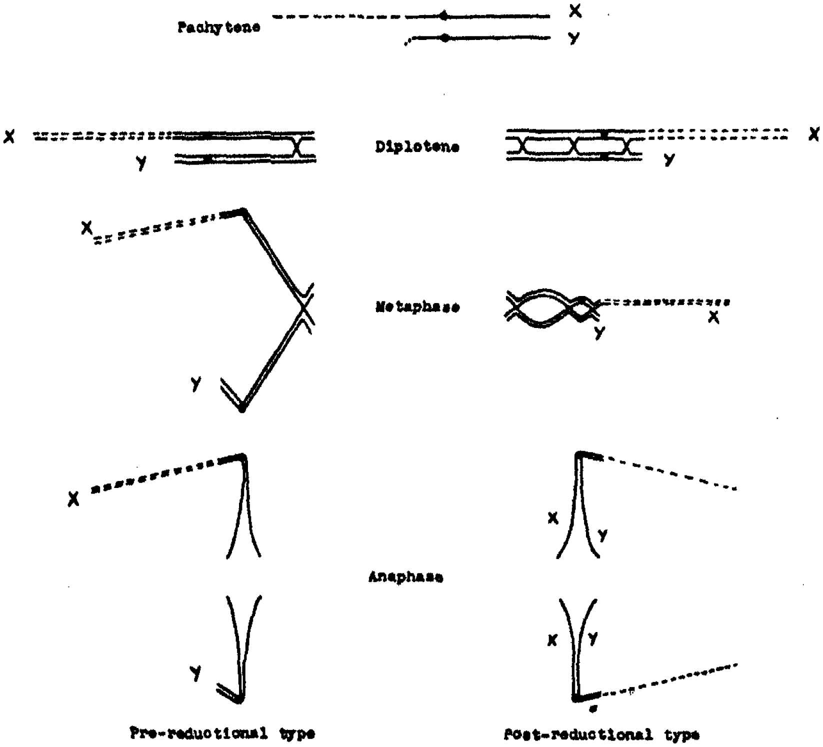


FIG. 12.—Diagram illustrating the structure of the XY bivalent.
——— = pairing segment; - - - - - - = differential segment.

It is accepted now (*cf.* Koller and Darlington, 1934) that the presence of a chiasma in the inner pairing segment of the XY bivalent leads to equational separation of the sex chromosomes during the first meiotic anaphase, and reduction of the X and Y takes place at the second meiotic anaphase. This type of sex bivalent is the post-reductional type, while the other, where chiasmata are formed only in the outer pairing segment, is the pre-reductional type, because the segregation of X and Y occurs at the first meiotic anaphase. These facts explain why two adjacent sister secondary spermatocytes, which are the product of division of the same primary spermatocyte, often exhibit difference in respect of chromosome morphology: as a result of post-reduction of the XY bivalent, one secondary spermatocyte has the X, the other the Y chromosome.

The cytological analysis given above shows that, in the sheep, chromosome behaviour, including that of the sex chromosomes, exhibits similar properties to those described in other mammals.

The author wishes to acknowledge his indebtedness to Professor F. A. E. Crew for his hospitality, and also to Dr P. C. Koller, under whose guidance the work was carried out.

SUMMARY.

1. The diploid chromosome number in the sheep is 54. The chromosomes differ in size and shape.

2. The sex chromosomes consist of a large X with a median centromere, and a smaller Y with a subterminal centromere.

3. There is a slight reduction in the number of chiasmata in diakinesis and metaphase, as compared with that at diplotene.

4. Two kinds of XY bivalents are formed during meiosis, the symmetrical and the asymmetrical. The former occurs twice as frequently as the latter.

5. Trivalents and univalents were seen in a few primary spermatocytes, indicating that a small terminal region is represented four times in the chromosome complement of one individual sheep.

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XXI.—The Swimming and Burrowing Habits of the Amphipod *Urothoë marina* (Bate).

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INTRODUCTION.

THE intertidal and shallow-water sands of the coasts of the British Isles contain, in many cases as the dominant amphipod fauna, species of the three genera *Haustorius*, *Urothoë*, and *Bathyporeia* of the family Haustoriidæ, often coexistent in the same habitat and obtainable in the same samples. The swimming and burrowing mechanisms of *Haustorius arenarius* have been described by Dennell (1933), and I have described (1939) the mechanisms in some species of the genus *Bathyporeia*. This paper deals mainly with *Urothoë marina*; no reference is made to the other species of the genus, although *U. brevicornis* has been examined and found to be similar to *U. marina*. Since all species of the genus *Urothoë* are very similar morphologically and live in similar habitats, it may be assumed that the following description of *U. marina* will apply to them also, but I have had no opportunity of examining the more southern species *U. grimaldii* or *U. pulchella*. Crawford (1937) states that the burrowing habits of *U. brevicornis* and *U. grimaldii* var. *poseidonis* are very similar to those of *Haustorius*.

This work was in part carried out at the Millport Marine Biological Station, where *U. marina* occurs abundantly in Balloch Bay and *U. brevicornis* in Kames Bay. The more widely distributed *U. elegans*, although recorded from the Clyde sea basin, does not occur intertidally on the island of Great Cumbrae. I am indebted to Mr R. Elmhirst, Director of the Scottish Marine Biological Station, Millport, for facilities afforded me there.

A COMPARISON OF THE SWIMMING HABIT OF *BATHYPOREIA*, *UROTHOË*, AND *HAUSTORIUS*.

The swimming movement of *Urothoë marina* is essentially similar to that described for *Haustorius* and *Bathyporeia*, and is dependent upon a metachronal rhythm of the three pairs of pleopods which draw a current of water through a ventral groove. The form of the pleopods in the three genera differs, and is directly related to the width and depth of the ventral groove, which is deep and narrow in *Bathyporeia* but comparatively shallow and broad in *Urothoë* and *Haustorius*. Thus the peduncle of the pleopods is short and broad in the latter two genera but long and narrow in the former. A pair of pleopods is linked together to act as one by coupling hooks (fig. 1, j_1) borne on the inner distal edge of the peduncle and by modified bristles on the basal joint of the inner ramus (fig. 1, j_2). The modified bristles are absent in *Haustorius*, due to an expansion of the peduncle on its inner border which separates the bases of the inner rami some distance from each other.

The ventral groove is formed in essentially the same manner in all three genera. In the anterior part of the peræon the sides of the groove are formed by the expanded coxal plates of the first and second gnathopods and first and second peræopods, and in the posterior part by the expanded "basis" of the third, fourth, and fifth peræopods, in the anterior region of the pleon by the epimeral plates of the first three pleonic segments. The groove is further defined in the peræon region by the expansion of some of the more distal segments of the limbs, and also by the development of an armature of feathered bristles. It is characteristic of the three genera that feathered bristles only occur, apart from those borne on the rami of the pleopods, in association with the formation of the ventral groove. They either assist in closing off the sides of the groove between the succeeding limbs or extend inwards to form an incomplete ventral wall to the groove. The segments of the limbs which are expanded, or carry feathered bristles, or combine both characters, vary in the three genera. *Bathyporeia* is the least specialised in this respect in that only the merus of the third peræopod is expanded, and it carries an edge row of feathered bristles which partly close off the groove ventrally in the peræon region, with a well-defined group of feathered bristles on the distal inner edge of the second gnathopod which close the groove ventrally and

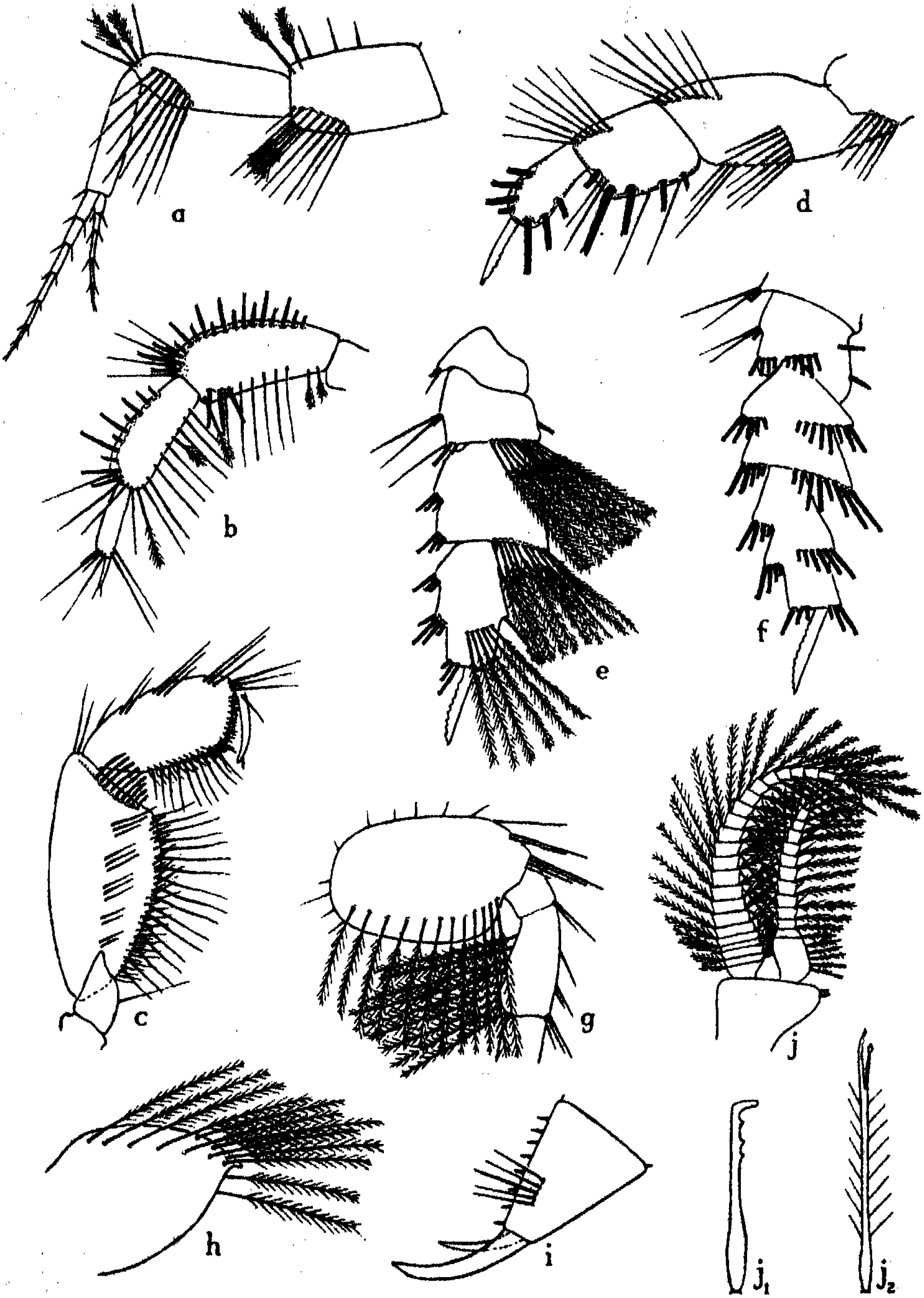

FIG. 1.—*Urothoë marina*: *a*, antennule; *b*, antenna of ♀; *c*, first gnathopod; *d*, first peræopod; *e*, third peræopod, inner face view; *f*, third peræopod, outer face view; *g*, basis and merus of fourth peræopod; *h*, second epimeral plate; *i*, first uropod; *j*, pleopod; j_1, coupling hook on basal joint; j_2, modified feathered bristle from base of inner margin of inner ramus.

anteriorly. *Urothoë marina* shows an advance in this specialisation of the ventral groove in that the third peræopod is expanded along its whole length, apart from the dactylus, and the ischium, merus, and carpus each carry a well-defined row of feathered bristles on their inner face (fig. 1, *e*), with a few feathered bristles on the distal inner edge of the basis of the second peræopod. In addition, the basis and merus of the fourth peræopod (fig. 1, *g*) carry on their posterior edges a row of feathered bristles which assist in closing off the space between this limb and the fifth peræopod. The space between the fifth peræopod and the body, due to the fact that this limb, like the fourth, is held at an angle diverged from the body, is closed by the development of feathered bristles on the second epimeral plate (fig. 1, *h*). *Haustorius* is still more specialised in this respect in that some of the peduncular segments of the antennules and antennæ are expanded and carry edge rows of feathered bristles which form an effective addition to the sides of the groove in its anterior part. The first and second peræopods also carry an edge row of feathered bristles on the basis, merus, and carpus. The third, fourth, and fifth peræopods are expanded almost along their whole length, and carry fringes of feathered bristles directed inwards to close the groove ventrally. This is particularly so on the fourth peræopod. The space which lies behind the fifth peræopod is closed by the development of feathered bristles on the second and third epimeral plates, and by bristles on the basis and merus of the fifth peræopod.

The water current drawn through this ventral groove by the beating of the pleopods in a metachronal rhythm enters, in all three genera, anteriorly and along two main paths. The primary path is from immediately in front of the head, between and ventral to the antennæ. In *Bathyporeia* this current passes in well beneath the mouth parts and between the right and left anteriorly directed second gnathopods and first and second peræopods. In *Urothoë* it passes partly between the first and second gnathopods, which are directed forwards and form, by the approximation of their right and left limbs, a channel immediately ventral to the mouth parts, and partly ventral to the gnathopods. In *Haustorius* the primary current passes immediately over the mouth parts between the right and left first and second gnathopods. The secondary current enters anteriorly but laterally, and at a different point in *Bathyporeia* from that in *Urothoë* and *Haustorius*. In *Bathyporeia* it enters between the anteriorly directed second peræopod and posteriorly directed third peræopod, whilst in *Urothoë* and *Haustorius* it enters between the anteriorly directed second gnathopod and posteriorly directed first peræopod. This difference arises from the different function performed by the first two pairs of peræopods in the burrowing of *Bathyporeia* and that of *Urothoë* and *Haustorius*.

The morphological differences referred to above in the degree of expansion of the segments of the limbs and the varying degrees of development of the armature of feathered bristles seem to determine the swimming position and the behaviour of the three genera in the water. *Bathyporeia* always swims with the dorsal surface uppermost with a rapidity of movement not approached by *Urothoë* or *Haustorius*. It immediately sinks if pleopod movement ceases and it has no buoyancy in the water. *Haustorius*, on the other hand, as Dennell says, "swims on its back . . . is able to swim quite slowly . . . it can hover in one place for a considerable time." It has considerable buoyancy in the water. *Urothoë* occupies a middle position; it may swim either way up, is not so rapid in the water as *Bathyporeia*, has some buoyancy, sinking somewhat slowly if pleopod movement ceases.

The general form of the body in the three genera further amplifies the differences between their behaviour in the water. *Bathyporeia* is narrow and elongate in outline, with the last three pleonic segments well developed and in the swimming position held extended and in line with the peræon. *Urothoë* is less elongated, with a broader body, arched from side to side, with the last three pleonic segments somewhat reduced and partly drawn under the three anterior segments. In both genera the third uropods are well-developed appendages carrying feathered bristles on their margins, and seem to serve as steering organs to the swimming movement in that they can be extended to the right or left as change in the direction of movement demands. In *Haustorius* the body is short and very broad, with the last three pleonic segments drawn in under the first three pleonic segments so that they are invisible when viewed from above. The third uropods are short stout appendages without an armature of feathered bristles. The steering function is performed by the fifth peræopod, which extends backwards beyond the animal.

THE BURROWING HABIT OF *UROTHOË MARINA*.

The burrowing movement may commence with the animal lying either on its dorsal or ventral surface. In the former case the burrowing movement is a direct continuation of the swimming movement; in the latter case swimming ceases before the burrowing movement begins. Thus *Urothoë* occupies a middle position between *Bathyporeia*, in which the burrowing movement is always a direct continuation of the swimming movement, and *Haustorius*, which has of necessity to perform righting movements to bring the dorsal surface uppermost before burrowing can begin.

The burrowing movement in *Urothoë* is dependent upon (*a*) the beating of the pleopods drawing through the ventral groove a current of water, the velocity of which is sufficient to throw sand particles from the groove backwards beyond the animal, and (*b*) the action of the first and second gnathopods. The first and second gnathopods are robust appendages almost equally developed and essentially similar. They are geniculate at the ischium, with the distal portions of the limbs directed forwards to lie immediately beneath the mouth parts. The second gnathopods are carried ventral to the first gnathopods and lie nearer together. The carpus and propodus joints are expanded on their anterior edges and bear a fringe of bristles directed inwards (fig. 1, *c*), the bristles of a pair interlocking. Thus a channel is formed between the gnathopods, roofed dorsally by the mouth parts, on the sides by the joints of the limbs, and ventrally by the interlocking setæ. Along this channel the water current, in part, enters the ventral groove, carrying with it sand particles. The gnathopods can be extended forwards and drawn back to assist in the displacement of the sand particles which pass directly under the mouth parts between the "bases" of the gnathopods into the main channel of the ventral groove, where they are caught in a further stream of water entering from the sides between the second gnathopod and first peræopod. No sand particles enter the ventral groove other than those from this gnathopod channel.

The burrowing movement is assisted by the action of the antennæ (fig. 1, *b*) and antennules (fig. 1, *a*), by the five pairs of peræopods and by the uropods. Immediately on touching the sand the antennæ commence a vertical up-and-down movement, the right and the left antenna alternating in the rhythm. The antennæ diverge at right angles to the body and are set wide apart outside the antennules. The fourth and fifth joints of the peduncle of the antenna carry a row of spines along the anterior border and a row of simple bristles on the posterior border. These spines and bristles are directed upwards into the sand, and form a platform on which the sand particles are pushed away from the immediate vicinity of the forefront end of the animal. The antennæ thus assist in clearing a space beneath the head in which the antennules can operate. The development of the flagellum in the adult male does not affect the movements of the peduncle. The antennules are set close together anteriorly with the flagellum carried well forward. The flagellum is rotated from its base in an anticlockwise direction, and this action sweeps sand particles backwards to within reach of the gnathopods. However, it cannot be imagined that this action is of much assistance in burrowing, but it is paralleled by that of the flagellum of the antenna in the females and young males of *Bathyporeia.*

The first (fig. 1, *d*) and second peræopods are essentially similar in form, function, and armature. They are held vertically beneath their corresponding segments with the carpus and propodus diverged outwards and forwards. Each limb is capable of a side-to-side movement and, to a less extent, a backward and forward movement. The carpus and propodus carry spines on their margins which are directed amongst the sand particles. They are thus adapted to push sand particles away from the sides of the forward moving animal to allow of a freer movement of the body through the sand. The third, fourth, and fifth peræopods are robust appendages with expanded joints carrying groups of spines on their anterior and posterior margins, and in the case of the third peræopod on the outer face (fig. 1, *f*). The distal portions of the limbs diverge, and they assist in propelling the animal forward through the sand by pushing against the particles, the spines serving to increase the purchase against the particles. As the animal moves forward the appendages are raised and moved forward to obtain a fresh purchase.

Normally, the pleon is held rigid and extended, and the first (fig. 1, *i*) and second uropods, which are poorly developed compared with those of *Bathyporeia*, play only a small part in the burrowing process. Occasionally, when an accumulation of sand particles beneath the pleon blocks the groove, a flexure of the pleon will take place and the groove is cleared. The third uropods are well developed, with the two rami of equal length and fringed with long feathered bristles. In forward movement they are held extended and in line with the pleon, and extended laterally so as to form a platform which prevents sand particles falling from above into the groove. When the pleon is flexed they are raised into the vertical plane, as in *Bathyporeia*, and their delicate structure remains undamaged by pleon flexure. A further function is indicated in that when the animal turns to the left or right, either in swimming or burrowing, the uropods converge in the direction of the turn, so that they assist in steering the animal much as a rudder steers a boat.

A COMPARISON OF THE BURROWING HABIT OF *BATHYPOREIA*, *UROTHOË*, AND *HAUSTORIUS*.

A comparison of the burrowing mechanism in the three genera reveals some points of similarity, particularly in the nature of the armature of the appendages. It is to be expected, since the three genera inhabit sands of similar mechanical composition, that similarities would occur, but as to how far such similarities are due to common descent or to convergence is debatable. Attention has already been drawn to their similarity in the formation of the ventral groove. This feature is, however, typical of gammarid amphipods in general. This study serves to indicate the function of some of the various forms of setal armature. (a) *The feathered bristles.* These are in all cases, apart from those borne on the pleopods, associated with the formation of the ventral groove and the water current entering the groove. They assist in closing the gaps between successive appendages and in forming a floor to the ventral groove; their interlocking setæ prevent the fall of sand particles into the groove and form an effective filter to any water current passing through them. Attention has already been drawn to the arrangement of the feathered bristles on various appendages in the three genera. (b) *The simple bristles.* These are borne on those appendages which draw sand particles into the ventral groove with the assistance of the water current. The bristles occur in groups mainly on the anterior edges of the two pairs of gnathopods in *Urothoë* and *Haustorius*, and on the second gnathopod and first two pairs of peræopods in *Bathyporeia.* The bristle appears to be a somewhat delicate structure for such a purpose, and apart from the water current would probably be unable to function. (c) *The spines.* These are developed at points where purchase against the sand particles is required for pushing the animal forward through the sand or pushing the sand particles sideways to assist in clearing a space for the forward moving animal. They are more numerous in the broader *Urothoë* and *Haustorius* than in the narrower *Bathyporeia.* Attention has been drawn to the presence of these spines on the first three pairs of peræopods in *Urothoë* and on the third peræopod in *Bathyporeia*, and to a less extent on the anterior margins of the last two peræopods in both genera. A detailed study of the appendages of *Haustorius* shows a distribution of the spines similar to that on the appendages of *Urothoë*. (d) *The plumose hairs.* These are developed on the ventral surface of the peduncular joints of the antennules and antennæ in all three genera, and hang downwards in such a position that the main water current which enters anteriorly between the antennæ must pass over them. They are delicate in structure and, from their position, probably sensory in function.

A comparison with the armature of *Gammarus chevreuxi*, as described by Sexton (1925), reveals that the Haustoriidæ show a specialisation in the more extensive development of certain types of setæ and bristles at various points for the purpose of burrowing, a specialisation that may be expected as an adaptation to a similar habitat.

A comparison of the burrowing mechanisms shows that there is a fundamental similarity between *Urothoë* and *Haustorius* in that the first and second gnathopods in both genera assist in drawing particles into the ventral groove. In *Haustorius* the two pairs of gnathopods are flexed at the ischium, and can be extended forwards with the second pair working ventral to and outside the first and out of rhythm with them. Sand particles drawn back by the gnathopods are immediately within the vortex of the pleopods, which reach far forward due to a shortening of the body. In *Bathyporeia*, however, the first gnathopod is small, closely applied to the mouth parts, and is a feeding appendage taking no part in the burrowing process, whilst the second gnathopod is assisted by the first two pairs of peræopods in loosening the sand particles and drawing them back into the ventral groove. In *Urothoë* and *Haustorius* the first two pairs of peræopods have a lateral movement which assists in pushing away sand particles from the sides of the body, and thus have a fundamentally different action and function to those of *Bathyporeia*. Further, the antennules and antennæ of *Urothoë* and *Haustorius* assist in clearing the sand particles away from the rounded anterior end of the body and carry an appropriate armature for this purpose; in *Bathyporeia* they have no armature for such a purpose, and the first joint of the antennule serves as a forward extension of the head, which is pointed and wedge-shaped for rapid penetration of the sand. Attention has already been drawn to the well-developed pleon region of *Bathyporeia* and its reduction in *Urothoë* and *Haustorius*, so that in the former genus the action of the first and second uropods is of very considerable importance in clearing the ventral groove. This is further shown by the fact that *Bathyporeia* can burrow when lying on moist sand, whilst *Haustorius*, as Dennell states, when "placed on the surface of the sand is incapable of burrowing and is completely helpless." *Urothoë* resembles *Haustorius* in this respect.

There is thus a fundamental similarity between *Urothoë* and *Haustorius*, and the two genera are closely allied. It is probable that the burrowing mechanism in *Bathyporeia* is paralleled more closely in other sand-dwelling amphipods, and that it will show a closer resemblance to that of other genera than to *Urothoë* and *Haustorius*. It may be noted that it is difficult to imagine the evolution of the more specialised mechanisms of *Urothoë* and *Haustorius* from the more primitive mechanism of *Bathyporeia*.

SUMMARY.

A detailed description of the burrowing mechanism of *Urothoë marina* is given, in which it is shown that burrowing is dependent upon a powerful water current set up by the pleopods and drawn through a ventral groove.

The first and second gnathopods form a channel beneath the mouth parts along which the sand particles loosened by their action are drawn into the ventral groove. The main water current enters anteriorly along the gnathopod channel, and a secondary current enters laterally between the second gnathopod and first peræopod. The first two pairs of peræopods serve to deflect sand particles laterally, and the last three pairs of peræopods assist, by the purchase of spines against the sand particles, in pushing the animal forward through the sand. The first and second uropods are only occasionally employed, by pleon flexure, in clearing the ventral groove. The third uropods are partly steering appendages, and also form a platform to prevent sand particles falling from above.

The swimming movement is generally similar to that of *Bathyporeia* and *Haustorius*, and there is a graded ability from the former, through *Urothoë* to the latter, in maintaining a swimming position in the water.

A study of the armature of *Bathyporeia*, *Urothoë*, and *Haustorius* shows that similar setæ and bristles are developed on those appendages which serve a similar purpose.

A comparison of the burrowing mechanisms of the three genera shows that *Urothoë* and *Haustorius* have essentially similar mechanisms and are probably closely allied genera, but that *Bathyporeia* may show closer affinity to other genera of sand-dwelling amphipods.

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XXII.—Random Paths in Two and Three Dimensions. By Professor W. H. McCrea, M.A., Ph.D., and F. J. W. Whipple, M.A., Sc.D., F.Inst.P. (With One Text-figure.)

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INTRODUCTION.

1. In a previous note (McCrea, 1936) [*] the following problem was enunciated: A rectangular lattice is given; a particle P moves from one lattice-point to another in such a way that, when it is at any interior point, it is equally likely to move to any of its four neighbouring points. P is liberated at any given lattice-point and it is required to find the probability that it will ultimately reach any stated point in the boundary of the lattice, assuming that on arrival at a boundary point its movement ceases.

Explicit solutions were given in the note only in certain cases of arrays consisting of one or two finite or infinite rows of points.

The first object of the present paper is to give the solution for the general case of a finite rectangular array. The various limiting cases of infinite arrays are then considered. The initial problem is meaningless for an array filling the whole plane, but it is shown that a modification of the problem can be stated and solved in this case, and that the solution can be used to recover that of the previous problem by an application of the method of images. Finally, the extension of the work to a three-dimensional array is studied, with special reference to the case of an array filling the whole space, which presents an essential difference from the analogous two-dimensional case.

After the publication of the previous note, the attention of its author was drawn to a paper (referred to as C.F.L.) (Courant, Friedrichs, and Lewy, 1928) in which the problem had already been stated, and some general properties of the solution had been established. The authors of that paper did not, however, seek explicit solutions, and we have not been able to find that any such have subsequently been given.

2. Let an array consist of m rows of n lattice-points in which P can wander, together with a border of points from which there is no return. The complete array consists then of $(m+2)$ rows of $(n+2)$ points; the

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[*] In that note "cells" took the place of the present lattice-points.

rows are numbered from 0 to $m+1$, the points of any row from 0 to $n+1$. P is assumed to start from the point (a, b).

Suppose for definiteness that P moves unit lattice distance per second. If it is at a specified interior point * at a particular instant, the chance that one second later it will be found at any given one of the four neighbouring points is $\frac{1}{4}$. The study of the probability of finding P at a particular point is then equivalent to the study of the distribution in the lattice of a fluid which moves with a velocity of one lattice distance per second and which diffuses through the lattice in such a way that at each point one-quarter of the fluid arriving there flows away along each of the four lattice-lines meeting at the point. More particularly, following C.F.L., we may suppose unit quantity of the fluid to be liberated at the point (a, b). Then the quantity which has accumulated after an indefinitely long time at a given boundary point measures the probability that P will ultimately reach that point.

Under these circumstances, let $F(p, q)$ be the total quantity of substance which leaves the point (p, q) during this time. The fact that $F(p, q)$ is finite is proved by C.F.L.; it may, of course, be greater than unity since some of the substance may revisit the point any number of times. It is clear that $F(p, q)$ is also the sum of the probabilities that P leaves the point (p, q) at least once, at least twice, at least three times, and so on, *i.e.* the probability that P leaves (p, q) precisely once, plus twice the probability that P leaves it precisely twice, plus three times the probability that P leaves it three times, etc. Or, again, $F(p, q)$ may be regarded as the *expectation* that P will visit (p, q) before finally emerging at a boundary point.

We have then the general difference-equation for $F(p, q)$,

$$F(p, q)=\tfrac{1}{4}[F(p-1, q)+F(p+1, q)+F(p, q-1)+F(p, q+1)], \qquad (2.1)$$

where (p, q) is an interior point distinct from the point (a, b). This merely expresses the fact that all the substance which flows from such a point must have reached it from one of the four neighbouring points. In the case of the point (a, b) we have in addition the fact that unit quantity of substance is initially introduced there from outside, which gives

$$F(a, b)=1+\tfrac{1}{4}[F(a-1, b)+F(a+1, b)+F(a, b-1)+F(a, b+1)]. \qquad (2.2)$$

Since no substance can, by hypothesis, leave a boundary point, we have

$$F(0, q)=0, \quad F(m+1, q)=0, \quad (0<q<n+1); \qquad (2.31)$$

$$F(p, 0)=0, \quad F(p, n+1)=0, \quad (0<p<m+1). \qquad (2.32)$$

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* *Point* will be taken to mean lattice-point; *neighbouring point* will be taken to mean a point at unit lattice distance from a given point.

Consider now any boundary point, say $(p, 0)$ in the boundary $q=0$; it possesses one and only one neighbouring interior point $(p, 1)$. All the substance which reaches $(p, 0)$ must therefore flow from $(p, 1)$, and must, in fact, represent one-quarter of the total amount flowing from $(p, 1)$. *Thus the probability that P ultimately reaches the boundary point in question is* $\frac{1}{4}F(p, 1)$. We note incidentally that P cannot reach the boundary points $(0, 0)$, $(m+1, 0)$, $(0, n+1)$, $(m+1, n+1)$ since these possess no neighbouring interior points.

The solution of (2.1) therefore leads to the solution of the problem originally enunciated, and gives, besides, the expectation that P will visit each lattice-point before reaching the boundary.

That the solution of (2.1), subject to the conditions (2.2), (2.3), exists and is unique follows by a standard argument as given, for example, by C.F.L. A general check on the foregoing interpretations is obtained by summing (2.1) for all p, q from 1 to m, 1 to n, respectively, except $(p, q)=(a, b)$, and using (2.2) for this point. We get

$$\sum_{p=1}^{m}\{\tfrac{1}{4}\mathrm{F}(p, 1)+\tfrac{1}{4}\mathrm{F}(p, n)\}+\sum_{q=1}^{n}\{\tfrac{1}{4}\mathrm{F}(1, q)+\tfrac{1}{4}\mathrm{F}(m, q)\}=1. \qquad (2.4)$$

From the interpretation just given, the L.H.S. is the total probability that P will reach the boundary. So (2.4) verifies that this probability is unity, as was to be expected.

3. The difference-equation (2.1) is satisfied by

$$\mathrm{F}(p, q)=\mathrm{A}\exp(ip\alpha+q\beta),$$

provided that the constants α, β satisfy the condition

$$\cos\alpha+\cosh\beta=2.$$

Hence we can construct solutions of (2.1) valid for $q<b$ and for $q>b$, and satisfying the boundary conditions (2.3), by writing

$$\mathrm{F}_1(p, q)=\sum_{r=1}^{m}\mathrm{C}(r)\sin\frac{pr\pi}{m+1}\sinh q\beta_r\sinh[(n+1-b)\beta_r], \qquad (q<b),$$

$$\mathrm{F}_2(p, q)=\sum_{r=1}^{m}\mathrm{C}(r)\sin\frac{pr\pi}{m+1}\sinh[(n+1-q)\beta_r]\sinh b\beta_r, \qquad (q>b),$$

where β_r is defined by

$$\cos\frac{r\pi}{m+1}+\cosh\beta_r=2.$$

The constants $\mathrm{C}(r)$ are to be chosen so that (2.1) is satisfied when $q=b$, $p\neq a$, and (2.2) is satisfied when $q=b, p=a$.

These conditions are that

$$\tfrac{1}{4}\sum_{r=1}^{m}\mathrm{C}(r)\sin\frac{pr\pi}{m+1}\sinh\beta_r\sinh(n+1)\beta_r = 0 \qquad \text{if } p\neq a,$$
$$= 1 \qquad \text{if } p = a.$$

Now

$$\sum_{r=1}^{m}\sin\frac{pr\pi}{m+1}\sin\frac{ar\pi}{m+1} = \tfrac{1}{2}\sum_{r=1}^{m}\cos\frac{a-p}{m+1}r\pi - \tfrac{1}{2}\sum_{r=1}^{m}\cos\frac{a+p}{m+1}r\pi$$
$$= 0 \qquad \text{if } p\neq a,$$
$$= \tfrac{1}{2}(m+1) \qquad \text{if } p = a.$$

It follows that the conditions are satisfied if

$$\mathrm{C}(r) = \frac{8}{m+1}\sin\frac{ar\pi}{m+1}\Big/\sinh\beta_r\sinh(n+1)\beta_r.$$

The required solution of our problem is therefore given by

$$\mathrm{F}_1(p,q) = \frac{8}{m+1}\sum_{r=1}^{m}\sin\frac{ar\pi}{m+1}\sin\frac{pr\pi}{m+1}\,\frac{\sinh q\beta_r\sinh[(n+1-b)\beta_r]}{\sinh\beta_r\sinh(n+1)\beta_r}, \qquad (3.11)$$

$$\mathrm{F}_2(p,q) = \frac{8}{m+1}\sum_{r=1}^{m}\sin\frac{ar\pi}{m+1}\sin\frac{pr\pi}{m+1}\,\frac{\sinh[(n+1-q)\beta_r]\sinh b\beta_r}{\sinh\beta_r\sinh(n+1)\beta_r}, \qquad (3.12)$$

the values of β_r *being defined by*

$$\cos\frac{r\pi}{m+1} + \cosh\beta_r = 2. \qquad (3.2)$$

F_1 *is the value of* F *when* $q \leqslant b$, F_2 *the value when* $q \geqslant b$.

The solution as found is unsymmetrical in the ways in which p, q appear. This is a familiar feature of the solution of a difference-equation which itself contains two symmetrically occurring variables. A formally different result is got by interchanging the rôles of p, q throughout the working, but by the uniqueness theorem the result is identical with that given, though the identity is not easy to establish directly.

The probability that P finally emerges to a boundary point, $(p, 0)$, say, is given by the equation

$$\tfrac{1}{4}\mathrm{F}_1(p,1) = \frac{2}{m+1}\sum_{r=1}^{m}\sin\frac{ar\pi}{m+1}\sin\frac{pr\pi}{m+1}\,\frac{\sinh[(n+1-b)\beta_r]}{\sinh(n+1)\beta_r}. \qquad (3.3)$$

The results of the previous note can be recovered from (3.3).

As a simple example of (3.1), take $m=n=5$, $a=b=3$; then $\mathrm{F}(3,3)=23/13 \doteqdot 1{\cdot}77$. So in this case the expectation of P revisiting its starting-point before reaching the boundary is 0·77 (approximately).

Further, if χ be the chance of revisiting the point at least once,

$$1+\chi+\chi^2+\ldots=23/13 \quad \text{and} \quad \chi=10/23 \doteqdot 0{\cdot}43.$$

INFINITE ARRAYS.

4.1. *Semi-infinite Strip.*—If the array extends to infinity in one direction only, say the q-direction, then $n \to \infty$, and the solution is seen from (3.1) to be given by

$$F_1(p, q)=\frac{8}{m+1}\sum_{r=1}^{m} \sin\frac{ar\pi}{m+1} \sin\frac{pr\pi}{m+1} \frac{\sinh q\beta_r \exp(-b\beta_r)}{\sinh \beta_r}, \tag{4.111}$$

$$F_2(p, q)=\frac{8}{m+1}\sum_{r=1}^{m} \sin\frac{ar\pi}{m+1} \sin\frac{pr\pi}{m+1} \frac{\sinh b\beta_r \exp(-q\beta_r)}{\sinh \beta_r}. \tag{4.112}$$

If, however, before proceeding to the limit, we first interchange the rôles of (p, q) in the solution, we get on letting $n \to \infty$,

$$F^1(p, q)=\frac{8}{\pi}\int_0^{\pi} \sin b\lambda \sin q\lambda \frac{\sinh p\mu \sinh[(m+1-a)\mu]}{\sinh \mu \sinh(m+1)\mu} d\lambda, \tag{4.121}$$

$$F^2(p, q)=\frac{8}{\pi}\int_0^{\pi} \sin b\lambda \sin q\lambda \frac{\sinh[(m+1-p)\mu] \sinh a\mu}{\sinh \mu \sinh(m+1)\mu} d\lambda, \tag{4.122}$$

where from (3.2)

$$\cos\lambda+\cosh\mu=2. \tag{4.13}$$

F^1 is the value of F when $p \leqslant a$, F^2 is the value when $p \geqslant a$.

From (4.11), (4.21) we have the identity

$$\int_0^{\pi} \sin b\lambda \sin q\lambda \frac{\sinh p\mu \sinh[(m+1-a)\mu]}{\sinh \mu \sinh(m+1)\mu} d\lambda$$
$$=\frac{\pi}{m+1}\sum_{r=1}^{m} \sin\frac{ar\pi}{m+1} \sin\frac{pr\pi}{m+1} \frac{\sinh q\beta_r \exp(-b\beta_r)}{\sinh \beta_r}, \tag{4.14}$$

where $p \leqslant a$, $q \leqslant b$, with similar results for the other cases. Such identities appear to be difficult to verify directly.

4.2. *Infinite Strip.*—If now the array extends to infinity in two opposite directions, b and q tend to infinity, but their difference $q-b=s$ (say) remains finite. Writing then $F(p, s)$ in place of $F(p, q)$, we have from (4.11)

$$F(p, s)=\frac{4}{m+1}\sum_{r=1}^{m} \sin\frac{ar\pi}{m+1} \sin\frac{pr\pi}{m+1} \frac{\exp(-|s|\beta_r)}{\sinh \beta_r}. \tag{4.21}$$

The forms (4.12) become

$$F^1(p, s)=\frac{4}{\pi}\int_0^{\pi} \cos s\lambda \frac{\sinh p\mu \sinh[(m+1-a)\mu]}{\sinh \mu \sinh(m+1)\mu} d\lambda, \tag{4.221}$$

$$F^2(p, s)=\frac{4}{\pi}\int_0^{\pi} \cos s\lambda \frac{\sinh(m+1-p)\mu \sinh a\mu}{\sinh \mu \sinh(m+1)\mu} d\lambda. \tag{4.222}$$

where (4.13) continues to hold. To establish (4.22) we use the fact that $\lim_{\nu \to \infty} \int_0^\pi \phi(x) \cos \nu x dx = 0$, if $\phi(x)$ is continuous and bounded and independent of ν, in $0 < x < \pi$. Combining (4.21), (4.22) we obtain identities analogous to (4.14).

An instructive check on (4.21) is got by using it to evaluate the total probability that P emerges on one or other of the infinite linear boundaries $p = 0$ and $p = m + 1$. These are found, after some not quite trivial calculations, to be

$$\sum_{s=-\infty}^{\infty} \tfrac{1}{4} \mathrm{F}(1, s) = \frac{m - a + 1}{m + 1}, \qquad \sum_{s=-\infty}^{\infty} \tfrac{1}{4} \mathrm{F}(m, s) = \frac{a}{m + 1}.$$

This result can be got directly from the fact that when P is in any row $p = p'$, the probability that on leaving this row it will pass into the rows $p = p' \pm 1$ is $\frac{1}{2}$ for each of these $(1 < p' < m)$. This was, in fact, the last example given in the previous note.

4.3. *Infinite Quadrant.*—If the array fills the whole quarter-plane $p, q > 0$, we have $m, n \to \infty$. The corresponding solution is got by letting $m \to \infty$ in (4.11), giving

$$\mathrm{F}_1(p, q) = \frac{8}{\pi} \int_0^\pi \sin a\lambda \sin p\lambda \frac{\sinh q\mu \exp(-b\mu)}{\sinh \mu} d\lambda, \quad . \quad (4.311)$$

$$\mathrm{F}_2(p, q) = \frac{8}{\pi} \int_0^\pi \sin a\lambda \sin p\lambda \frac{\sinh b\mu \exp(-q\mu)}{\sinh \mu} d\lambda, \quad . \quad (4.312)$$

where

$$\cos \lambda + \cosh \mu = 2. \quad . \quad . \quad . \quad . \quad . \quad (4.32)$$

A similar result, with the rôles of p, q interchanged, follows by letting $m \to \infty$ in (4.12).

4.4. *Half-plane.*—If the array fills the whole half-plane $p > 0$, the corresponding solution is got by letting $m \to \infty$ in (4.21), which gives

$$\mathrm{F}(p, s) = \frac{4}{\pi} \int_0^\pi \sin a\lambda \sin p\lambda \frac{\exp(-|s|\mu)}{\sinh \mu} d\mu. \quad . \quad . \quad (4.41)$$

A different form of the result follows from letting $m \to \infty$ in (4.22); we get

$$\mathrm{F}^1(p, s) = \frac{4}{\pi} \int_0^\pi \cos s\lambda \frac{\sinh p\mu \exp(-a\mu)}{\sinh \mu} d\lambda, \quad . \quad . \quad (4.421)$$

$$\mathrm{F}^2(p, s) = \frac{4}{\pi} \int_0^\pi \cos s\lambda \frac{\sinh a\mu \exp(-p\mu)}{\sinh \mu} d\lambda. \quad . \quad . \quad (4.422)$$

The probability that P ultimately escapes to a boundary point (0, s) of the half-plane is, using (4.421),

$$\tfrac{1}{4}F(1, s)=\frac{1}{\pi}\int_0^\pi \cos s\lambda\, e^{-a\mu}d\lambda, \quad . \quad . \quad . \quad (4.43)$$

where, from (4.32),

$$e^{-\mu}=2-\cos\lambda-\{(2-\cos\lambda)^2-1\}^{\frac{1}{2}}. \quad . \quad . \quad . \quad (4.44)$$

Some numerical values of (4.43), with $a=1$, are

$$\tfrac{1}{4}F(1, 0)=1-2/\pi=0\cdot363, \qquad \tfrac{1}{4}F(1, 1)=-\tfrac{1}{2}+2/\pi=0\cdot137,$$
$$\tfrac{1}{4}F(1, 2)=-1+10/3\pi=0\cdot061.$$

These were got by elementary integration. Further values can be written down from Table I by using the method of images explained in § 10.

In a case like this, if the probability of escape only is required, it can be got most easily by forming directly the difference-equation satisfied by it as a function, in the present instance, of a and s, and solving with the appropriate boundary condition.

5. *Infinite Plane.*—The most interesting case is that in which the array fills the whole plane, extending to infinity in every direction. Before dealing with this case we notice an alternative interpretation of the function F. Suppose first, in the case of the fluid diffusing through the lattice as in § 2, that, instead of having unit quantity liberated once and for all at (a, b), we have unit quantity liberated per second at this point. Then it is clear that $F(p, q)$ now gives the quantity of substance which leaves the point (p, q) per unit time. Suppose next that particles P move from point to point in the random manner postulated in §§ 1, 2, the moves of all the particles in the lattice being simultaneous. Also suppose that at the point (a, b) one new particle is introduced at each move. Then comparison with the case of the fluid shows that *the average number of particles at the point* (p, q) when a steady state has been attained (except when the moves are in process of taking place) *is* $F(p, q)$.

In the cases already dealt with, the chance that a particle remains inside the array after a sufficiently large number of moves is arbitrarily small, and the number of particles at any point remains finite. In the case of the infinite plane array, on the other hand, there is no limit to the number of particles at any point. This can be seen if we attempt to let $a, p \to \infty$ in (4.42), $p-a=r$ (say) remaining finite; we should get formally

$$\frac{2}{\pi}\int_0^\pi \cos s\lambda\,\frac{\exp(-|r|\mu)}{\sinh\mu}d\lambda, \quad . \quad . \quad . \quad . \quad (5.1)$$

which does not converge at $\lambda=0$ (*i.e.* $\mu=0$).

We therefore deal with differences, introducing the function $G(r, s)$ defined by

$$G(r, s) = F(a, b) - F(p, q). \quad . \quad . \quad . \quad . \quad (5.2)$$

We recall that the point (a, b) at which the particles are introduced is the point $r=0$, $s=0$. G can be calculated directly by writing down the difference-equations for the steady state with appropriate boundary conditions, but we shall here use the results already obtained for F. We are now interested in the case where $m, n, a, b, p, q \rightarrow \infty$, while $p-a=r$, $q-b=s$ remain finite. Letting $m, n, b, q \rightarrow \infty$, we have from (4.422)

$$F(a, b) - F(p, q) \rightarrow \frac{4}{\pi}\int_0^\pi \frac{\sinh a\mu\, e^{-a\mu} - \cos s\lambda \sinh a\mu\, e^{-p\mu}}{\sinh \mu} d\lambda \qquad (p \geqslant a).$$

Letting then $a, p \rightarrow \infty$,

$$F(a, b) - F(p, q) \rightarrow \frac{2}{\pi}\int_0^\pi \frac{1 - \cos s\lambda\, e^{-r\mu}}{\sinh \mu} d\lambda \qquad (r \geqslant 0).$$

The like formula for $r < 0$ follows from (4.421). So

$$G(r, s) = \frac{2}{\pi}\int_0^\pi \frac{1 - \cos s\lambda \exp(-|r|\mu)}{\sinh \mu} d\lambda, \quad . \quad . \quad (5.3)$$

with, as before,

$$\cos \lambda + \cosh \mu = 2. \quad . \quad . \quad . \quad . \quad . \quad (5.4)$$

(5.3) is now the required result for the infinite plane. We note that, unlike (5.1), the integral in (5.3) converges, *i.e.* $G(r, s)$ is finite for all r, s.

We may remark that the function G may be used to derive a quantity having a simple physical meaning. For consider a lattice-line joining two neighbouring points, say (p, q), $(p, q+1)$. Under the circumstances envisaged in this paragraph, the average number of particles flowing per second from (p, q) along this line is $\frac{1}{4}F(p, q)$, and the average number flowing from $(p, q+1)$ is $\frac{1}{4}F(p, q+1)$. Hence the net flow per second in the direction (p, q) to $(p, q+1)$ is

$$\tfrac{1}{4}F(p, q) - \tfrac{1}{4}F(p, q+1) = \tfrac{1}{4}G(r, s+1) - \tfrac{1}{4}G(r, s),$$

using (5.2). This flow is now seen to be finite, as we should expect, in all cases. The problem studied here is thus seen to be closely related to the study of current networks (as was pointed out to one of the authors by Dr E. T. S. Walton).

6. *Symmetry of G(r, s).*—In virtue of the equivalence of the rows and columns of the infinite lattice, we must have

$$G(r, s) = G(s, r). \quad . \quad . \quad . \quad . \quad . \quad (6.1)$$

It is of interest, however, to prove this directly:

Theorem.
$$\int_0^\pi \frac{1-\cos s\lambda\, e^{-r\mu}}{\sinh\mu}\,d\lambda=\int_0^\pi \frac{1-\cos r\lambda\, e^{-s\mu}}{\sinh\mu}\,d\lambda, \qquad (r,\ s>0), \quad (6.2)$$

where

$$\cos\lambda+\cosh\mu=2. \qquad (6.3)$$

In (6.3) we take the solution for which $\mu/\lambda \to 1$, as $\lambda \to 0$. This determines unambiguously one continuous solution for μ as a function of λ in $0 \leqslant \lambda \leqslant 2\pi$; μ has the same value for λ and $2\pi-\lambda$. We can then derive from (6.3) for this solution

$$\sin\tfrac{1}{2}\lambda=\sinh\tfrac{1}{2}\mu, \qquad (6.4)$$

$$\sinh\tfrac{1}{2}(\mu-i\lambda)=-i\tanh\tfrac{1}{2}(\mu+i\lambda), \qquad (6.5)$$

and using these,

$$\frac{d\lambda}{\sinh\mu}=\frac{d\mu}{\sin\lambda}=\frac{d(\mu+i\lambda)}{2\sinh\frac{1}{2}(\mu+i\lambda)}. \qquad (6.6)$$

Let $\mu=\alpha$ when $\lambda=\pi$; so that, from (6.4),

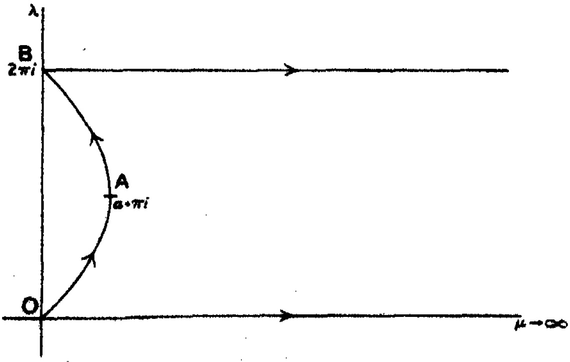
$$\alpha=2\sinh^{-1}1=1{\cdot}7627. \qquad (6.7)$$

Consider now

$$2\mathrm{I}=\int_0^{2\pi}\left(e^{-s\mu-ir\lambda}-e^{-r\mu-is\lambda}\right)\frac{d\lambda}{\sinh\mu}$$
$$=\int_0^{2\pi i} e^{-\frac{1}{2}(r+s)(\mu+i\lambda)}\,\frac{\sinh\left[\frac{1}{2}(r-s)(\mu-i\lambda)\right]}{\sinh\frac{1}{2}(\mu+i\lambda)}\,d(\mu+i\lambda),$$

using (6.6).

The path of integration is determined by the solution of (6.3) which we are using, and is indicated (qualitatively) by the arc OAB in the figure.



Then by (6.5) we have on OAB,

$$\sinh\left[\tfrac{1}{2}(r-s)(\mu-i\lambda)\right]=f\{\sinh\tfrac{1}{2}(\mu-i\lambda)\}=f\{-i\tanh\tfrac{1}{2}(\mu+i\lambda)\}.$$

Therefore, writing $z=\mu+i\lambda$,

$$2\mathrm{I}=\int_{\mathrm{OAB}} e^{-\frac{1}{2}(r+s)z} f(-i\tanh\tfrac{1}{2}z)\,\frac{dz}{\sinh\frac{1}{2}z}.$$

From the nature of the functions involved we have then by an application of Cauchy's theorem,

$$2\mathrm{I}=\int_0^{\infty}-\int_{2\pi i}^{\infty+2\pi i} e^{-\frac{1}{2}(r+s)z} f(-i\tanh\tfrac{1}{2}z)\,\frac{dz}{\sinh\frac{1}{2}z}, \qquad (6.8)$$

the integrals being taken along the lines shown in the figure.

Now

$$e^{-\frac{1}{2}(r+s)2\pi i}=\pm 1, \qquad \sinh\tfrac{1}{2}(x+2\pi i)=-\sinh\tfrac{1}{2}x,$$
$$-i\tanh\tfrac{1}{2}(x+2\pi i)=-i\tanh\tfrac{1}{2}x.$$

Also f is an odd function of its argument, so it is purely imaginary in both integrals in (6.8). Hence both integrals are purely imaginary; therefore I is purely imaginary.

Now by changing λ to $2\pi-\lambda$ in the range π to 2π in the definition of I and recalling that this does not change the relation between λ and μ, we obtain

$$\mathrm{I}=\int_0^{\pi}(e^{-s\mu}\cos r\lambda-e^{-r\mu}\cos s\lambda)\,\frac{d\lambda}{\sinh\mu}. \qquad (6.9)$$

Hence I is purely real.

Combining these results we have therefore

$$\mathrm{I}=0,$$

which on account of (6.9) proves the theorem, and so verifies (6.1).

7. We proceed to find a formula by which G can be evaluated when $r=0$. The method of integration is suggested by consideration of the case $s=1$. We introduce a variable ψ defined by

$$1-\cos\lambda=\cosh\mu-1=2\cos\psi,$$

so that

$$\sin\lambda=2\sin\tfrac{1}{2}\psi(2\cos\psi)^{1/2},$$
$$\sinh\mu=2\cos\tfrac{1}{2}\psi(2\cos\psi)^{1/2},$$

and

$$\frac{(1-\cos\lambda)}{\sinh\mu}d\lambda=-d\psi. \qquad (7.1)$$

When λ goes from 0 to π, ψ goes from $\frac{1}{2}\pi$ to 0. So using (5.3), (7.1),

$$\mathrm{G}(0,s)=\frac{2}{\pi}\int_0^{\pi/2}\frac{1-\cos s\lambda}{1-\cos\lambda}d\psi. \qquad (7.2)$$

Now $\cos s\lambda$ is the coefficient of x^s in the expansion of $\Phi(x)$, where

$$\Phi(x) = (1 - x\cos\lambda)/(1 - 2x\cos\lambda + x^2)$$
$$= \sum_{t=0}^{\infty}(-4x\cos\psi)^t(1-x)^{-1-2t} - \tfrac{1}{2}\sum_{t=1}^{\infty}(-4x\cos\psi)^t(1-x)^{-2t}.$$

Hence

$$1 - \cos s\lambda = -s\sum_{t=1}^{s}\frac{(s+t-1)!}{(s-t)!\,(2t)!}(-4\cos\psi)^t,$$

and

$$\frac{1-\cos s\lambda}{1-\cos\lambda} = 2s\sum_{t=0}^{s-1}\frac{(s+t)!}{(s-t-1)!\,(2t+2)!}(-4\cos\psi)^t. \quad . \quad . \quad (7{\cdot}3)$$

Using

$$\frac{2}{\pi}\int_0^{\pi/2}\cos^t\psi d\psi = \frac{(t-1)(t-3)\ .\ .\ .\ 1}{t(t-2)\ .\ .\ .\ 2} \qquad \text{when } t \text{ is even,}$$
$$= \frac{2}{\pi}\,\frac{(t-1)(t-3)\ .\ .\ .\ 2}{t(t-2)\ .\ .\ .\ 3} \qquad \text{when } t \text{ is odd,}$$

it follows from (7.2), (7.3) that

$$G(0, s) = 2s\sum_{(0)}\frac{(s+t)!\,4^t}{(s-t-1)!\,(2t+2)!}\,\frac{(t-1)(t-3)\ .\ .\ .\ 1}{t(t-2)\ .\ .\ .\ 2}$$
$$-\frac{4s}{\pi}\sum_{(1)}\frac{(s+t)!\,4^t}{(s-t-1)!\,(2t+2)!}\,\frac{(t-1)(t-3)\ .\ .\ .\ 2}{t(t-2)\ .\ .\ .\ 3}, \qquad (s \geqslant 0), \quad (7{\cdot}4)$$

where $\Sigma_{(0)}$, $\Sigma_{(1)}$ are taken over all even and odd values of t, respectively, from 0 to $s - 1$.

TABLE I.—VALUES OF G(r, s).

$s \diagdown r$	0.	1.	2.	3.	4.	5.	6.	7.	8.
0.	0	1	$4 - \frac{8}{\pi}$	$17 - \frac{48}{\pi}$	$80 - \frac{736}{3\pi}$	$401 - \frac{3760}{3\pi}$	$2084 - \frac{98104}{15\pi}$	$11073 - \frac{521696}{15\pi}$	$59712 - \frac{19696256}{105\pi}$
1.	1	$\frac{4}{\pi}$	$-1 + \frac{8}{\pi}$	$-8 + \frac{92}{3\pi}$	$-49 + \frac{480}{3\pi}$	$-280 + \frac{13292}{15\pi}$	$-1569 + \frac{74040}{15\pi}$	$-8752 + \frac{2887748}{105\pi}$	
2.	$4 - \frac{8}{\pi}$	$-1 + \frac{8}{\pi}$	$\frac{16}{3\pi}$	$1 + \frac{8}{3\pi}$	$12 - \frac{472}{15\pi}$	$97 - \frac{4472}{15\pi}$	$672 - \frac{220944}{105\pi}$		
3.	$17 - \frac{48}{\pi}$	$-8 + \frac{92}{3\pi}$	$1 + \frac{8}{3\pi}$	$\frac{92}{15\pi}$	$-1 + \frac{144}{15\pi}$	$-16 + \frac{5988}{105\pi}$			
4.	$80 - \frac{736}{3\pi}$	$-49 + \frac{480}{3\pi}$	$12 - \frac{472}{15\pi}$	$-1 + \frac{144}{15\pi}$	$\frac{704}{105\pi}$				

The series in (7.4) have been summed for all values of s up to 8, and the results are set out in the first line of Table I. That G(0, 0) = 0 follows from the definition (5.2), and that G(0, 1) = 1 follows directly from (7.2). Table II gives the same quantities to four decimal places.

TABLE II.—VALUES OF G(r, s) (FIVE FIGURES).

$s \backslash r$	0.	1.	2.	3.	4.	5.	6.	7.	8.
0.	0	1	1·4535	1·7211	1·9080	2·0516	2·1685	2·2670	2·3523
1.	1	1·2732	1·5465	1·7615	1·9296	2·0650	2·1776	2·2737	
2.	1·4535	1·5465	1·6977	1·8488	1·9839	2·1012	2·2032		
3.	1·7211	1·7615	1·8488	1·9523	2·0558	2·1528			
4.	1·9080	1·9296	1·9839	2·0558	2·1342				

8. Another case in which G can be evaluated exactly occurs when $r = s$.

From (5.3), (6.6), (6.7),

$$\begin{aligned} G(r+1, r+1) - G(r, r) &= \frac{2}{\pi}\int_0^{\pi}[\cos r\lambda\, e^{-r\mu} - \cos (r+1)\lambda\, e^{-(r+1)\mu}]\frac{d\lambda}{\sinh \mu} \qquad (r > 0) \\ &= \frac{2}{\pi}R\int_0^{\alpha+\pi i}[e^{-rz} - e^{-(r+1)z}]\frac{dz}{2 \sinh \frac{1}{2}z} \\ &= \frac{2}{\pi}R\int_0^{\alpha+\pi i} e^{-(r+\frac{1}{2})z}dz \\ &= \frac{4}{\pi}R\frac{1 \pm ie^{-(r+\frac{1}{2})\alpha}}{2r+1} \\ &= \frac{4}{\pi}\frac{1}{2r+1}. \end{aligned}$$

Hence, using G(0, 0) = 0, we have

$$G(r, r) = \frac{4}{\pi}\left[\frac{1}{1} + \frac{1}{3} + \ . \ . \ . + \frac{1}{2r-1}\right]. \qquad (8.1)$$

We may then obtain the asymptotic form of (8.1) for large r; we have

$$\begin{aligned} G(r, r) &= \frac{4}{\pi}\left[\sum_1^{2r}\frac{1}{t} - \frac{1}{2}\sum_1^{r}\frac{1}{t}\right] \\ &= \frac{4}{\pi}\left[\log 2r + \gamma + o\left(\frac{1}{r}\right) - \frac{1}{2}\log r - \frac{1}{2}\gamma + o\left(\frac{1}{r}\right)\right] \qquad \text{as } r \to \infty \\ &= \frac{1}{\pi}\log (2r^2) + C + o\left(\frac{1}{r}\right), \qquad (8.2) \end{aligned}$$

where

$$C = \frac{1}{\pi}[3 \log 2 + 2\gamma] = 1{\cdot}02937,$$

γ being Euler's constant.

Values of $G(r, r)$ in accordance with (8.1) have been inserted in Tables I and II. The remaining entries could be derived from these values of $G(r, r)$ and the value of $G(1, 0)$, which is unity, by the use of the scale of relation,

$$4G(r, s) = G(r-1, s) + G(r+1, s) + G(r, s-1) + G(r, s+1). \quad . \quad (8.3)$$

The equation 7.4 serves as a check.

9. We proceed to generalize 8.2 and obtain an asymptotic formula for $G(r, s)$.

We have

$$\frac{\pi}{2}[G(r, s) - G(r, t)] = \int_0^\pi \frac{\cos t\lambda - \cos s\lambda}{\sinh \mu} e^{-r\mu} d\lambda, \quad . \quad . \quad (9.1)$$

and it is known that

$$\int_0^\infty \frac{\cos t\lambda - \cos s\lambda}{\lambda} e^{-r\lambda} d\lambda = \tfrac{1}{2} \log \frac{s^2 + r^2}{t^2 + r^2}. \quad . \quad . \quad (9.2)$$

When r is large the important part of each of these integrals occurs with small values of λ. Accordingly we divide the range of integration at ϵ, defined by $\epsilon = r^{-1/3}$.

Now it can be established that

$$\int_\epsilon^\pi \frac{\cos t\lambda - \cos s\lambda}{\sinh \mu} e^{-r\mu} d\lambda = O[r^{1/3} \exp(-r^{2/3})] \quad . \quad . \quad (9.3)$$

$$\int_\epsilon^\infty \frac{\cos t\lambda - \cos s\lambda}{\lambda} e^{-r\lambda} d\lambda = O[r^{-2/3} \exp(-r^{2/3})] \quad . \quad . \quad (9.4)$$

and

$$\int_0^\epsilon (\cos t\lambda - \cos s\lambda)\left(\frac{e^{-r\lambda}}{\lambda} - \frac{e^{-r\mu}}{\sinh \mu}\right) d\lambda = O\left(\frac{1}{r^2}\right). \quad . \quad . \quad (9.5)$$

9.5 holds whatever the values of t and s may be. If t and s remain finite as $r \to \infty$ the stronger expression $O\left(\frac{1}{r^4}\right)$ can be substituted.

Combining (9.1)–(9.5), we have therefore

$$G(r, s) - G(r, t) = \frac{1}{\pi} \log \frac{s^2 + r^2}{t^2 + r^2} + o\left(\frac{1}{r}\right), \qquad \text{as } r \to \infty.$$

Putting $t = r$ and using (8·2), we have as $r \to \infty$,

$$G(r, s) = \frac{1}{\pi} \log (r^2 + s^2) + C + o\left(\frac{1}{r}\right). \quad . \quad . \quad . \quad (9.6)$$

Table III gives the values of $\frac{1}{\pi}\log(r^2+s^2)+C$ for the values of r, s included in Table II. It shows that this expression actually gives a good approximation to $G(r, s)$ when r or $s > 2$.

TABLE III.—VALUES OF $\frac{1}{\pi}\log_e(r^2+s^2)+C$.

$$C=\frac{1}{\pi}[3\log_e 2+2\gamma]=1{\cdot}02937.$$

$_s\diagdown^r$	0.	1.	2.	3.	4.	5.	6.	7.	8.
0.	..	1·0294	1·4706	1·7288	1·9119	2·0540	2·1700	2·2682	2·3532
1.	1·0294	1·2500	1·5417	1·7623	1·9312	2·0665	2·1788	2·2746	
2.	1·4706	1·5417	1·6913	1·8458	1·9829	2·1012	2·2036		
3.	1·7288	1·7623	1·8458	1·9494	2·0540	2·1518			
4.	1·9119	1·9312	1·9829	2·0540	2·1326				

The dominant term $\frac{1}{\pi}\log(r^2+s^2)$ in (9.6) can, of course, also be got by passing over from (9.1) to the corresponding differential equation, which is just Laplace's equation.

METHOD OF IMAGES.

10. In the interpretation followed in § 5 it is natural to call the point (a, b) a unit *point-source*. We shall now write $F(p, q;\ a, b)$, $G(p, q;\ a, b)$ instead of the previous $F(p, q)$, $G(r, s)$, thus indicating explicitly the position of the source. Further, we may suppose that $p, q;\ a, b$ are referred to any convenient lattice-lines as "axes."

Now suppose for the moment that F were finite in the case of the infinite plane. Consider the problem of deriving the solution in the case of a unit source (a, b) with $a > 0$ in the *half-plane* $p \geqslant 0$. A procedure naturally suggested by the hydrodynamical analogy would be to add the solutions for the infinite plane for the cases of a unit *source* at (a, b) and a unit *sink* at the image point $(-a, b)$, giving

$$F(p, q;\ a, b)-F(p, q;\ -a, b). \quad . \quad . \quad . \qquad (10.1)$$

Now this can be written formally as

$$-[F(a, b;\ a, b)-F(p, q;\ a, b)]+[F(-a, b;\ -a, b)-(Fp, q;\ -a, b)]$$

$$=-G(p, q;\ a, b)+G(p, q;\ -a, b) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10.2)$$

$$=-\frac{2}{\pi}\int_0^\pi\frac{1-\cos s\lambda\exp[-(a-p)\mu]}{\sinh\mu}d\lambda+\frac{2}{\pi}\int_0^\pi\frac{1-\cos s\lambda\exp[-(a+p)\mu]}{\sinh\mu}d\lambda \quad (p < a)$$

$$=\frac{4}{\pi}\int_0^\pi\cos s\lambda\,\frac{\sinh p\mu\; e^{-a\mu}}{\sinh\mu}d\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10.3)$$

using (5.3). This is precisely the result already given in (4.421) for the half-plane.

As a second example, consider the problem of deriving the solution for the case of a unit source (a, b) with $a, b > 0$ in the *quadrant* $p, q > 0$. Here the necessary images are unit *sinks* at $(-a, b)$, $(a, -b)$, and a unit *source* at $(-a, -b)$. So in place of (10.2) we must take

$$-G(p, q;\ a, b) + G(p, q;\ -a, b) + G(p, q;\ a, -b) - G(p, q;\ -a, -b)$$

$$= \frac{4}{\pi}\int_0^\pi \cos (b-q)\lambda \, \frac{\sinh p\mu \, e^{-a\mu}}{\sinh \mu} d\lambda - \frac{4}{\pi}\int_0^\pi \cos (b+q)\lambda \frac{\sinh p\mu \, e^{-a\mu}}{\sinh \mu} d\lambda \quad (p \leqslant a,\ q \leqslant b)$$

$$= \frac{8}{\pi}\int_0^\pi \sin b\lambda \sin q\lambda \, \frac{\sinh p\mu \, e^{-a\mu}}{\sinh \mu} d\lambda,$$

using (10.3). This is precisely the result (4.311) (with the rôles of p, q interchanged) already given for the quadrant.

It can now be seen how all the previous results could be recovered in this way, but in the case of arrays with two parallel boundaries an infinite number of images must in general be used. However, as we have given it, the procedure is purely formal since in expressions like (10.1) F is not finite. It would not be difficult to render this part of the work rigorous, thus making it possible to base all the results on the function G for the infinite array. G would then be regarded as given directly by the solution of (9.1), subject to the necessary boundary conditions. Nevertheless these boundary conditions, and problems of uniqueness, present difficulties unless we deal with the infinite array as the limit of a finite array. So it has seemed to us preferable to adopt the development followed in this paper.

THREE-DIMENSIONAL LATTICE.

11. Suppose now P moves at random in a three-dimensional lattice of points (p, q, r) with $0 \leqslant p \leqslant l+1$, $0 \leqslant q \leqslant m+1$, $0 \leqslant r \leqslant n+1$. Then if P is at (p, q, r) the chance that it will move along one of the six lattice-lines meeting at this point is 1/6 for each of them. Let P start from the point (a, b, c). We may define the function $F(p, q, r)$ in the same way that $F(p, q)$ was defined for the two-dimensional array.

Working analogous to that in §§ 2, 3 gives

$$F_1(p, g, r) = \frac{24}{(l+1)(m+1)} \sum_{s=1}^{l} \sum_{t=1}^{m} \sin \frac{sa\pi}{l+1} \sin \frac{sp\pi}{l+1} \sin \frac{tb\pi}{m+1} \sin \frac{tq\pi}{m+1} \frac{\sinh r\beta_{st} \sinh [(n+1-c)\beta_{st}]}{\sinh \beta_{st} \sinh (n+1)\beta_{st}}, \quad (11.11)$$

$$F_2(p, q, r) = \frac{24}{(l+1)(m+1)} \sum_{s=1}^{l} \sum_{t=1}^{m} \sin\frac{sa\pi}{l+1} \sin\frac{sp\pi}{l+1} \sin\frac{tb\pi}{m+1} \sin\frac{tq\pi}{m+1} \frac{\sinh[(n+1-r)\beta_{st}] \sinh c\beta_{st}}{\sinh\beta_{st} \sinh(n+1)\beta_{st}}, \quad (11.12)$$

the values of β_{st} being defined by

$$\cos\frac{s\pi}{l+1} + \cos\frac{t\pi}{m+1} + \cosh\beta_{st} = 3. \quad . \quad . \quad . \quad (11.2)$$

F_1 is the value of F when $r \leqslant c$, F_2 the value when $r \geqslant c$.

Now let $l, m, n, a, b, c, p, q, r \to \infty$, $p - a = u$, $q - b = v$, $r - c = w$ remaining finite. Denoting the limit of F by $F(u, v, w)$, we find

$$F(u, v, w) = \frac{3}{\pi^2}\int_0^\pi\int_0^\pi \cos u\lambda \cos v\mu \frac{e^{-|w|\nu}}{\sinh\nu} d\lambda d\mu, \quad . \quad . \quad (11.3)$$

where from (11.2)

$$\cos\lambda + \cos\mu + \cosh\nu = 3. \quad . \quad . \quad . \quad . \quad (11.4)$$

Thus *in the three-dimensional case of a lattice filling all space the function F has the finite limit* (11.3). This constitutes the essential difference from the two-dimensional case.

In particular the expectation that P will return to its starting-point is $F(0, 0, 0) - 1$, where from (11.3)

$$F(0, 0, 0) = \frac{3}{\pi^2}\int_0^\pi\int_0^\pi \frac{d\lambda d\mu}{\sinh\nu} = \frac{3}{\pi^2}\int_0^\pi\int_0^\pi \frac{d\lambda d\mu}{\{(3 - \cos\lambda - \cos\mu)^2 - 1\}^{1/2}}. \quad (11.5)$$

12. The integral in (11.5) can be evaluated by using elliptic functions, as follows:—

To compute

$$J = \int_0^\pi\int_0^\pi \frac{d\lambda d\mu}{\{(3 - \cos\lambda - \cos\mu)^2 - 1\}^{1/2}}.$$

Write

$$I = \int_0^\pi \frac{d\mu}{\{(\alpha - \cos\mu)^2 - 1\}^{1/2}}$$

$$= \int_0^\pi \frac{\sec^2\frac{1}{2}\mu d\mu}{\{[\alpha + (\alpha+2)\tan^2\frac{1}{2}\mu][(\alpha - 2) + \alpha\tan^2\frac{1}{2}\mu]\}^{1/2}},$$

where $\alpha = 3 - \cos\lambda$. Put

$$\frac{\alpha}{\alpha - 2}\tan^2\tfrac{1}{2}\mu = \tan^2\psi, \quad \text{so that} \quad \tfrac{1}{2}\sqrt{\left(\frac{\alpha}{\alpha-2}\right)}\sec^2\tfrac{1}{2}\mu d\mu = \sec^2\psi d\psi.$$

Then

$$I = \frac{2}{\alpha}\int_0^{\pi/2} \frac{d\psi}{\left[1 - \frac{4}{\alpha^2}\sin^2\psi\right]^{1/2}},$$

$$= kK,$$

where

$$k=\frac{2}{a}=\frac{2}{3-\cos\lambda}\quad\text{and}\quad K=\int_0^{\pi/2}\frac{d\psi}{(1-k^2\sin^2\psi)^{1/2}}.$$

Now let $k=\sin\theta$, giving

$$\cos\lambda=3-2\operatorname{cosec}\theta\quad\text{and}\quad -k\,d\lambda=\left(\frac{1+\sin\theta}{2\sin\theta-1}\right)^{1/2}d\theta.$$

When λ goes from 0 to π, θ goes from $\pi/2$ to $\pi/6$. Therefore

$$J=\int_0^{\pi}kK\,d\lambda=\int_{\pi/6}^{\pi/2}K\,.\left(\frac{1+\sin\theta}{2\sin\theta-1}\right)^{1/2}d\theta.$$

Tables of K as a function of θ are available, so J can be evaluated by numerical integration. Intervals of 5° in θ were used. Difficulties arise owing to the singularity of K at $\pi/2$ and that of $(2\sin\theta-1)^{-1/2}$ at $\pi/6$, though J converges at both limits. These were overcome by using the asymptotic formula $K\sim\log 4-\log\cos\theta$, as $\theta\to\pi/2$, in the range 85° to 90°, and by using a constant value of K in the range 30° to 35°. Smaller ranges could have been used here had high accuracy been demanded. The approximate value of J actually obtained was

$$J=5{\cdot}02.$$

This gives in (11.5)

$$F=1{\cdot}53.$$

Thus if P visits any point of the lattice *the expectation of subsequent visits* $=\cdot 53$ *and the probability of at least one subsequent visit* $=\cdot 53/1{\cdot}53=\cdot 34$. For, if P visits a given point, that point is, of course, the starting-point for its subsequent wanderings, which are not affected by what has happened before.

13. *Half-space.*—Owing to the finiteness of $F(u, v, w)$, there is now no difficulty in the immediate use of the method of images. Let now any convenient system of "axes" for p, q, r be taken, and consider the lattice filling the half-space $r \geqslant 0$. Let a unit source be at the point (a, b, c) with $c>0$. Then putting unit sink at the image-point $(a, b, -c)$, and using (11.3), the appropriate solution is $f(u, v, r)$, where

$$\begin{aligned}f(u,v,r)&=F(u,v,r-c)-F(u,v,r+c)\\&=\frac{3}{\pi^2}\int_0^{\pi}\int_0^{\pi}\cos u\lambda\cos v\mu\frac{e^{(-c+r)\nu}-e^{(-c-r)\nu}}{\sinh\nu}d\lambda d\mu\qquad(r\leqslant c)\\&=\frac{6}{\pi^2}\int_0^{\pi}\int_0^{\pi}\cos u\lambda\cos v\mu\,\frac{\sinh r\nu\,.\,e^{-c\nu}}{\sinh\nu}d\lambda d\mu,\quad .\quad .\quad (13.1)\end{aligned}$$

with r, c interchanged if $r \geqslant c$.

In particular, the probability of escape to a boundary point (u, v, 0) is

$$\tfrac{1}{6}f(u, v, 1) = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \cos u\lambda \cos v\mu \, e^{-cv} d\lambda d\mu, \qquad (13.2)$$

where, from (11.4),

$$e^{-v} = 3 - \cos\lambda - \cos\mu - \{(3 - \cos\lambda - \cos\mu)^2 - 1\}^{\frac{1}{2}}. \qquad (13.3)$$

(13.1)–(13.3) are the three-dimensional analogues of (4.42)–(4.44), and can be derived also by methods analogous to those giving the latter results.

Some numerical values of (13.2) for the case $c = 1$ are

u \ v	0.	1.	2.
0.	0·210	0·050	0·015
1.	0·050	0·023	0·010
2.	0·015	0·010	0·005

They were got by rather rough numerical integration. They are given merely to show how, in the case where P starts at the interior point adjacent to a given boundary point, the probability of emergence is distributed over the nearby points.

The following estimates of F are derived from the above values of f and the scales of relation:—

F(0, 0, 0) = 1·53
F(0, 0, 1) = ·53
F(0, 1, 1) = ·35
F(1, 1, 1) = ·28
F(0, 0, 2) = ·27
F(0, 1, 2) = ·23
F(0, 2, 2) = ·18
F(1, 1, 2) = ·21
F(1, 2, 2) = ·17
F(2, 2, 2) = ·15

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(*Issued separately September* 30, 1940.)

XXIII.—An Analysis of the Influence of Weather upon a Migratory Movement of Birds. By Professor James Ritchie, M.A., D.Sc., Department of Zoology, University of Edinburgh. (With Six Text-figures.)

(MS. received May 17, 1940. Revised MS. received June 28, 1940.
Read June 3, 1940.)

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I. INTRODUCTORY.

The fundamental impulse determining the migration of birds is connected with the reproductive cycle; but when that has been recognised it must also be admitted that there are secondary conditions of environment which regulate the actual movements of which a migration is compounded. Of these weather has long been looked upon as one of the most important. There is still, however, much divergence of opinion as to the extent and nature of the influence of weather, and indeed in recent years statistical analysis has led workers, such as Bretscher (1915), to doubt whether any satisfactory connection between meteorological conditions and migration can be demonstrated, and Lincoln to announce boldly that "the state of the weather at any point has little if anything to do with the time of arrival of migratory birds" (1935, p. 59).

The evidence has been so well summarised in two modern general works on migration, by Lucanus (1922, Ch. 6) and Landsborough Thomson (1926, Ch. 2 and 6), although they reach conclusions somewhat different, that it will be sufficient here to state that different workers have attributed migratory movements to temperature, barometric pressure, and wind, individually, or collectively as "fine weather," while others can see no relation between the amount of migration and any aspect of weather conditions.

The aim of the present paper is, by the aid of wide collection of data and simple statistical analysis, to examine the part played by weather conditions in a definite and pronounced migratory movement, and further to elucidate the weather elements concerned in certain appearances of migratory birds in the British Isles.

It must be noted, however, that the series of movements discussed here belongs to the limited and somewhat irregular type of migration, less definite in character and probably more intimately associated with the immediate demands of hunger, than the movements of the regular long-distance migrants. Conclusions drawn from the analysis, therefore, may not necessarily be applicable to the migrations of the latter.

To Dr A. Landsborough Thomson I am indebted for reading the manuscript and making helpful suggestions, and my thanks are due also to the Carnegie Trust for the Universities of Scotland for a grant towards the cost of the illustrations and tables.

II. THE WAXWING MIGRATION IN THE WINTER OF 1921.

The material upon which the analysis is based concerns the notable southward migration of Waxwings (*Bombycilla garrulus*) which took place in the winter of 1921. The Waxwing breeds in sub-Arctic regions, in Europe in the pine-woods of the extreme north of Scandinavia, Finland, and Russia. After the breeding season it undertakes in autumn a somewhat limited southward migration which seldom extends beyond Central Europe, the general southern limit ranging from Denmark through Northern Prussia. Stragglers may occur in northern Italy, in Holland and Belgium, but these countries lie beyond the normal annual range of the mass of Waxwings.

The exceptional movement in the autumn of 1921 extended southwards beyond the normal limit. But the farther a bird is found from its starting-point the more have its movements become complicated by interferences such as physical obstacles, stoppages for feeding, resting and the like, so that the factors associated with the commencement of a migration are more likely to be revealed the nearer observations are made to the point of origin. On this account the farther south records have been ignored in this analysis, and attention has been concentrated upon two series of observations made respectively in Great Britain and Denmark, in each of which countries large numbers of Waxwings were observed.

The Danish records were collected in a valuable paper by P. Skovgaard (1922), who gathered all the information available in a series of notes dealing with the three provinces of Denmark. The British immigration was described by the writer in a paper based upon all the records available for Great Britain, but it was noteworthy that the mass of the immigrant birds landed in Scotland (Ritchie, 1922–23). To these papers readers may turn for general accounts of the movements of the Waxwings in Denmark and Britain; here they shall be referred to only in so far as their contents bear upon the problem of the present analysis.

III. THE STARTING-POINT OF THE MIGRATION.

Attempts have often been made after a notable immigration of birds to associate the movement with the weather conditions in the area in which they have been observed to arrive. This method has been followed notably by K. Bretscher (1915) who, in dealing with some 9000 records of immigration to central Switzerland, endeavoured to correlate them, unsuccessfully, with meteorological data derived mainly from the area of arrival. Moreover, he dealt with a long series of bird species whose only association lay in their arrival at the same place and whose starting-points may have been in quite different areas. It is obvious that such attempts start at the wrong end of the journey, and that if any correlation is to be discovered between climatic factors and the elements of a migratory movement, it must be sought at the place of origin.

A first effort, therefore, was made to determine the area from which the Waxwing migration in the late autumn of 1921 set out. From the habits shown by the Waxwings after their arrival in Britain it would appear that their movements consist not so much of a great migratory sweep, as of a series of southward stages in search of food. Dismissing as improbable the possibility that our migrants came directly from the great Arctic forests of Siberia or North-western Europe, I confined my attention to the conditions in Scandinavia in the autumn of 1921. The evidence, given in the paper already mentioned (Ritchie, 1922–23, pp. 193, 194), need not be detailed here. It is sufficient to say that in Norway in 1921 the crop of berries—cranberries, cowberries, blaeberries and the like—was a complete failure owing to a severe protracted drought and cold spring, accompanied by hard frosts in June. Lord Salvesen estimated that on the hills on which he shoots annually in Norway there was not 1 per cent. of the ordinary berry crop.

Since this crop is the staple food-supply of the Waxwings in autumn, general considerations suggest that the birds which bred in Scandinavia would be compelled, by lack of food at each place of arrival on their short stages southward, to make a further journey; so that eventually they would tend to congregate in mass in the southernmost parts of Scandinavia.

Actual observations confirmed this general surmise. I have been unable to discover evidence of massing in Southern Norway where observers are scarce, but from Sweden the evidence is clear. Lindner (1921) and Bellander (1922) state that from October 15 to 28 large flocks were observed in Western Götaland and in the neighbourhood of Stockholm. Later they were noted farther south at Helsingborg in the extreme south-west of Sweden, where M. Sjöbeck saw a flock of 20 on November 2, on the following day several flocks on hawthorn and rowan trees, and on the 5th great flocks in the town. He says that flocks were seen up to 300 in number, and that in these days there were thousands in the environs of the town. By November 12 all the bushes had been cleared of their berries, and by the 26th the mass of the birds had disappeared and only about 20 remained in the town.

This massing in the end of October and the first half of November in Southern Sweden confirms the general deduction, and warrants the supposition that Waxwing flocks were present in suitable areas throughout Southern Scandinavia; the disappearance of the birds between November 12 and 26 corresponds with the period of the great influx of Waxwings to Denmark and Great Britain.

To weather conditions in Southern Scandinavia, then, we must look for the factors, if any, which set agoing the great migration. It would be well to distinguish at this point a difference in the kind of migratory movement necessitated by the grouping of the birds in this region. Up to this stage the migration had probably consisted of short land flights which scarcely brought the birds within the influence of the wind. They were now hemmed in by the sea on all sides in the direction of their advance, and their next southward movement, at any rate for individuals not on the Swedish side of the narrow strait of Ore Sund, would compel them to take the air in a serious flight, and to expose themselves to the full force of the wind.

IV. WAXWING NUMBERS UPON WHICH COMPARISONS ARE BASED.

There is no method by which, in the case of such a migration as is considered here, an absolute count of the birds arriving can be made. In Britain I gathered information, as Skovgaard did in Denmark, from ornithological journals, from private correspondents, and from letters to newspapers, and by appeals for records through the correspondence columns of the daily press. I have summed up for each day the total numbers of Waxwings seen in whatever part of Britain they were observed. Each record has been entered, except where correspondence of date, locality, and numbers showed that two observers had seen the same flock of birds, and where indefiniteness of date made a record valueless. In the few cases where "few" or "several" birds were mentioned, the numbers 3, 4, or 5 have been entered as the context of the letter suggested.

The number of individual records used from Scotland was 87, recording 625 Waxwings; the records from England numbered 28 and the birds 200; so that the full number of immigrants to Britain dealt with is in the neighbourhood of 825. Observations later than December 10, 1921, and they continued intermittently up to April 10, 1922, do not appear here, for there is no indication of a fresh immigration in mass, and the birds seen, generally singly or in pairs, are presumably the relics of the main movement.

As regards Denmark, Skovgaard classified his records in three geographical groups: from Zealand, Fünen, and Jutland. Following the method adopted for the British material, I have summed these three groups of records, which refer to 1291 birds, in daily totals.

The total number of Waxwings referred to is 2116. From a statistical point of view these numbers have obvious weaknesses. In the first place, it is impossible that all the immigrants could have been observed; in the second place, the records represent counts of restless birds and therefore cannot be strictly accurate; and in the third place, the movements of birds from one place to another after their arrival may have brought it about that birds seen in one place on one day may enter the list subsequently from another place. Such objections must apply to any biological problem of this nature, but in the case of the Waxwing they have somewhat less relevance than must usually be the case, first because the bird forces itself upon notice on account both of its conspicuous appearance and of its noisy chatter, and, secondly, because although the immigrants arrive in flocks, which again are conspicuous, these quickly break up and scatter in small feeding parties, less liable to be noticed. So that records tend to concentrate upon the groups of newly arrived birds.

V. COMPARISON OF THE IMMIGRATIONS IN GREAT BRITAIN AND DENMARK.

The numbers of Waxwings seen daily in Great Britain and Denmark are shown graphically in fig. 1, A. In neither country before November 7 were Waxwings noticeably present, except perhaps a few isolated individuals. On November 10 in both countries the great movement began, and in both it appears to have been made up of a series of waves of different magnitudes. In Denmark a succession of waves occurred every other day from November 11 until a climax was reached on the 17th, when 523 birds were recorded. Thereafter it tailed rapidly away, with a spurt on the 20th, and no observations made after that date suggest any fresh immigration.

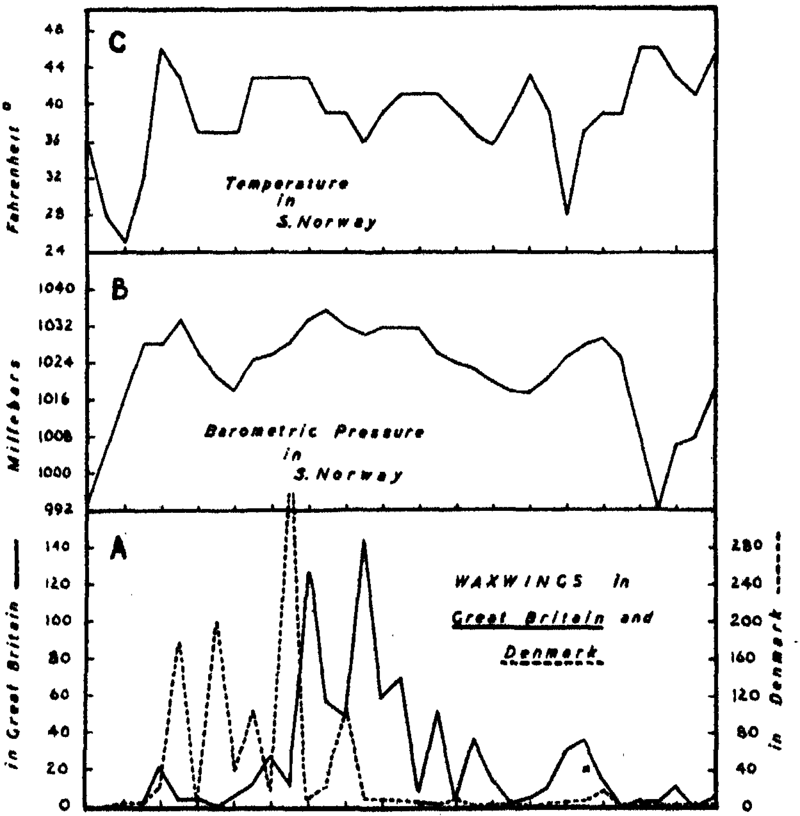

FIG. 1.—Relationship between appearance of Waxwings in Great Britain and Denmark, and atmospheric pressure and temperature in Southern Scandinavia.

A. Numbers of Waxwings seen daily in Great Britain and Denmark.
B. Daily records of atmospheric pressure in Southern Scandinavia, in millibars.
C. Daily records of temperature in Southern Scandinavia, in Fahrenheit °.

In Great Britain the first observations of numbers were made upon November 10, but no further accessions worth noting were made till the 16th, when a relatively small number of birds appears to herald the great wave of the 18th, which was succeeded by another three days later. Thereafter the observations tailed gradually off, with diminished peaks upon November 23, 25, and 27, until the last noticeable influx occurred on December 2 and 3.

The graph indicates a relationship between the observations in Denmark and in Britain. To take the clearest examples of fresh influxes, the Danish climax of November 17 corresponds to the British climax of the 18th, and the lag of one day in the British peak can be accounted for by the time taken in the migratory flight across the North Sea. The rate of flight for small passerine birds averages under 30 miles per hour, as Meinertzhagen has reckoned (1921), so that the passage of some 400 miles could be accomplished by direct flight in still air in about 13 to 20 hours, or in the absence of direct flight by favouring winds of equivalent velocity. On the other hand, as regards the Danish records, Jutland is separated from Southern Norway by about 80 miles across the Skagerrak, while Zealand is almost contiguous with Southern Sweden, separated only by the narrow Ore Sund, so that the greatest crossing time to direct flight can be only 3 to 4 hours.

Admitting a lag of one day in the British arrivals, the Danish peaks of November 15, 17, and 20 have their counterpart in Britain on the 16th, 18th, and 21st, though it need not be assumed that the Danish and British migrants set out from exactly the same place, which they probably did not. While the migrations began about the same time in both countries, the Danish movements of November 11 and 13 are unrepresented in Britain, and I suggest as the reason that at that time winds unfavourable for the east to west passage of the North Sea were blowing (p. 316). On the other hand, the declining peaks in the British numbers on November 23, 25, and 27, which are not represented in Denmark, may indicate either fresh immigrations or simply the residue of the great influx. But the fact that 65, out of 67, birds recorded for November 23 form the first large group from England, and that the vast majority of the records on November 25 are from the coast, suggest that fresh arrivals had taken place. In Denmark a residue of the main immigration would not be represented, for there, according to Skovgaard (1922, p. 121), lack of food compelled the flocks to disappear in the latter days of November and the first days of December. In this connection it may be recalled that in Germany Waxwings were seen on November 1, 7, 22, and 28 (*fide* Skovgaard, 1922, p. 121); the latter groups most likely represented the southward drift of the Danish invaders.

VI. RELATION BETWEEN WEATHER AND THE MIGRATION.

The starting of migrations has been attributed to responses set in action by conditions of atmospheric pressure, temperature, wind, or clear weather. Defant held the decided view that it was impracticable and scientifically improper to attempt to correlate immigration times with such isolated meteorological factors, since only the co-ordination of all the meteorological elements we call weather had any determining influence upon movements (1913, pp. 135, 136), and Lucanus strongly supported that view (1922, p. 135). But admitting that the totality of weather conditions must be considered, it is needful also to analyse the elements of "weather" in order to determine whether or not one or other may be predominant or negligible in the general effect. Therefore having traced the starting-point of the Waxwing migration of November and December 1921 to Southern Scandinavia, I have compared the presumed waves of the migration with the weather in that area as regards these four factors. The information regarding the weather was obtained from the *Daily Weather Reports (International Section) of the Meteorological Office, London*, issued by the Air Ministry. Having compared the records of Meteorological Stations in Southern Scandinavia giving a complete series of readings for the whole period of the migration, I fixed upon those made at Skudesnaes in Southern Norway as representing fairly the conditions over the area.

1. *Atmospheric Pressure and Migration.*

The relation between daily atmospheric pressure in Southern Scandinavia and the daily numbers of Waxwings seen in Great Britain and Denmark is shown in fig. 1, A and B, the barometric readings being in millibars corrected to mean sea-level.

A general correlation is noticeable, that the whole series of migratory waves took place within a period of very high atmospheric pressure, preceded and succeeded by periods of low pressure. Before the migration began the barometer stood at 984 mb. (Nov. 5) and after it ended at 992 mb. (Dec. 7). Migration began on November 9, when the barometer had risen in four days continuously and rapidly from 984 to 1028, and during the period of movement the maximum reached 1035 mb. and did not fall below 1018 mb. No noticeable migration followed the later rise in pressure to 1017 mb. on December 10, because by that time the majority of the birds had already left Scandinavia.

In October 1928 Professor W. Rowan observed a great migration of ducks in Central Alberta, which set out with an "exceptionally high barometer," though there is no indication in the few figures he gives of a sudden great rise (Rowan, 1929).

The Waxwing migration, because of its compactness and clear-cut limits of start and finish, offered an exceptional opportunity of testing with precision the influence of changing pressure and pressure gradients upon bird movements. A comparison has therefore been made between the day-to-day occurrences and the general atmospheric conditions over North-Western Europe as shown by the Weather Charts.

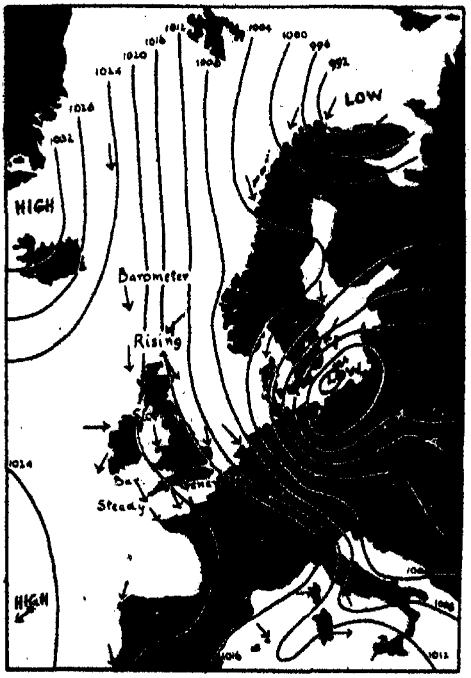

FIG. 2.—Atmospheric conditions before the migration began, November 7, 1921, 7ʰ G.M.T. Isobars at 4 millibars distance; arrows indicate the direction of the wind.

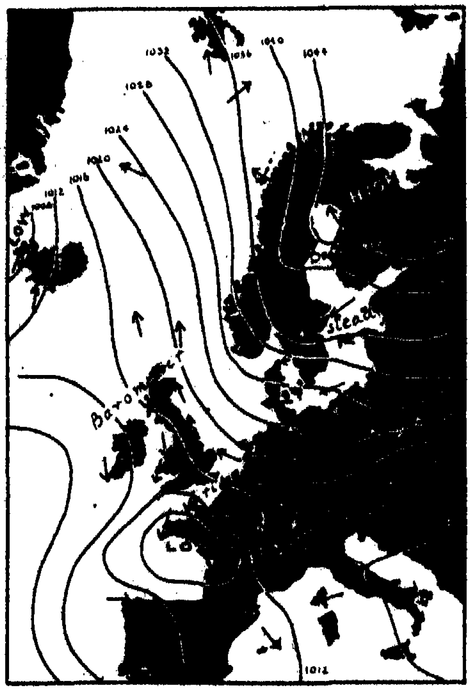

FIG. 3.—Atmospheric conditions when the migration was at its height, November 17, 1921, 7ʰ G.M.T.

Previous to the beginning of the migration, cyclonic conditions had prevailed in the Scandinavian region: on November 5, 6, 7, and 8 cyclones were centred in Southern Scandinavia or its neighbourhood. A typical condition occurred on November 7 and is shown in fig. 2. Centres of low pressure lay over Northern Scandinavia and the southern Baltic Sea, and of high pressure over Iceland and in the Atlantic Ocean west of Spain and Portugal. Isobars ran in a general north and south direction between Britain and Scandinavia, and the pressure gradient over the North Sea descended fairly steeply from west to east. Although, as the arrows in fig. 2 show, the winds in the area covered by the migration were northerly and therefore "favourable," in the ordinary sense, to a southward movement, no migration took place on that day, although it is known that birds were present in abundance, at any rate in Southern Sweden.

It can therefore be said that migration did not take place, although the wind would have been behind such a movement, along lines of equal pressure, that is parallel to the isobars (from Sweden to Denmark), nor did it take place against the pressure gradient falling from W to E (from Scandinavia to Britain).

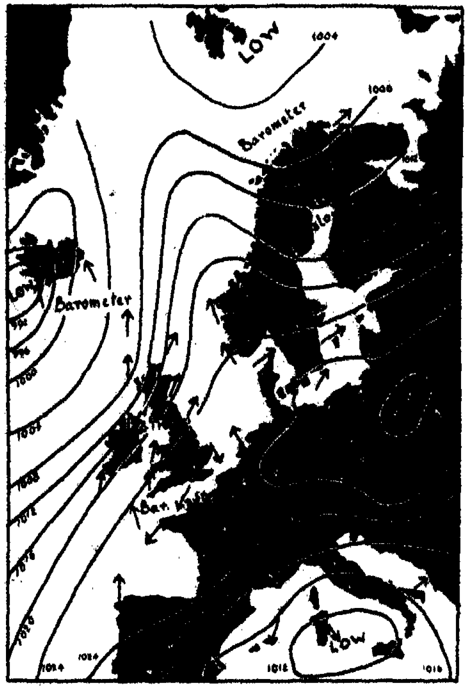

Fig. 4.—Atmospheric conditions after the migration ceased, December 5, 1921, 7h G.M.T.

On November 9 cyclonic conditions disappeared and anticyclones were centred over South England and Holland (Nov. 9 and 10), moving to Southern Sweden on November 11, the date of the first notable influx of Waxwings to Denmark. Throughout the period of the migrations anticyclones were centred on some part of Scandinavia or on Denmark, apart from a period of indeterminate conditions from November 24 to 28. In fig. 3 is shown the distribution of pressure on the day when the greatest number of Waxwings appear to have left Scandinavia, November 17 (523 seen in Denmark and 127 in Britain on the following day). The centre of high pressure had moved northward to Finland, and centres of low pressure lay over the Bay of Biscay and Iceland. The pressure gradient fell steeply from Southern Scandinavia to Britain and Denmark, and the wind in Southern Scandinavia and Denmark was from the east.

On November 7 and 17 (*cf.* figs. 2, 3) the isobars between Scandinavia and Great Britain were remarkably similar in direction, but on the former day no migration took place and on the latter it was intense. The difference was that the pressure gradient from Scandinavia to Britain and Denmark was a rising one on the earlier date, while on the latter date it was descending. The birds, then, began to move, and the movement reached its height when the conditions were such that they could proceed from a region of high pressure to a region of much lower pressure.

Finally on December 5 anticyclonic conditions in Scandinavia failed and were replaced on December 7 by a cyclone centred on Southern Norway. The pressure distribution when migration came to a stop on December 5 is shown in fig. 4. The centre of high pressure had moved southwards to Poland, and low-pressure areas lay over Spitsbergen, Iceland, and the Mediterranean Sea. In the area of the migration, isobars were now running in a general W to E direction, the pressure gradient was descending from Central Europe towards north and west, and winds were generally south-westerly. As before the movement, no migration took place parallel to the isobars, that is through areas of equal pressure, or across the isobars from areas of lower to higher pressure; but the fact that the majority of the birds had already left Scandinavia makes deduction less reliable.

Taking these evidences together it may be said (*a*) that the Waxwing migration took place from areas of high to areas of low atmospheric pressure when the pressure gradient was regular and fairly steep, that is the birds moved across, or at right angles, to the isobars; (*b*) that no migration took place through regions of equal pressure, that is parallel to the isobars; and (*c*) that no migration took place across isobars against a descending pressure gradient.

These results confirm Eagle Clarke's insistence upon the favourable influence for migration between Britain and Europe of anticyclonic conditions in North-Western Europe and the adverse effect of cyclonic conditions in that area (1912, vol. i, p. 173) in so far as migration from Scandinavia to Britain and Denmark is concerned. But the further statement of Eagle Clarke that the same disposition of atmospheric pressure equally favours migration in the reverse direction, that is from Britain to Europe (see explanation of Plate V, vol. i, p. 172), cannot be tested by the data at my disposal and would controvert the proposition that movement follows a falling pressure gradient. In this respect it would also contradict Defant's results, with which the interpretation of the Waxwing records agrees. Defant's minute examination of the effect of atmospheric conditions on selected dates upon the arrival of one or two species of birds in Austria during spring migration, although the numbers of arrivals upon the particular dates are relatively small, indicates that migration proceeded from areas of high pressure towards areas of low pressure, that is across the isobars, but never parallel to them (Defant, 1913, p. 155).

The records were further examined to see whether there existed any relationship between the successive waves of migration and specific changes in atmospheric pressure (see fig. 1, B). Of the thirteen possible waves, eight set out when the barometer, already high, was rising, namely, those reaching their destinations on November 10 (Br.), 11 (Den.), 15 and 16 (Den. and Br.), 17 and 18 (Den. and Br.), 23 (Br.), and December 2–3 (Br.). Four set out when the barometer was falling slightly—November 13 (Den.), 20 and 21 (Den. and Br.), and 27 (Br.), while that of November 25 (Br.) started when the barometer had been steady for three days and was on the point of falling.

It may be said therefore that within a period of high pressure the commencement of a wave is not definitely related to a specific change in atmospheric pressure, although the conditions associated with a rising barometer seem to be rather more favourable than those accompanying a falling one.

2. *Temperature and Migration.*

The relationship between fluctuations of temperature in Southern Scandinavia and the numbers of birds seen daily in Great Britain and Denmark is shown graphically in fig. 1, A and C. The mass of the migration took place when, after dropping from 36° F. on November 5 and 6 to 25° F. on November 8, the temperature rose in two days to 46° F. and remained, with fluctuations of from 3° to 9°, at or over 36° F. for three weeks. This does not suggest that the Waxwing migrations were due to a falling temperature, as is said to be the rule generally for autumn migratory movements (Eagle Clarke, 1912, vol. i, p. 179; Thomson, 1926, p. 105), and is stated by Eagle Clarke to be the specific forerunner of individual movements: "These movements [autumn migrations at Kentish Knock Lightship], I found, did not begin in earnest until a decided fall in temperature took place in Western Central Europe, and this important factor was the precursor of each of the pronounced movements observed" (1912, vol. ii, p. 27). On the contrary, the Waxwing movements as a whole occurred during a period of normal temperature for the season and area, although it followed a sharp and temporary fall.

As regards the relation of the individual waves of migration to changes in temperature, if the usual assumption be made that the Danish immigrants set out upon the day on which they were observed and the British on the day before observation, then six waves set out when the temperature had been stationary for two or three days—November 13 (Den.), 17 and 18 (Den. and Br.), 20 and 21 (Den. and Br.), and 25 (Br.); four set out when the temperature was rising—November 10 (Br.), 15 and 16 (Den.

and Br.), and 23 (Br.); and three when the temperature was falling—November 11 (Den.), 27 (Br.), and December 2–3 (Br.).

On the whole therefore, while changes of temperature do not appear to be the immediately determining influence which sets a wave of migration in motion, the most favoured condition for departure seems to be a temperature which has remained steady for two or three days. This condition may be associated with settled weather.

3. *Moisture Conditions and Migration.*

It is impossible from the data available to analyse in a numerical way the local conditions of visibility and moisture content which everyday language describes as "the weather," fine or bad. But Table I shows the dates when the different waves of immigration were observed in Great Britain and Denmark, and the weather conditions in Southern Scandinavia when the waves are presumed to have set out—the same day as arrival in Denmark, the day before arrival in Great Britain.

TABLE I.—WEATHER CONDITIONS IN SOUTHERN SCANDINAVIA WHEN WAVES OF MIGRATION SET OUT.

Apparent Waves of Migration.		Weather Conditions in Scandinavia.
Britain.	Denmark.	
November 10		Cloudy, rain.
	11	Overcast.
	13	,,
	15	,,
16		,,
	17	,,
18		,,
	20	,,
21		,,
23		Cloudy, unusual visibility.
25		Overcast.
27		Cloudy.
December 2–3		Blue sky.

With the exception of the last apparent wave, which set out under "blue sky," all the movements started under a cloudy or "overcast" sky. The Beaufort notation, in which the data are recorded in the Meteorological Records, sets out a graded series of weather conditions as follows:—

1. Blue Sky (not more than a quarter covered).
2. Sky partly cloudy (one-half covered).
3. Generally cloudy (three-quarters covered).
4. Overcast sky.
5. Drizzle.

During the 24 days covered by the migration as a whole the weather available was made up of 3 days of "blue sky," 4 days "generally cloudy," and 17 days "overcast." Since the days on which departure probably took place were 9 "overcast," 3 "generally cloudy," and 1 with "blue sky," "fine" weather in the human sense cannot have been a determining stimulus.

Indeed the indication is that days on which the sky is wholly or mainly obscured by cloud are most favourable for migration. The statement that fine weather is a predisposing factor is, however, frequently made: Rowan states that the great migration of ducks observed in Alberta in October 1928 set out in fine weather and "in the main clear skies, though even at Edmonton the weather was cloudy on and off" (1929, p. 91). Yet the table he gives of weather conditions during five days of that migration, October 26 to 30, shows 3 days cloudy, 1 cloudy in the afternoon and clear in the forenoon, and 1 recorded as clear (p. 90). Instead of selecting clear days, the Alberta ducks, like the Waxwings, favoured cloudy weather for the commencement of their movements. If this be found to be generally true of the commencement of migrations (the weather during the remaining part of a migratory flight is beyond the birds' knowledge), a physiological explanation may be offered for the choice, namely, that the higher moisture content of the atmosphere under cloudy conditions would enable the lung surface to be kept sufficiently moist for respiratory exchange, without undue sacrifice of the water-content of the body, during a period, which might extend into many hours, when replenishment would be impossible.

4. *Wind and Migration.*

Do birds make use of a favouring wind on setting out on migration? Although in the absence of landmarks birds may be unconscious of the direction of the wind when they are launched in a moving mass of air, as Acworth (1929) has emphasised, before setting out they are aware of the local direction and force of the wind. The general course of the Waxwing migration during the time it was traversing Scandinavia was towards the south, and that direction is most likely to have been retained. Winds favouring this course in the ordinary sense would be northerly, north-easterly, or north-westerly.

In Table II the apparent waves of migration in Great Britain and Denmark are set out along with the direction and force of the wind in Southern Scandinavia on the day of their departure. The wind-force is given in miles per hour, the mean of the Beaufort notation of the

Meteorological Reports, and the usual assumption is made that the Danish birds set out on the day when they were observed, and the British birds the day before they were observed.

TABLE II.—DIRECTION AND SPEED OF WIND ON DATES OF DEPARTURE.

Apparent Waves of Migration.		Wind in S. Scandinavia on Dates of Departure.	
Britain.	Denmark.	Direction.	Speed, m.p.h.
November 10		E	10
	11	SE	22
	13	NE	5
	15	SSE	22
16		SSE	22
	17	E	22
18		E	22
	20	E	16
21		E	16
23		E	22
25		SE *	43 *
27		SSE	5
December 2–3		SSE	22

* A discrepancy occurs in the official records for wind at Skudesnaes on November 24, 7ʰ G.M.T.: while the table gives direction SE and force in Beaufort notation 8, the chart indicates a NW wind of force 4. Since the latter is quite out of keeping with the run of the isobars, I have adopted the written record.

From these data it will be seen that only one of thirteen possible waves set out with a wind favouring the southward journey, that of November 13. Of the other movements six departed in an east wind and six in winds SE to SSE which were in varying degree opposed to a southwards flight, mostly "moderate" winds in the Beaufort scale, although one was of "gale" force (Nov. 25) and one was only a "slight breeze" (Nov. 27).

At first glance it might be concluded that since, as a rule, the migrants did not commence their movements in winds favouring a southward journey, they avoided starting in a tail wind of any strength and selected a beam wind or a head wind, and such a conclusion would be at variance with the view of Defant (1913, p. 155) and favour that of Eagle Clarke, Wetmore (1926, p. 55), and other writers. While I disagree with the statement, recently repeated by Wetmore (1926) and Lincoln (1935, p. 61), that strong winds blowing in the direction in which the birds are travelling disarrange feathers when the birds are in the air, for if the bird's movement did not exceed that of the wind it would stall, nevertheless a reasonable suggestion may be made for the selection of a head wind. Birds in taking off from the ground invariably rise into the wind, and it may be that the difficulty of taking off in a tail wind induces them to wait for a wind which permits their rising in the direction in which the migration movement is to proceed.

Unfortunately no reliable conclusion can be reached from the data in the case of the Waxwings, for during the whole period covering the movement the wind in Southern Norway remained in the E to S quadrant, with the exception of the NE wind of November 13, so that during that period the birds had no opportunity of choice. It is worth noting, however, that on the three days preceding November 9, on which the first batch appears to have set out, the wind held persistently from the N and NNE. Departure began when the wind veered to E.

The safest finding in this case is that these easterly and south-easterly winds are concomitants of the anticyclonic conditions which, as has been shown, favour migratory movement, and that being so the data do not permit of a decision as to the part played by the direction of wind of itself in contributing to the start of a migration.

As regards the velocity of the wind, that appears to be a factor of little significance, the range covering from 5 to 43 m.p.h. Seven of the waves set out in a wind moving at 22 m.p.h., a high speed relative to that of the bird's flight, which almost certainly does not exceed 25 to 30 m.p.h., and one appears to have set out in a SE wind of 43 m.p.h., although the indication of the chart is that this was a local wind confined to the neighbourhood of Skudesnaes.

5. *Influence of Wind upon British Immigrations.*

In view of a current theory that many birds on migration follow definite migration routes, and that one such route or broad highway passes from Southern Norway across the North Sea to the shores of Britain, it becomes of interest to determine whether this route existed for the Waxwings and was followed by them. The essence of the theory of migration routes is that the birds select by instinct or otherwise more or less definite tracks. And the question to be examined is whether the Waxwings made a definite bid to reach Britain or whether their arrival there was entirely undetermined by them, and so far as they were concerned was purely a matter of chance.

In endeavouring to answer this question I must at the same time test another disputed point, the influence of the direction of the wind upon a migration. Eagle Clarke says: "The importance of winds in connection with bird-migration has been much over-estimated, and their bearing upon the phenomenon, such as it is, greatly misunderstood. Their direction, apart from the weather conditions to which they are due, has no significance whatever on their movements. Thus, if a migrant were crossing the North Sea from Norway to Britain, one fails to see why a north-west wind should not be as suitable for a passage as a south-east, for both would be beam winds, and yet during the prevalence of the former, such migration is practically at a standstill, and during the latter at a climax" (1912, vol. 1, p. 178).

Whether or not the *setting out* of a migratory movement may be dependent upon a wind favouring taking off from the ground in the direction of migration, the arrival or resultant of a migration in relation to wind direction is a different matter, and it seems to me that Eagle Clarke has misapprehended the problem. He happens to cite the very case, migration from Norway to Britain, which the Waxwing movements may help to settle.

An attempt has therefore been made to plot daily wind direction and force against the arrivals of the birds in Great Britain. This demands certain assumptions which will be checked later. I assume that the instinct of the Waxwings during the autumn migration is to move in a southerly direction, and that, if that be so, their appearance in Britain can be due only to the influence of wind carrying them out of their ordained path. A due north or south wind cannot affect their assumed direction of movement, an easterly wind, if strong enough, would blow them to Britain, a westerly wind, of whatever inclination or strength, would prevent their appearance in Britain.

Accordingly, arbitrary numbers have been given to wind directions (*plus* values for easterly winds, *negative* values for westerly), N and S being regarded as negligible (0), E and W as having most influence (30), positive or negative, and intermediate directions having intermediate values. But the force of the wind is of importance only second to its direction, for the ultimate line of flight is a resultant of the bird's own speed and southerly direction together with the speed and direction of the wind. In order to bring the force of the wind into the calculation I have added a number representing the speed in miles per hour, since this appears to give a fair working value relative to the arbitrary direction numbers.

It is not enough, however, to reckon the wind only at the place of origin of the migration, for winds of different direction and force may be encountered by the birds after leaving Scandinavia. Assuming that the crossing of the North Sea would occupy a period of twelve or more hours, I have added a factor embodying, on the same scheme, the direction and force of the wind on the east coast of Britain, at the most appropriate time recorded in the data of the Meteorological Charts, namely, eleven hours later than the data used for Scandinavia.

Two examples will illustrate the method employed:

Date.		Scandinavia.		Britain.			Final "Wind Value."
		Direction.	Force.	Direction.	Force.		
Nov. 8:	Chart reading	NNE	1 *	NW	3 *		
	Values allotted	+(10 +	2 †)	−(20 +	10)	=	−18
Nov. 20:	Chart reading	E	4	SE	5		
	Values allotted	+(30 +	16)	+(20 +	22)	=	+88

* = Beaufort notation.
† = Miles per hour.

These wind values are arbitrary, but they have relative significance, and that is all that is claimed for them.

Calculated for the days covering the period of the migration, they are plotted, along with the daily numbers of birds seen in Great Britain, in fig. 5. When the wind value falls below the horizontal broken line at zero, birds cannot have been carried to Britain by wind. If they arrived there under these conditions their course must have been determined by an impulse which compelled them to battle against contrary winds, and if that were so it would indicate that they were following a predetermined "migration route."

But no arrival in Britain occurred under these conditions. Indeed, assuming that generally the birds were seen here on the day after they travelled, it is apparent that the great influxes coincided with winds strongly favouring the drift of the birds from Southern Scandinavia to Britain, and few birds were seen on the day following adverse winds. The tailing off of the influxes, particularly the modest peak of December 2–3, which occurred when wind values were favourable for the E to W drift, can be attributed to the likelihood that the reservoirs of birds in Southern Scandinavia were by this time almost depleted of their numbers, as we know was the case in Southern Sweden.

Further, non-arrival in Britain was not due to the fact that the birds were not migrating, for on November 11 and 13 they appeared in considerable numbers in Denmark, 179 and 197 respectively on the two days. The explanation suggested by the graph is that on these days winds were not favourable for drift to Britain.

Correspondence between the wind values and the appearance of birds in Britain can be tested in another way. The more favourable the wind is, in direction and force, for carrying birds westwards across the North Sea, the farther north on the coast of Britain will they land. I have added to the graph (fig. 5) of the numbers of birds recorded from Britain, a second graph representing the numbers recorded from England alone. The general inference is apparent, that on days most favourable for the westerly crossing, the whole or the vast majority of the birds were recorded from Scotland (*see* peaks of November 16, 18, 21). The English birds formed the major part of the contingent notably on days when the wind values were distinctly lower (*see* peaks of November 23, 25). The peak of December 2–3, while it gives a fair interpretation of the arrivals over a few days, is not reliable from day to day, since some of the records were rather vaguely stated as "about Dec. 1" and "first week in December," and the inaccurate distribution of these records may have distorted the daily relationship of numbers. As it stands, it is the only exception to the inference that the more northerly arrivals came with the more favourable winds.

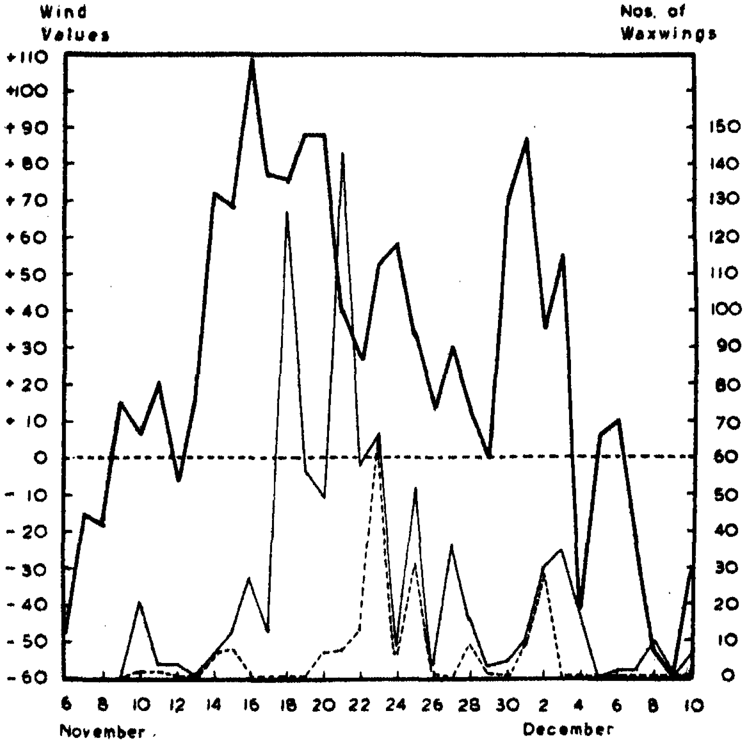

FIG. 5.—Daily "Wind Values" across North Sea (see text) and records of Waxwings seen in Great Britain and separately in England.

▬▬▬ Wind values.
——— Waxwings in Great Britain.
– – – – Waxwings in England separately.

A final test may be made of the original assumption that the flight of the birds themselves was southerly and that the westerly inclination was imposed by the wind. One or two of the migration waves appear to have landed mainly in a localised area. This was notably the case on November 21, when of 142 birds recorded about a hundred were seen in Berwickshire and ten in the neighbouring county of Haddington. On the morning of the previous day the wind in the south of Norway was E at about 16 miles per hour, and in the evening on the coast of Britain SE at about 22 m.p.h. The average direction may be taken as ESE and the average speed 19 m.p.h. If we assume that the birds move south at 25 m.p.h., and impinge upon that movement a steady force from ESE at 19 m.p.h., the resultant, as shown in fig. 6, will represent the actual track of the migration, namely, a movement to the SW at a speed of 25 m.p.h. This result can be only approximate, for we do not know exactly the speed of the birds, nor can we be sure that the winds on the E and W side of the North Sea shared equally in influencing a journey across that Sea. Nor do we know exactly whence the birds set out; but if the apex of the triangle of forces be placed on Rogaland and the supposed original line

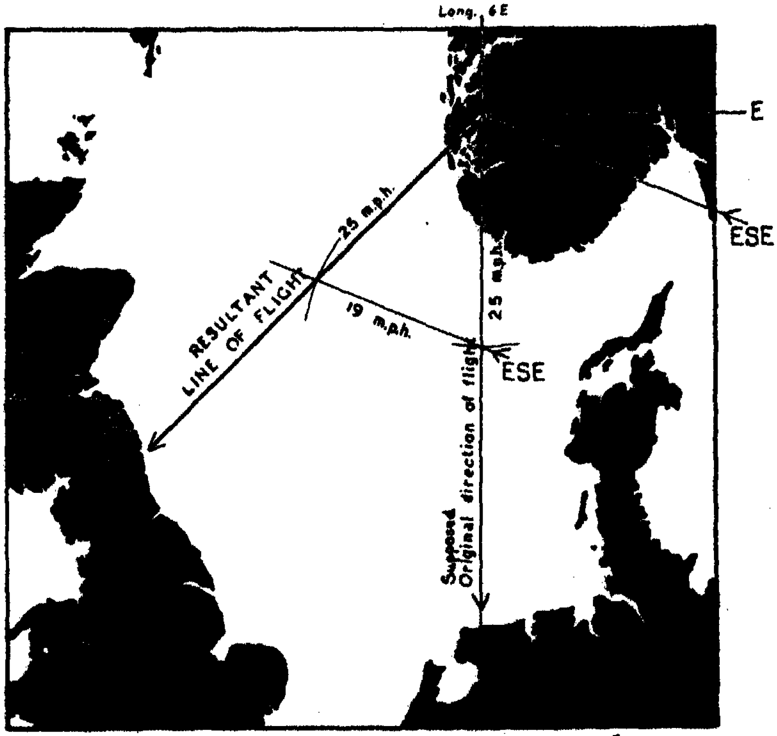

Fig. 6.—Effect of winds across North Sea (November 20, 1921) on the supposed southerly flight of Waxwings from Norway. Each white dot in Britain represents a Waxwing seen on November 21, 1921 (the Berwickshire birds were mostly near the coast).

of flight of the birds be taken as along Long. 6 E, then the resultant line of flight (as fig. 6 shows) would theoretically have landed the birds in Northumberland. The actual observations of the Waxwings are indicated by white spots on the figure (although the Berwickshire group has expanded far beyond the coastal region where it mainly occurred); and these observations are sufficiently close to the expected result to suggest that the assumptions made are not far from the truth.

Since then, by all the tests applied, the observations of the Waxwing numbers fit generally with "wind values" allotted on the assumption that the migration journey is a southwards movement, into which wind may bring a modified resultant direction, it may be taken that the assumption is correct. Accordingly, the appearance of the Waxwings in Britain is due, not to the choice or instinct of the birds, but to the hazard of winds; in other words, for Waxwings there is no "migration route" in any significant sense from Scandinavia to Britain.

If this finding about the Waxwings applies to the more regular migrants in autumn, Eagle Clarke's denial of the influence of wind upon migrants is mistaken, and the stoppage of immigration to Britain during a NW wind and its climax during a SE are direct results of wind influence.

While, however, Waxwings arrive in Britain because the wind (impinged upon their own flight) carries them there, it cannot be said that they are carried on "favouring" winds. During the whole period of the migration, from November 9 to December 3, favourable winds, that is winds bearing from some northerly direction, appear in the records of the Norwegian Station on only three occasions—in the morning of November 13 and the evenings of November 20 and 21—so that the majority of the movements must have taken place in winds opposed in some degree to southward migration.

Such conditions of migration in opposing winds are very different from those described by Defant for the northward migration of birds arriving in Austria in spring, when different species were found to arrive with, and according to Defant to "make use of," a favouring south-east wind: "Aus der ganzen Art and Weise, wie der Wind den Frühlingszug nach Norden beeinflusst, erscheint es also als feststehend, dass die Vögel zu ihrer Wanderung sich der südöstlichen Winde bedienen, somit stets fast nur mit dem Winde fliegen und nicht gegen ihm" (1913, p. 155). It may be that the discrepancy between Defant's results and mine is due to some difference between the character of spring and autumn migration.

While I regard the appearance of the Waxwings in Britain to have been determined by the wind, I do not imagine that the birds are carried passively like balloons; they exercise some control over their ultimate destination. This is indicated by the fact that the resultants, based upon the direction and speed of the wind and of the birds, tend to place the Waxwings to the south of their actual landing areas, a discrepancy which I interpret as meaning that whenever the birds perceive land they change direction and fly towards it,* and on the approach to Britain from Scandinavia such a change of course would land them to the north of the calculated position. Further, there is an indication that a fresh influx of birds was noted in England on November 23 (fig. 5), but on the evening of the previous day a slight southerly to south-westerly breeze (5 m.p.h.) was blowing off the coast of Britain, and this, had the birds been entirely passive, would have prevented their arrival. Moreover, it is a matter of observation that migrants may arrive on the coast flying against the wind.

But the end of a journey is no criterion of the dominant conditions under which it was carried out, and these in the case of the Waxwing migrations show that the wind was a major factor in accounting for their appearance in Britain.

VII. SUMMARY.

1. An extensive migratory movement of Waxwings (*Bombycilla garrulus*) in 1921 offered special opportunity for testing with precision the influence of weather upon migration.

2. Large numbers of Waxwings were recorded in Great Britain (825 records) and Denmark (1219 records), and the daily records suggest that the migration was composed of successive waves. Some waves were common to both countries, a day's lag in the British arrivals being attributed to the time taken in crossing the North Sea.

3. The starting-point of the migration is traced to Southern Scandinavia, and accordingly weather conditions there are tested against the succession of movements.

4. The migration was confined to a period of about three weeks when barometric pressure in Scandinavia was very high. It was preceded and succeeded by very low pressures.

5. Migration proceeded from an area of high atmospheric pressure towards an area of low pressure during anticyclonic conditions. It began with the setting in of such conditions and ceased when they failed.

6. The setting out of the migration waves was not definitely related to a specific change in atmospheric pressure, although the conditions associated with a rising barometer seem to have been more favourable.

7. The whole migration occurred over a period of normal temperature

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* A similar visual stimulus might well have induced Waxwings located in the neighbourhood of Helsingborg in the south of Sweden to cross the Ore Sund westwards to Denmark instead of continuing the southern trend of migration.

for the season following a sudden temporary drop. Migration waves most often set out when the temperature had been steady for a day or two.

8. With one exception the thirteen possible migration waves set out under cloudy or overcast skies, and a physiological explanation is suggested for such an association.

9. Only one wave appears to have set out with a favouring, that is a rear wind; in all other cases the wind was easterly or in some degree opposed to a southward journey.

10. Analysis of wind directions and force across the North Sea suggests that the Waxwings instinctively moved in a southerly direction and that their appearance in Britain was due to their being blown there.

11. The Waxwing migrations to Britain did not take place, like those described by Defant (spring migrations), before the wind, but always in some degree against the wind.

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(*Issued separately September* 30, 1940.)

XXIV.—Early Glacial Remains of Reindeer from the Glasgow District.

By **M. Macgregor,** M.A., D.Sc., H.M. Geological Survey (Scotland), and **Professor James Ritchie,** M.A., D.Sc., Department of Zoology, University of Edinburgh. (With One Plate and Three Text-figures.)

(MS. received May 29, 1940. Read June 3, 1940.)

In the early part of 1937, during the course of excavations for a sewage extension tunnel at Queen's Park, on the south side of Glasgow, some pieces of bone were obtained by the workmen from a bed of gravel at a depth of 70 feet from the surface. Through the kindness of Mr T. Somers, Master of Works and City Engineer to the Corporation of Glasgow, these remains were loaned to the Geological Survey and, again with his sanction, submitted for examination to Professor James Ritchie of Edinburgh University, one of the collaborators in this note. Professor Ritchie reported that the fragments of bone fitted together to form the greater part of the right radius and ulna of a reindeer. At a later stage of the excavations a portion of an antler was disinterred from the same bed. It seems desirable to record these occurrences in some detail, and for convenience this communication has been divided into two parts. In the first of these the geological position and age of the fossiliferous deposit are briefly discussed, while the second part is devoted to a description of the remains themselves and to a comparison with similar remains elsewhere.

1. GEOLOGICAL POSITION AND AGE. By M. Macgregor, M.A., D.Sc.

The sketch-map forming text-fig. 1 shows the distribution of the various superficial deposits at the locality, the line of the tunnel, and the sites of the manholes to which reference will be made in the course of this account (Nos. 4, 5, 6, and 7). These deposits consist from above downwards of

Recent alluvium of the River Cart.
100-Ft. Raised Beach deposits (clays, sandy clays, sands).
Boulder Clay.

Text-fig. 2 is a horizontal section along the line of the tunnel between Manholes Nos. 4 and 7. This section is based on a large-scale one, kindly supplied by the City Engineer's Department, but incorporates certain direct observations made by the writer. The more easterly of the two crosses shown on the line of the tunnel indicates where the fragments of bone were discovered, at a depth of 70 feet; the more westerly one marks the locality where the portion of antler was later obtained, at a depth of 53 feet. The first visit paid to the site was at the end of February 1937, shortly after the bone fragments had been found, and thereafter visits were made from time to time, as opportunity offered. The observations made on these occasions are recorded below. The writer would like to acknowledge his great indebtedness to Mr R. A. Lyall, the Corporation engineer in charge of the excavations, for the assistance he rendered in the investigations.

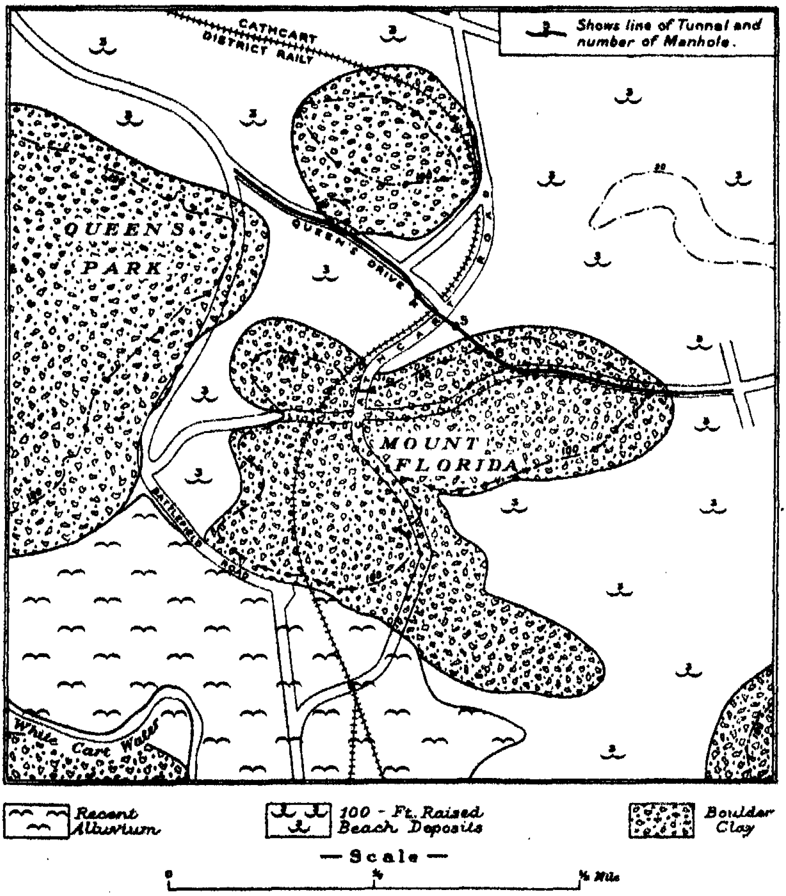

FIG. 1.—Sketch-map to show site of excavation.

Manhole No. 4 (75 feet above O.D.).—This manhole showed, according to Mr Lyall's records,

	ft.
Turf and soil . .	6
Brown clay . .	13
Soft plastic clay . .	21

In the tunnel itself, at the foot of this manhole and for some distance to east and west, stiff, dark-coloured, stony boulder clay was noted by the writer. The material thrown out of the upper part of the manhole was a laminated, somewhat dark, muddy clay containing sandy layers and, near the top, irregular wisps of vegetable matter. It yielded a few ostracods and an occasional fragment of shell. These beds and the underlying soft plastic clays undoubtedly belong to the deposits of the 100-Ft. Raised Beach; soft plastic clays, often reddish in colour, are characteristic of the lower part of the deposits of this period in the Glasgow-Paisley area. The junction of these clays with the stony boulder clay below was not observed.

Manhole No. 5 (91·4 feet).—This manhole, which was never completed, showed, according to the engineer's records,

	ft.
Surface material . .	1
Forced material . .	10
Muddy clay and dark running sand . .	17

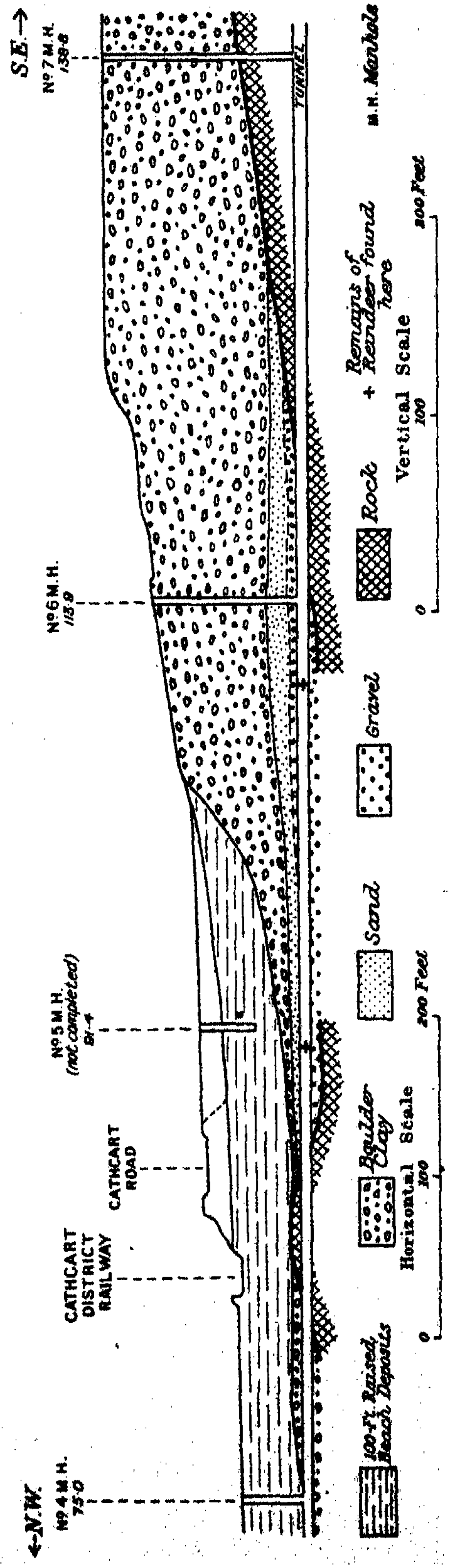

FIG. 2.—Horizontal section along part of line of excavation.

The upper 11 feet or so represent waste material filling in the hollow that formerly existed here on the east side of Cathcart Road. This artificially filled-in area is indicated by the unshaded portion of text-fig. 2. The muddy clay and dark running sand may be taken as 100-Ft. Raised Beach deposits laid down near the margin of the Mount Florida drumlin.

Manhole No. 6 (113·9 feet).—The section of this manhole reads as follows:—

	ft.
Surface material	2
Brown sandy clay	8
Boulder clay	2
Sand	4½
Boulder clay	33½
Sand, muddy, very wet	12
Gravel	17

It was not possible to verify this section in detail, but all the excavated and dumped material examined by the writer was boulder clay. The gravel bed was also examined in the tunnel close to the foot of the manhole. The 8 feet of brown sandy clay at the surface may be regarded as weathered boulder clay, while the 4½-feet layer of sand is no doubt one of the lenticular beds of sandy material not infrequently encountered in the boulder clay of the district. The gravel deposits at the base of the section are referred to later.

Manhole No. 7 (138·8 feet).—This manhole, 103 feet deep, showed superficial deposits to a depth of 70 feet as follows:—

	ft.	in.
Surface (*i.e.* road material)	1	3
Brown clay and boulders	4	..
Dark boulder clay	38	..
Reddish clay and stones	8	..
Dark boulder clay	10	..
Grey sand	..	9
Dark boulder clay	8	..

The rest of the manhole was in solid rock.

The Tunnel Section.—Text-fig. 2 illustrates the nature of the material encountered in the tunnel. As already stated, boulder clay was noted by the writer close to the foot of Manhole 4. The tunnel was driven in this clay for 100 feet and then cut solid rock, consisting mainly of sandstone, for another 100 feet. The eastern face of this rock-ridge was considerably steeper than the western; beyond it the tunnel entered sand and gravel. These deposits were met with up to a point beyond Manhole No. 6. Sections in the gravel were examined by the writer at three points: (1) close to where the reindeer bones were found; (2) a little west of where the antler fragment was got; and (3) a little east of No. 6 Manhole. Section (2) may be recorded:

	ft.	in.
Roof of Tunnel:		
Slightly reddish, laminated clayey sands, with irregular base	3	..
Gravel and sand, bedded; pebbles, many water-worn, up to 3 or 4 in. in diameter	4	2

In section (3) rock was first seen in the floor of the tunnel about 13 feet east of Manhole No. 6 and showed irregularly in the floor for a short distance before reaching the roof at 111 feet from the manhole. In general the gravel was a rudely bedded deposit with a good deal of fine sandy and clayey material in it. Some of the included stones were definitely water-worn, others more angular. Among them were igneous rock types (porphyritic basalts, fine-grained basalts), together with more local sedimentary types. As already stated, the gravel was examined only at three points, but its continuity along this part of the tunnel is shown by the records of the engineer in charge.

Age of the Gravel.—When the gravel was encountered in the western part of the tunnel it was thought to underlie the Raised Beach deposits directly and to be accordingly of Late Glacial age. But later investigations and the fuller information available from the completed tunnel showed that this view is not tenable. The following points should be noted:—

(1) At Manhole No. 6 the gravel is overlain by at least 50 feet of boulder clay.

(2) It is a continuous bed resting against solid rock at both its eastern and western ends.

(3) While there is no evidence as to what underlies it along the greater part of its extent, it appears to occupy a hollow in the rock-surface.

(4) It extends east of Manhole No. 6 for approximately 111 feet.

These data show that it must be regarded as of earlier date than the boulder clay of the drumlin. It may be inferred that it represents part of a stream-gravel that has here been protected and preserved owing to its occurrence in a rock-basin. Its age cannot be directly determined as there is no evidence as to what underlies it along the greater part of its course. The authors suggest, however, that it corresponds in age with the fossiliferous beds found near Kilmaurs, in North Ayrshire, beneath the boulder clay of the district. These beds, which have yielded remains of the mammoth and reindeer, are generally accepted as interglacial in age. As regards the deposit now being considered, it is noteworthy that the character and composition of the gravel suggest derivation from a pre-existing boulder clay. The questions of correlation and age are, however, more fully dealt with in the following section.

2. THE REINDEER RELICS. By Professor James Ritchie, M.A., D.Sc.

These consisted of part of an antler and three fragments of limb bones. Although the antler and the limb bones were found more than 200 feet away from each other, they occurred in the same deposit.

(a) *The Antler.*

The antler fragment (Pl., fig. 1) consists of portion of a beam, 60 cm. long measured on the outside of the curve, with a circumference of 112 mm. at the distal end and 109 mm. at the proximal end. The latter is almost circular in section, the former rather flattened, but the greatest flattening occurs at the curve, on the outer margin of which is the base of a small tine which has been broken off. The surface is smooth, except for a certain amount of pitting especially on the inner aspect.

The dimensions of the beam, its smooth surface and wide curve, and particularly the presence of a small back tine, are typical of antlers of Reindeer, *Rangifer tarandus.*

In the accompanying table a comparison between the Glasgow fragment and the antler of a modern reindeer of the *arcticus* type suggests that the fossil specimen was rather stouter than a typical modern antler.

	Glasgow Fragment.	Modern Antler.
Circumference of beam (*a*) above bez tine	109 mm.	106 mm.
„ „ „ (*b*) 6 in. above small back tine .	112 „	104 „
Length of antler along outside of curve	..	112 cm.

Some of the pitting on the antler is due to crushing of the surface by incisor and canine teeth of a carnivore. The size and the distance between some of the pits suggest that they were made by a wolf, but no bite is sufficiently complete to make identification sure.

(b) *The Limb Bones.*

These consisted of three fragments of a fore-limb which fitted together at their fractures to form the upper or proximal two-thirds of the right radius and ulna of a Reindeer (Pl., figs. 5, 6). They show that their owner was adult but of rather small size.

The bones are pale brown in colour and are well preserved, having a hard, polished surface and splintering with sharp fractures. The fractures appear to be recent, but there is some abrasion at the natural free edges of the articular surface of the radius and the free end of the olecranon process of the ulna, suggesting a certain amount of water-rolling. There are no striations suggesting the action of ice, and the only superficial markings are a few abrasions and many minute scratches which have probably been scored on the surface by the teeth of carnivorous animals.

Bones of reindeer, other than antlers, have very seldom been found in Scottish deposits, and on this account a comparison has been made between the Glasgow bone and corresponding bones found about 1886 in a rock-fissure at Green Craig, near Dreghorn, in the Pentland Hills, and now preserved in the Anatomical Museum of the University of Edinburgh. The latter were associated with remains of a moderately sized horse, a wolf, and a fox (J. Simpson, 1887). These radii and ulnæ, recorded by Simpson as belonging to the Cervidæ, are now definitely attributed to the reindeer, and so fall into line with the other bones which he identified specifically.

COMPARISON OF FOSSIL REINDEER BONES FROM GLASGOW AND THE PENTLAND HILLS, WITH MODERN REINDEER AND RED DEER.

	Reindeer.				Red Deer. Modern.§
	Glasgow Fossil.	Pent. Hills Fossil.*	Pent. Hills Fossil.†	Modern.‡	
Radius:					
Length . .	185+ mm. (*c.* 235 when complete)	285	..	263	225
Proximal end—					
long diam. .	43+	51	46	47	47
short diam. .	25	28	26	28	25
Neck, diam.. .	29	35	31	31	23
Ulna:					
Olecranon process—breadth . .	44+	50	..	47	40

* No. 19 in Simpson's paper—a left radius with olecranon process detached.

† No. 21 in Simpson's paper—a fragment, the proximal articular surface with a small portion of the shaft of a right radius.

‡ Right radius and ulna of skeleton in Royal Scottish Museum.

§ Right radius and ulna of Scottish Red Deer skeleton in Royal Scottish Museum.

It is impossible from the characters of the Glasgow bones to suggest the race of Reindeer to which the individual belonged; all that can be said is that the proportions agree with those of the individuals represented in the Pentland Hills rock-fissure and of a modern Reindeer, but that it was considerably less in size.

Since confusion might arise between the radii of reindeer and red deer, measurements of the latter are included in the table. The most noticeable difference is that the reindeer bone is more robust, since the shaft does not show the strong narrowing at the neck, below the proximal articulating surface or head, which marks the red deer radius. Roughly it may be said that the breadth of the neck of the radius in the red deer is less than half the greater diameter of the head of the radius, while in the reindeer it considerably exceeds one half. In the specimen examined the relationship, stated as the ratio of breadth of "neck" to greater diameter of "head," is 0·49 in the red deer and 0·66 to 0·68 in the reindeer.

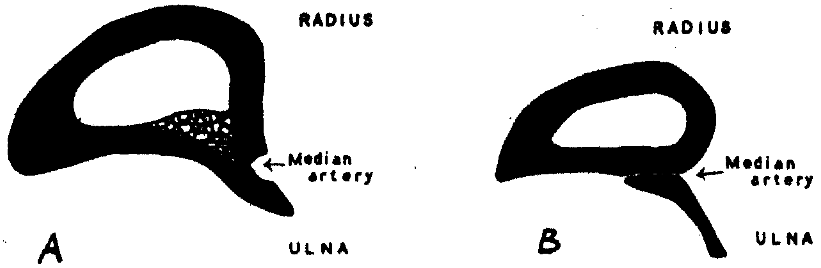

FIG. 3.—Comparison of transverse sections of radius and ulna of A, Fossil Reindeer, and B, Modern Red Deer. Natural size.

Another distinction lies in the relationship of the ulna to the radius (text-fig. 3 and Pl., figs. 3, 5). In the reindeer the distal two-thirds of the ulna is fused to the radius so that the surfaces of the two are continuous and the position of the junction of the two bones is indicated only by a groove in which lay the median artery and nerve. In a transverse section the thoroughness of the fusion is apparent (text-fig. 3, A) and the resulting modification of shape in the neighbourhood of the junction has given rise to an area of cancellous bone which impinges upon the cavity of the radius. In the red deer the distal portions of ulna and radius are closely and firmly opposed, but there is no intimate fusion and the two bones meet at an open angle in which the median artery and nerve lay (text-fig. 3, B).

(c) *Remarks regarding those Remains, and other Glacial Records of Reindeer from Scotland.*

Although the race of reindeer cannot be determined from these fragments, the indications of the limb bones are that the individual to which they belonged was a small animal, whereas the antler was rather more robust than normal. It is possible, therefore, that the two relics found 200 feet apart represent two individuals.

Remains of reindeer have hitherto been found in some twenty Scottish localities (Reynolds, 1933, gives records up to that date) in deposits ranging from the Ice Age to the periods of the brochs of Caithness and Sutherland, although Simpson (1887), Henderson (1887), and the Abbé Breuil (1922), all upon very little evidence, have cast doubts upon the authenticity or significance of the relics from the brochs. Five of these earlier records have been attributed to the Glacial Period, but for one the evidence is unsatisfactory.

The reindeer bones from a rock-fissure in the Green Craig, in the Pentland Hills near Dreghorn, have already been referred to. They are described by Gregory and Currie (1928, p. 18) as "probably not later than the Upper Palæolithic," and in a table (p. 21) are included in the mammoth-reindeer period associated with the "middle boulder clay." * This determination was based upon the statement that the bones were "gnawed by hyena" and must therefore "date from the occupation of Scotland by the hyena" (p. 18). I have examined the bones in the Anatomical Museum, and although several are split and a few show indentations caused by the teeth of some carnivore, there are no tooth-marks which can be identified as those of the hyæna, and there is no record of the occurrence of the hyæna in Scotland to support such an identification. Furthermore, the bone-splitting habit of the hyæna is so invariable that, according to Boyd Dawkins (1874, p. 317), "their dens are characterised by the absence of perfect long-bones and skulls, and consequently when these occur in a cave it is certain proof that it was not occupied by these animals." Quite a number of unbroken long bones were found in the Dreghorn rock-fissure.

Simpson in his original account does not definitely state that the work was that of a hyæna: the bones, he writes, "are quite evidently gnawed and split up in the manner so characteristic of the bones of the prey of such carnivores as the hyæna" (p. 294). The presence of the bones of a horse is against the early date assigned by Gregory and Currie, but these authors suggest that as these bones are not stated to have been hyæna-gnawed they may have been imported to the fissure at a later date. In appearance, however, they seem to be of the same age and are split like some of the others, and in the absence of any record that they were obtained at a higher level in the deposit than the reindeer remains, they cannot safely be regarded as belonging to so different a period. The indications are, therefore, that the Dreghorn reindeer remains do not belong to the early interglacial period attributed to them by Gregory and Currie, and may not even be of glacial date at all.

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* The table indicates that the Dreghorn bones are in the Hunterian Museum, Glasgow, but they are in the Anatomical Museum of the University of Edinburgh.

Of the remaining four earlier records, two appear to be of Late Glacial age.

The large collection of antlers and bones, representing at least 400 reindeer, found in the "Reindeer Cave" during the excavations at Allt nan Uamh, near Inchnadamph, in Sutherland, I have correlated with Magdalenian times in the Late Glacial period (Ritchie, 1928, p. 193). In the Late Glacial period also may be placed the fragment of antler found at Croftamie,* near Loch Lomond, in a bed of laminated clay under boulder clay which appears to belong to the district glaciation of the 100-Ft. Beaches (Gregory and Currie, 1928, p. 20).

To an earlier epoch belong the antlers found in a bed of peaty clay, under a bed with Arctic marine shells and boulder clay, at Woodhill Quarry, Kilmaurs, where they were associated with remains of the mammoth. These are assigned to the interglacial period which preceded the last great extension of the ice-sheet.† To the same age probably belongs, according to Gregory and Currie, the antler fragment with well-marked glacial striæ found in boulder clay at Raesgill, near Carluke, which Reynolds (1933, p. 27) erroneously records as of "Iron Age."

The reindeer remains from Glasgow described in the present paper, found in a stream gravel underlying boulder clay which itself is overlaid by deposits of 100-Ft. Beach times, may tentatively be ascribed to the same interglacial period.

Our thanks are due to the Carnegie Trust for the Universities of Scotland for a grant towards the cost of the illustrations.

DESCRIPTION OF PLATE.

Fig. 1. Part of beam of antler of Reindeer from Glasgow deposit, showing position of back tine and indentations caused by teeth of a carnivore. About ¼ natural size.

Fossil Reindeer bones from Glasgow and the Pentland Hills. Slightly under ⅓ natural size.

- Fig. 2. Pentland Hills, rock-fissure near Dreghorn—olecranon process of left ulna.
- Fig. 3. Same locality—corresponding radius and part of ulna.
- Fig. 4. Same locality—proximal fragment of right radius.
- Fig. 5. Glasgow deposit—parts of right radius and ulna.
- Fig. 6. Same locality—corresponding olecranon process.

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* Wrongly recorded as Croftaurie in Woodward and Sherborn, 1890, p. 379.

† This corresponds to James Geikie's "Second Interglacial Period" (1914, pp. 253–257).

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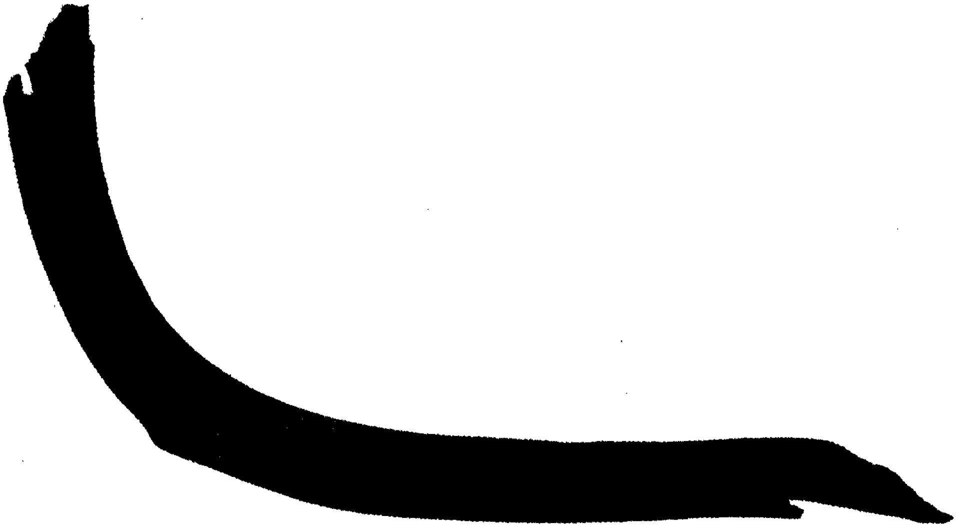

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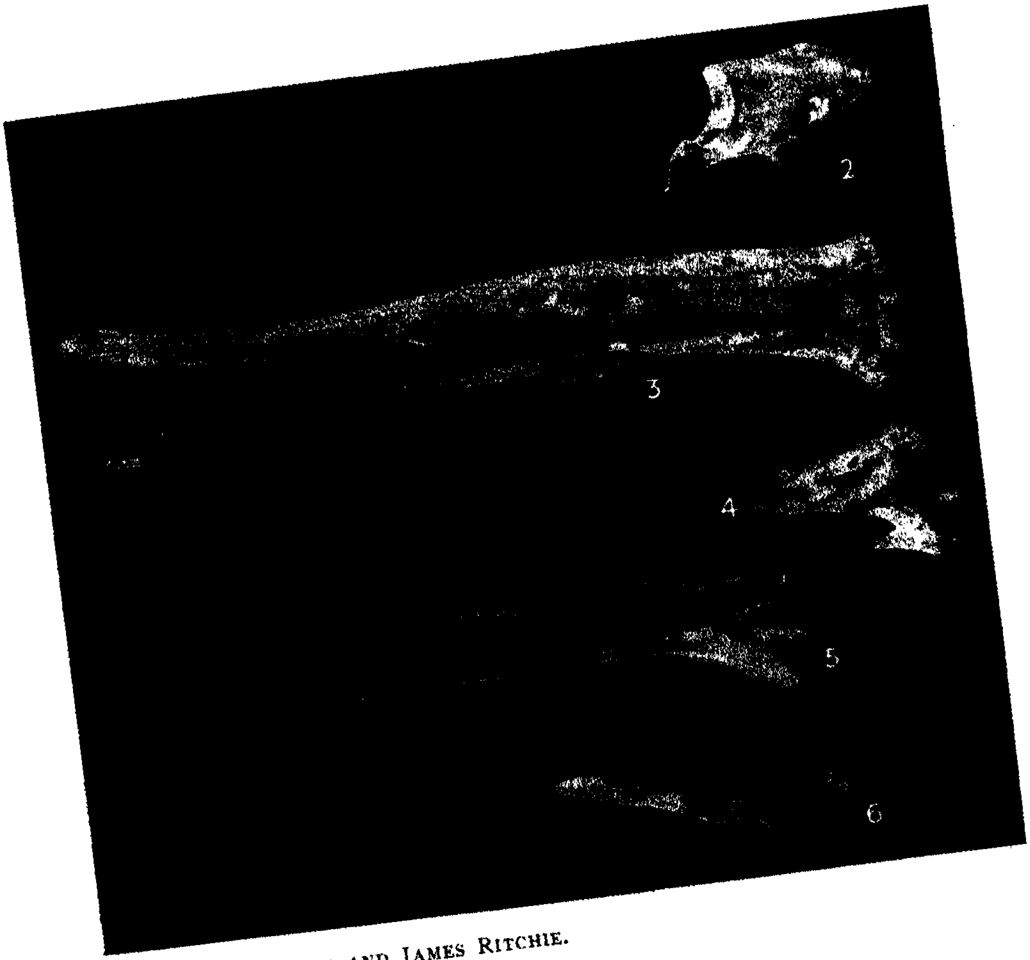

M. Macgregor and James Ritchie.

XXV.—The Effect of Increased Daily Illumination and of Reversed Day and Night on the Œstrus Cycle of the Mouse (*Mus musculus*).

By **R. A. R. Gresson,** Ph.D., Department of Zoology, University of Edinburgh. *Communicated by* Professor J. RITCHIE, M.A., D.Sc.

(MS. received June 19, 1940. Read July 1, 1940.)

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I. INTRODUCTION.

IN the course of a recent investigation on the fertilised egg of the mouse (Gresson, 1940 *a*, *b*) it was necessary to keep a record of the time at which pairing took place. This revealed that copulation, especially in the winter, was very infrequent during the hours of daylight, and usually took place at about 12 midnight or between midnight and early morning. Similar observations were made by Lewis and Wright (1935), who found that in the autumn and winter mice seldom copulated during the day, and that while the majority of pairings took place between 10 p.m. and 1 a.m., several of their mice paired between 1 a.m. and 7 a.m., whereas in the spring pairing occurred more readily during all hours of the day.

In an attempt to obtain pairings during the day the writer decided to keep some female mice in total darkness during the day and under bright illumination during the night. It is well known that extra illumination accelerates the onset of sexual activity of certain mammals. In the ferret œstrus has been induced during the winter months by exposure to extra evening illumination (Bissonnette, 1932; Bissonnette and Bailey, 1936; Allanson, Rowlands and Parkes, 1934). Marshall and Bowden (1936) and Marshall (1940) found that ferrets treated with ultra-violet light usually remain on heat for a longer period than those subjected to light irradiation. Acceleration of the onset of sexual activity was induced in male and female raccoons by the application of increased daily periods of illumination (Bissonnette and Csech, 1937).

The shortening of the daily period of light from 15 to 9 hours almost prevents reproduction of the field mouse, *Microtus agrestis* (Baker and Ranson, 1932). Whitaker (1936) accelerated sexual activity in the white-footed mouse, Peromyscus, and later (1940), by varying the periods of artificial illumination, found that the highest percentage of white-footed mice in breeding condition occurred during the "long day" periods.

Although a considerable amount of work has been carried out on the effect of light on reproduction, there appears to be only one record of an attempt to induce copulation during the day in an animal which normally pairs at night. Hemmingsen and Krarup (1937) kept 16 female rats in complete darkness during the day and under illumination during the night. The animals normally pair at about 12 midnight. An examination of the experimental animals showed that the spontaneous muscular activity, mating instincts, and cyclic vaginal changes were shifted 12 hours —that is, a complete reversal of œstrus was produced, the new œstrus rhythm being established about 10 days after the reversal of the lighting conditions. The authors conclude "that the direct action of darkness does not immediately and directly produce the manifestations of œstrus. But once a certain rhythm of darkness and light has become established, the œstrus phenomena centre about the middle of the dark periods." Hemmingsen and Krarup found that alternation of 8 hours dark and 8 hours light does not abolish the 24-hour activity rhythm, and that constant light stimulated œstrus in 4 females in which a previous reversal of sexual activity had been obtained.

The experiments described in the present paper were undertaken in order to obtain cytological preparations of the stages of fertilisation. They show that a "long day" period and reversed lighting affect the sexual activity of the mouse.

II. METHODS.

The experiments described in the present paper were carried out between November 1937 and July 1938. The stock was a mixed one of white and coloured mice.

Female mice were housed in a constant-temperature room maintained at 74° F. and illuminated by electric light for 16 to 17 hours a day commencing at about 6 p.m. The light was turned off in the morning so that the animals were kept in total darkness during 7 to 8 hours of the solar day. The conditions approximated to those of summer in which day and night were reversed. The mice were provided with a small quantity of fine wood shavings for bedding placed in one corner of the cage, and the cages were placed in such a position that all parts were brightly illuminated when the light was turned on.

The controls were kept with the general stock, in an inside animal room, and were housed and fed in exactly the same manner as the experimental animals. The room was heated and the temperature maintained as close as possible to that of the constant-temperature room; it was impossible, however, to avoid slight fluctuations, but these were so slight as to be unimportant. The room was illuminated by daylight and was kept in darkness during the night.

As male mice will pair at any time of the day throughout the year, the males were kept with the general stock.

The males were introduced into the experimental cages, containing females on œstrus, at intervals during the day beginning at about 11 a.m., and were finally removed at about 5 p.m. The females were examined every half-hour for vaginal plugs, and the time of copulation determined to the nearest half-hour. In some cases the exact time of pairing was recorded.

Males were introduced into the control cages at intervals between about 11 a.m. and 5 p.m. When pairings took place the time was recorded.

My thanks are due to Professor James Ritchie for placing the facilities necessary for the carrying out of the experiments at my disposal, and for reading the manuscript.

III. Results of Experiments.

In the majority of the experiments a careful control was kept, consisting of litter mates or, where this was impossible, of mice of the same age. In the first experiment, however, no special controls were kept, the general stock, consisting of about 12 mice, being used as control.

An examination of Table I shows that 7 mice, 4 weeks old, were subjected to reversed lighting on November 19. One of these paired after 17 days at 5.45 p.m., and in 4 of the others a complete reversal of the time of pairing was established in between 27 and 68 days. The remaining 2 failed to pair and were removed from the experimental cage slightly later. Three mice were added on January 17; these showed a reversal of œstrus in 4 to 47 days. In this experiment 6 mice or, if the individual which paired at 5.45 p.m. be excluded, 5 out of 10 mice underwent a reversal of œstrus in December and January, a time of year when no pairings were recorded, during the day or night, in the general stock.

Table II represents a second series of experiments set up in March and April: the general stock was used as control. Nine of the ten mice paired during the day in 9 to 38 days after the beginning of the experiment. Amongst the stock animals pairings were more frequent during the daytime than in the preceding 2 months.

Tables III, A and B, represent the first experiment of which a careful control, consisting of litter mates, was kept. Three of the five experimental animals paired within 32 days, at times varying from 11 a.m. to 4.15 p.m., and two controls paired at 12 noon in 24 and 34 days.

In the experiment dealt with in Tables IV, A and B, all the experimental mice paired during the daytime, while only 2 of the 5 controls did so although the experiment was continued until after May 18, and thus extended into a period in which copulation during the day is more frequent than during the winter months.

In an experiment commenced in March (Table V, A) all the mice paired during March, April, and May, while all the controls paired in April, May, and June (Table V, B). Only one of the experimental animals paired in an appreciably shorter time than the controls; it must be remembered, however, that these pairings took place during the spring and early summer.

An experiment was started on April 25 (Tables VI, A, B). All the experimental animals paired within 38 days. Two of the controls paired in 44 and 49 days; the remaining 2 did not pair although the control was continued until the middle of June.

Seven of the nine mice dealt with in Table VII, A, paired within 25 days, while none of the 7 controls did so although kept until the end of June.

Table VIII deals with 8 mice which had been kept as controls for various periods, and were then subjected to reversed light and dark periods. None of these animals had paired while in the control groups. The 3 mice transferred on March 3 paired within 35 days; 2 of the five transferred on June 27 paired in 9 days' time. The experiment was discontinued about the middle of July.

In Tables I–VII, A, each entry represents one mouse.

TABLE I.

1. Age.	2. Beginning of Experiment.	3. Date of Pairing.	4. Days Elapsed between 2 and 3.	5. Time of Day.
4 weeks	Nov. 19	Dec. 6	17	5.45 p.m.
,,	,, 19	,, 16	27	12.0 noon
,,	,, 19	,, 23	34	11·0 a.m.
,,	,, 19	,, 31	42	11.15 ,,
,,	,, 19	Jan. 25	68	11.0 ,,
,,	,, 19	Did not pair	..	..
,,	,, 19	Did not pair	..	..
,,	Jan. 17	Feb. 10	25	11.40 a.m.
Adult	,, 17	Jan. 21	4	11.0 ,,
,,	,, 17	Mar. 4	47	1.30 ,,

Total No. of mice 10. Paired 8.

TABLE II.

1. Age.	2. Beginning of Experiment.	3. Date of Pairing.	4. Days Elapsed between 2 and 3.	5. Time of Day.
4 weeks	Mar. 4	April 4	31	11.0 a.m.
,,	,, 4	,, 5	32	3.45 p.m.
,,	,, 4	Did not pair	..	..
3 weeks	April 4	May 12	38	1.0 p.m.
,,	,, 4	,, 12	38	11.30 ,,
,,	,, 11	,, 3	22	12.0 noon
,,	,, 11	,, 7	26	12.30 p.m.
,,	,, 11	,, 14	31	12.0 noon
Adult	,, 4	April 13	9	1.40 p.m.
,,	,, 4	May 2	28	12.30 ,,

Total No. of mice 10. Paired 9.

TABLE III.

A.

1. Age.	2. Beginning of Experiment.	3. Date of Pairing.	4. Days Elapsed between 2 and 3.	5. Time of Day.
4 weeks	Dec. 17	Dec. 31	14	11.0 a.m.
,,	,, 17	Jan. 13	27	4.15 p.m.
,,	,, 17	,, 18	32	12.0 noon
,,	,, 17	Did not pair	..	..
,,	,, 17	Did not pair	..	..

Total No. of mice 5. Paired 3.

B.

Control.

4 weeks	Dec. 17	Jan. 10	24	12.0 noon
,,	,, 17	,, 20	34	12.0 ,,
,,	,, 17	Did not pair	..	..

Total No. of mice 3. Paired 2.

TABLE IV.

A.

1. Age.	2. Beginning of Experiment.	3. Date of Pairing.	4. Days Elapsed between 2 and 3.	5. Time of Day.
4 weeks	Feb. 3	Mar. 10	35	12.10 p.m.
,,	,, 3	,, 17	42	3.0 ,,
,,	,, 7	,, 15	37	2.15 ,,
3 weeks	,, 7	April 8	58	2.0 ,,
,,	,, 7	May 18	98	4·30 ,,

Total No. of mice 5. Paired 5.

B.

Control.

4 weeks	Feb. 3	Did not pair	..	..
,,	,, 3	Did not pair	..	..
,,	,, 7	May 2	82	1·0 p.m.
3 weeks	,, 7	Mar. 28	27	12–1.0 ,,
,,	,, 7	Did not pair	..	..

Total No. of mice 5. Paired 2.

TABLE V.

A.

1. Age.	2. Beginning of Experiment.	3. Date of Pairing.	4. Days Elapsed between 2 and 3.	5. Time of Day.
3 weeks, 3 days	Mar. 4	April 13	40	1.40 p.m.
3 weeks, 3 days	,, 4	May 4	61	11.45 ,,
8 weeks	,, 7	Mar. 17	10	2.45 ,,
,,	,, 7	May 18	72	10.45 a.m.
,,	,, 7	,, 27	81	12.0 noon

Total No. of mice 5. Paired 5.

B.

Control.

3 weeks, 3 days	Mar. 4	April 11	38	12.0 noon
3 weeks, 3 days	„ 4	May 27	84	12–1.0 p.m.
8 weeks	„ 7	April 11	35	11.30 a.m.
„	„ 7	May 12	66	1.0 p.m.
„	„ 7	June 2	87	12.0 noon

Total No. of mice 5. Paired 5.

TABLE VI.

A.

1. Age.	2. Beginning of Experiment.	3. Date of Pairing.	4. Days Elapsed between 2 and 3.	5. Time of Day.
4 weeks	April 25	May 20	25	12.0 noon
„	„ 25	„ 28	33	1.0 p.m.
„	„ 25	„ 30	35	12.35 „
„	„ 25	June 2	38	12.0 noon

Total No. of mice 4. Paired 4.

B.

Control.

3 weeks	April 25	June 6	44	12.0 noon
„	„ 25	„ 13	49	1.0 p.m.
„	„ 25	Did not pair	..	..
„	„ 25	Did not pair	..	..

Total No. of mice 4. Paired 2.

TABLE VII.

A.

1. Age.	2. Beginning of Experiment.	3. Date of Pairing.	4. Days Elapsed between 2 and 3.	5. Time of Day.
3 weeks	May 2	May 20	18	12.0 noon
„	„ 2	„ 20	18	11.0 a.m.
„	„ 2	„ 23	21	10.15 „
4 weeks (approx.)	„ 26	June 6	11	4.30 p.m.
4 weeks (approx.)	„ 26	„ 9	14	12.0 noon
4 weeks (approx.)	„ 26	„ 9	14	1.0 p.m.
4 weeks (approx.)	„ 26	„ 20	25	12.0 noon
4 weeks (approx.)	„ 26	Did not pair	..	..
4 weeks (approx.)	„ 26	Did not pair	..	..

Total No. of mice 9. Paired 7.

B.

Control.

3 weeks (2 mice)	May 2	Did not pair	..	..
4 weeks (5 mice)	,, 26	Did not pair	..	..

Total No. of mice 7. Paired 0.

TABLE VIII.

1. Age.	2. Time in Control.	3. Transferred on.	4. Date of Pairing.	5. Days Elapsed between 3 and 4.	6. Time of Day.
4 months (approx.)	3 months	Mar. 3	Mar. 24	6	12.20 p.m.
10 weeks	6 weeks	,, 3	April 8	21	12.0 noon
,,	,,	,, 3	,, 22	35	12.20 p.m.
2 months (2 mice)	1 month	June 27	July 5	9	11.0 a.m. and 12.30 p.m.
2 months (3 mice)	,,	,, 27	Did not pair	..	..

Total No. of mice 8. Paired 5.

IV. DISCUSSION.

In considering the effect of reversed light conditions on the sexual activity of the mouse, the results of the present work, as indicated by the tables, are somewhat unconvincing if taken by themselves; but if considered in relation to the time of year certain important effects are evident.

The majority of animals represented in Table I were first subjected to reversed day and night conditions on November 19. The conditions as regards length of "day" and temperature approximated to those of summer. When it is remembered that under ordinary conditions pairing is infrequent during mid-winter, it is seen that the "long day" accelerated sexual activity, bringing 4 of the 7 mice into œstrus during December and 1 in January. The oviducts of these mice were subsequently sectioned, and the ova were found to have been fertilised. That the acceleration was not due to temperature, but to the long light period, was indicated by the absence of pairing during December and January amongst the animals in the general stock. When mice are illuminated during the short winter day by natural light only, œstrus seldom occurs. Such pairing as normally takes place during mid-winter would appear to be due to the extensive use of electric light during the whole or part of the working day.

The present findings on the effect of the "long day" period of artificial light on the reproduction of the mouse agrees with the observations of Bissonnette (1932) and others referred to in the introduction to this paper. Yet increased daily lighting periods appear to have no effect on the reproduction of some mammals. A summary of the work on mammals will be found in Whitaker's paper (1940).

That reversed day and night brings about a reversal of the time of pairing is evident from an examination of Table I. Seven of the 8 mice which copulated did so between 11 a.m. and 1.30 p.m. It is possible that the animal which paired at 5.45 p.m. may have also undergone reversal, as mice are known to copulate in the early hours of the morning.

In the experiment set up on December 17 (Tables III, A, B) 2 of the 3 controls paired in January. As the number of mice used in the control group was small, and in view of the fact that no daylight pairings occurred at this time of the year amongst the 12 animals of the general stock, it is unsafe to draw any definite conclusions from this experiment, but it may be that some strains pair more readily in the daytime than others.

The experiments begun in the spring may be considered together (Tables II; V, A, B; VI, A, B). Nine of the mice referred to in Table II paired during the hours of daylight within 38 days. Tables V and VI give more precise information, as careful controls of these experiments were kept. The 9 experimental animals and 7 of the 9 controls paired during the daytime. This confirms the findings of Lewis and Wright (1935) that in the spring mice pair more readily during the day. Reversed lighting, however, appears to have accelerated daytime pairing, as the majority of the experimental animals copulated at an earlier date than the corresponding controls.

Tables VII, A, B, indicate that during late May and June pairing does not usually occur during the day, but that this may be induced by subjecting the animals to reversed day and night.

Table VIII is of interest in that it shows that daytime pairing can be induced in animals which have previously spent a considerable time as controls. Unfortunately this experiment had to be discontinued at an earlier date than was intended.

As the proportion of daytime pairings amongst the experimental animals is higher, at all times of the year, than amongst the corresponding controls it is concluded that reversed day and night, and not the "long day" conditions, is the determining factor in increasing the proportion of daytime pairings. It is concluded that (1) mice can be brought into œstrus in mid-winter when subjected to "long day" conditions; (2) reversed lighting during the winter induces daytime pairing; (3) in the spring pairing takes place more frequently during the hours of daylight than at other times of the year; under reversed lighting conditions the proportion of pairings during the daytime is slightly higher than under natural conditions; (4) daytime pairings are not common during late May and June, but may be induced by reversed lighting.

Hemmingsen and Krarup (1937) found that in the rat œstrus is reversed in about 10 days from the beginning of the experiment. The time taken to bring about reversal in the mouse varies considerably. In some cases reversal is established in less than 10 days, while some mice did not pair until more than 60 days had elapsed. It should be noted that only 2 of the controls paired in less than 30 days from the beginning of the experiment. In considering the number of days required to bring about reversal it should be remembered that the majority of the mice were 3 to 4 weeks old when first subjected to reversed lighting, and that the mice used in these experiments usually came on the first œstrus when about 5 weeks old. An examination of the tables shows that few of the young mice underwent a reversal at the first œstrus. Six of the 12 adult mice used in the course of the present investigation paired within 10 days of the beginning of the experiment.

No record was made of the spontaneous activity of the experimental mice, but observations indicated that, under reversed day and night, a reversal of activity took place.

It is possible that some strains of mice may pair more readily during the day than others. Further experiments, using a large number of pedigree mice, would have to be carried out in order to obtain information on this point, but this was not possible during the present investigation.

V. SUMMARY.

1. Female mice were kept in darkness for 7 to 8 hours during the day, and were kept under bright illumination (electric light) for 16 to 17 hours beginning at about 6 p.m. The general stock was used as control for some of the experiments, while for others, controls consisting of mice of the same age were used.

2. The "long day" conditions accelerated œstrus and induced pairing in mid-winter. Reversed day and night conditions induces daytime pairing. In the spring the mice in the controls paired more readily during the daytime than at other times of the year, but the proportion of daytime pairings among the experimental animals was somewhat higher than among the control mice.

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XXVI.—Some Confluent Hypergeometric Functions of Two Variables.

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INTRODUCTION.

1. This paper is the continuation of a former one (Erdélyi, 1939), and deals with the integration of the system of two partial linear differential equations of the second order

$$\left.\begin{aligned} x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\partial y}+(\gamma-x)\frac{\partial z}{\partial x}-\beta z=0,\\ x\frac{\partial^2 z}{\partial x\partial y}+y\frac{\partial^2 z}{\partial y^2}+(\gamma-y)\frac{\partial z}{\partial y}-\beta' z=0.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

The former paper will be referred to as I; all the notations of I will be retained without further explanation.

The integration of (1) is dealt with not only because of its importance, but mainly because it is a typical example of a method of integrating systems of linear partial differential equations by definite integrals, which I believe to be powerful. A certain justification of this belief will be found in the fact that the results presented in I and here contain the first complete theory of solutions of any hypergeometric system of two variables. There seems to be only one system whose integration has been accomplished with a degree of completeness comparable to that attained here—namely the system of Appell's series F_1, of which (1) is a limiting case—and it is noteworthy that the results on F_1 have been obtained essentially by the same method. I have not been able to consult the Thèse of Le Vavasseur; from the account of it given by Appell and Kampé de Fériet (1926, p. 53 and *seq.*) it seems that Le Vavasseur, as well as Picard, did not complete the integration of the system of F_1, and thus did not arrive at the understanding of the equivalence of different systems of partial differential equations. Later (Borngässer, 1933, p. 30 and *seq.*) it has been found that the system of F_1 possesses solutions of the type of Horn's series G_2. It is doubtful if anyone has realised hitherto the *full equivalence* of the two seemingly different systems of partial differential equations satisfied by F_1 and G_2 respectively. To bring out connections of this kind is a further advantage of the method used here.

In the present paper I first obtain a great number of convergent expansions of the solutions z_0, z_1, z_2 and z_3 defined in I. Moreover, three new solutions are defined and several convergent expansions for these solutions obtained.

Having thus thirteen different solutions represented by contour integrals, the transformation theory of these solutions can be obtained in the usual way. For the sake of brevity only an outline of the transformation theory is given. Next, the monodromic group of the system (1) is discussed.

In the discussion of the connection between Horn's results and the present ones, the equivalence of certain known systems of linear partial differential equations with (1) is pointed out. It is believed that this equivalence has not been recognised until now; to have exhibited it clearly is another proof of the usefulness of the method used here.

EXPANSIONS OF z_0.

2. In I (29) three different expansions of z_0 in series of the type Φ_2 have been given. Eighteen more expansions will be given here. They involve Humbert's series (Humbert, 1920–21, p. 75; see also Appell and Kampé de Fériet, 1926, p. 126)

$$\Psi_1(\alpha, \beta, \gamma, \gamma'; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_m(\gamma')_n m! n!} x^m y^n \qquad (|x| < 1), \qquad (2)$$

Borngässer's series (Borngässer, 1933, p. 9, footnote)

$$H_{11}(\alpha, \beta, \gamma, \delta; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m-n}(\beta)_n(\gamma)_n}{(\delta)_m m! n!} x^m y^n \qquad (|y| < 1), \qquad (3)$$

and a confluent hypergeometric series of higher order of two variables which will be denoted in this paper by

$$\Phi_6(\alpha, \beta, \gamma, \delta; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_n}{(\gamma)_{m+n}(\delta)_n} \frac{x^m}{m!} \frac{y^n}{n!}. \quad . \quad . \qquad (4)$$

This series is convergent for all finite values of x and y.

We have the integral representation I (13)

$$\Phi_2(\beta, \beta', \gamma; x, y) = \frac{\Gamma(\gamma)}{2\pi i} \int_{-\infty}^{(0+, x+, y+)} e^s s^{\beta+\beta'-\gamma}(s-x)^{-\beta}(s-y)^{-\beta'} ds. \quad . \qquad (5)$$

In this let us use three different expansions of $(s-y)^{-\beta'}$:

$$(s-y)^{-\beta'}=(s-x)^{-\beta'}\sum_{n=0}^{\infty}\frac{(\beta')_n}{n!}\left(\frac{y-x}{s-x}\right)^n$$

$$=(s-x)^{-\beta'}\left(\frac{x}{y}\right)^{\beta'}\sum_{n=0}^{\infty}\frac{(\beta')_n}{n!}\left(\frac{s}{s-x}\,\frac{y-x}{y}\right)^n$$

$$=s^{-\beta'}\left(\frac{x}{x-y}\right)^{\beta'}\sum_{n=0}^{\infty}\frac{(\beta')_n}{n!}\left(\frac{s-x}{s}\,\frac{y}{y-x}\right)^n. \qquad (6)$$

With each of these expansions use

$$(s-x)^{-\alpha}=s^{-\alpha}\sum_{m=0}^{\infty}\frac{(\alpha)_m}{m!}\left(\frac{x}{s}\right)^m,$$

α being equal to $\beta+\beta'+n$ in the first two cases of (6) and equal to $\beta-n$ with the last expansion. Assuming $|y| \gtrless |y-x|$ with the second and third expansion (6) respectively (no further assumption is necessary with the first expansion), the path of integration in (5) can be deformed in such a way that the double series for $(s-x)^{-\beta}(s-y)^{-\beta'}$ is uniformly convergent along it. Hence term-by-term integration is permissible and yields series of the types Φ_6, Ψ_1 and H_{11} respectively.

Now (5) remains unaltered when x and y and at the same time β and β' are interchanged. Performing these operations on the series obtained, three more expansions for Φ_2 appear. Also we have six similar expansions for each of the series in the second and third line of I (29) respectively. Thus eighteen new expansions of z_0 are obtained which, together with the three obtained in I, yield the following table of twenty-one convergent expansions of z_0:—

$$z_0=\Phi_2(\beta,\beta',\gamma;\ x,y)$$

$$=e^{x}\Phi_2(\gamma-\beta-\beta',\beta',\gamma;\ -x,y-x)$$

$$=e^{y}\Phi_2(\gamma-\beta-\beta',\beta,\gamma;\ -y,x-y)$$

$$=\left(1-\frac{x}{y}\right)^{\gamma-\beta-\beta'}e^{y}\Psi_1\left(\gamma-\beta',\gamma-\beta-\beta',\gamma-\beta',\gamma;\ \frac{x}{y},x-y\right) \qquad (|x|<|y|)$$

$$=\left(1-\frac{y}{x}\right)^{\gamma-\beta-\beta'}e^{x}\Psi_1\left(\gamma-\beta,\gamma-\beta-\beta',\gamma-\beta,\gamma;\ \frac{y}{x},y-x\right) \qquad (|x|>|y|)$$

$$=\left(1-\frac{x}{y}\right)^{-\beta}e^{y}\Psi_1\left(\gamma-\beta',\beta,\gamma-\beta',\gamma;\ \frac{x}{x-y},\ -y\right) \qquad (|x|<|x-y|)$$

$$=\left(\frac{x}{y}\right)^{-\beta}\Psi_1\left(\beta+\beta',\beta,\beta+\beta',\gamma;\ 1-\frac{y}{x},y\right) \qquad (|x|>|x-y|)$$

$$=\left(1-\frac{y}{x}\right)^{-\beta'}e^{x}\Psi_1\left(\gamma-\beta,\beta',\gamma-\beta,\gamma;\ \frac{y}{y-x},\ -x\right) \qquad (|y|<|x-y|)$$

$$z_0=\left(\frac{y}{x}\right)^{-\beta'}\Psi_1\left(\beta+\beta', \beta', \beta+\beta', \gamma;\ 1-\frac{x}{y}, x\right) \qquad (|y|>|x-y|)$$

$$=\left(1-\frac{x}{y}\right)^{\gamma-\beta-\beta'}e^{x}\mathrm{H}_{11}\left(\beta', 1-\beta', \gamma-\beta-\beta', \gamma;\ y-x,\ -\frac{x}{y}\right) \qquad (|x|<|y|)$$

$$=\left(1-\frac{y}{x}\right)^{\gamma-\beta-\beta'}e^{y}\mathrm{H}_{11}\left(\beta, 1-\beta, \gamma-\beta-\beta', \gamma;\ x-y,\ -\frac{y}{x}\right) \qquad (|x|>|y|)$$

$$=\left(1-\frac{x}{y}\right)^{-\beta}\mathrm{H}_{11}\left(\beta', 1-\beta', \beta, \gamma;\ y, \frac{x}{y-x}\right) \qquad (|x|<|x-y|)$$

$$=\left(\frac{x}{y}\right)^{-\beta}e^{y}\mathrm{H}_{11}\left(\gamma-\beta-\beta', \beta+\beta'-\gamma+1, \beta, \gamma;\ -y, \frac{y}{x}-1\right) \qquad (|x|>|x-y|)$$

$$=\left(1-\frac{y}{x}\right)^{-\beta'}\mathrm{H}_{11}\left(\beta, 1-\beta, \beta', \gamma;\ x, \frac{y}{x-y}\right) \qquad (|y|<|x-y|)$$

$$=\left(\frac{y}{x}\right)^{-\beta'}e^{x}\mathrm{H}_{11}\left(\gamma-\beta-\beta', \beta+\beta'-\gamma+1, \beta', \gamma;\ -x, \frac{x}{y}-1\right) \qquad (|y|>|x-y|)$$

$$=e^{x}\Phi_6(\gamma-\beta, \beta', \gamma, \gamma-\beta;\ -x, y)$$

$$=e^{y}\Phi_6(\gamma-\beta', \beta, \gamma, \gamma-\beta';\ -y, x)$$

$$=\Phi_6(\beta+\beta', \beta', \gamma, \beta+\beta';\ x, y-x)$$

$$=e^{y}\Phi_6(\gamma-\beta', \gamma-\beta-\beta', \gamma, \gamma-\beta';\ x-y,\ -x)$$

$$=\Phi_6(\beta+\beta', \beta, \gamma, \beta+\beta';\ y, x-y)$$

$$=e^{x}\Phi_6(\gamma-\beta, \gamma-\beta-\beta', \gamma, \gamma-\beta;\ y-x,\ -y).$$

EXPANSIONS OF z_1, z_2, z_3.

3. Each of the solutions z_1, z_2, z_3 can be represented by twelve convergent series. This depends on twelve different convergent representations of Φ_1. Beside series already encountered, Humbert's series (Humbert, 1920–21, p. 75; see also Appell and Kampé de Fériet, 1926, p. 126)

$$\Xi_1(\alpha, \alpha', \beta, \gamma;\ x, y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m}{(\gamma)_{m+n}m!\,n!}x^m y^n \qquad (|x|<1), \tag{7}$$

Horn's series (Horn, 1931, p. 384)

$$\mathrm{H}_2(\alpha, \beta, \gamma, \delta;\ x, y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m-n}(\beta)_m(\gamma)_n}{(\delta)_m m!\,n!}x^m y^n \qquad (|x|<1), \tag{8}$$

and also two hypergeometric series of two variables of higher order appear.

In the integral representation I (22),

$$\Phi_1(\alpha, \beta, \gamma;\ x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-xu)^{-\beta}e^{uy}\,du, \qquad (9)$$

three different expansions of the exponential can be used, namely

$$e^{uy} = \sum_{n=0}^{\infty}\frac{(uy)^n}{n!} = e^{y}\sum_{n=0}^{\infty}\frac{(1-u)^n(-y)^n}{n!} = e^{y/x}\sum_{n=0}^{\infty}\frac{(1-xu)^n}{n!}\left(-\frac{y}{x}\right)^n. \qquad (10)$$

Each of these expansions may be combined with any of the following four expansions (in the case of the third expansion (10), β and γ have to be replaced by $\beta - n$ and $\gamma - n$ respectively):—

$$\begin{aligned}(1-xu)^{-\beta} &= \sum_{m=0}^{\infty}\frac{(\beta)_m}{m!}(xu)^m = (1-x)^{-\beta}\sum_{m=0}^{\infty}\frac{(\beta)_m}{m!}\left(x\frac{1-u}{x-1}\right)^m \\ &= (1-xu)^{-\gamma-n}\sum_{m=0}^{\infty}\frac{(\gamma-\beta+n)_m}{m!}\left(\frac{-xu}{1-xu}\right)^m \\ &= (1-x)^{\gamma-\beta+n}(1-xu)^{-\gamma-n}\sum_{m=0}^{\infty}\frac{(\gamma-\beta+n)_m}{m!}\left(x\frac{1-u}{1-xu}\right)^m. \qquad (11)\end{aligned}$$

Thus twelve expansions of the right-hand side of (9) into hypergeometric series of two variables are obtained, which are recorded in the following table:—

$$\begin{aligned}&\Phi_1(\alpha, \beta, \gamma;\ x, y) \\ &\qquad = e^{y}\Xi_1(\alpha, \gamma-\alpha, \beta, \gamma;\ x, -y) \\ &\qquad = e^{y/x}\mathrm{H}_2\left(\beta, \alpha, 1-\beta, \gamma;\ x, \frac{y}{x}\right) \\ &\qquad = (1-x)^{\gamma-\alpha-\beta}e^{y/x}\Psi_1\left(\gamma-\beta, \gamma-\alpha, \gamma, \gamma-\beta;\ x, y\frac{x-1}{x}\right) \\ &\qquad = (1-x)^{\gamma-\alpha-\beta}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma-\alpha)_m(\alpha)_n(\gamma-\beta+n)_m}{(\gamma)_{m+n}m!\,n!}x^m y^n \\ &\qquad = (1-x)^{\gamma-\alpha-\beta}e^{y}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma-\alpha)_{m+n}(\gamma-\beta+n)_m}{(\gamma)_{m+n}m!\,n!}x^m\{y(x-1)\}^n \\ &\qquad = (1-x)^{-\beta}e^{y}\Phi_1\left(\gamma-\alpha, \beta, \gamma;\ \frac{x}{x-1}, -y\right) \\ &\qquad = (1-x)^{-\beta}\Xi_1\left(\gamma-\alpha, \alpha, \beta, \gamma;\ \frac{x}{x-1}, y\right) \\ &\qquad = (1-x)^{-\beta}e^{y/x}\mathrm{H}_2\left(\beta, \gamma-\alpha, 1-\beta, \gamma;\ \frac{x}{x-1}, y\frac{1-x}{x}\right)\end{aligned}$$

$$\begin{aligned}
\Phi_1(\alpha, \beta, \gamma;\ x, y) &= (1-x)^{-\alpha} e^{y/x} \Psi_1\left(\gamma-\beta, \alpha, \gamma, \gamma-\beta;\ \frac{x}{x-1}, -\frac{y}{x}\right) \\
&= (1-x)^{-\alpha} e^{y} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_m (\gamma-\alpha)_n (\gamma-\beta+n)_m}{(\gamma)_{m+n}\, m!\, n!} \left(\frac{x}{x-1}\right)^m (-y)^n \\
&= (1-x)^{-\alpha} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n} (\gamma-\beta+n)_m}{(\gamma)_{m+n}\, m!\, n!} \left(\frac{x}{x-1}\right)^m \left(\frac{y}{1-x}\right)^n. \quad . \quad (12)
\end{aligned}$$

The first six series are convergent for $|x| < 1$, the other six for $\mathbf{R}(x) < \frac{1}{2}$, all twelve for all finite values of y. With the series of the type H_2 and Ψ_1 the additional assumption $x \neq 0$ is necessary.

The identity of the first two lines (and similarly of the seventh and eighth line) of (12) is well known (Humbert, 1920–21, p. 77, where there is a misprint, Ξ_1 having the variable y instead of $-y$; see also Appell and Kampé de Fériet, 1926, p. 127 (17)); the equality of the first and the seventh line is the transformation formula I (23). The rest of the transformations of Φ_1 seem to be new.

Applying (12) to z_1, z_2 and z_3 we get twelve different representations for each of these solutions, six of them being convergent in each of the domains of I (24), (25), (26). Thus we have obtained thirty-six different series representing three different (though not linearly independent) solutions.

OTHER SOLUTIONS IN CONVERGENT SERIES.

4. The remark of I, p. 232, that z_0, z_1, z_2 and z_3 are the only solutions of (1) expressible by convergent hypergeometric series of two variables, is erroneous. There are three more solutions, namely

$$z_{10} = \frac{\Gamma(\beta+\beta')\Gamma(2-\gamma)}{\Gamma(\beta+\beta'-\gamma+1)2\pi i} \int_0^{(x+,\, y+)} e^s s^{\beta+\beta'-\gamma} (s-x)^{-\beta} (s-y)^{-\beta'} ds, \quad . \quad (13)$$

$$z_{11} = \frac{\Gamma(\gamma-\beta)\Gamma(2-\gamma)}{\Gamma(1-\beta)2\pi i} \int_x^{(0+,\, y+)} e^s s^{\beta+\beta'-\gamma} (s-x)^{-\beta} (s-y)^{-\beta'} ds, \quad . \quad (14)$$

$$z_{12} = \frac{\Gamma(\gamma-\beta')\Gamma(2-\gamma)}{\Gamma(1-\beta')2\pi i} \int_y^{(0+,\, x+)} e^s s^{\beta+\beta'-\gamma} (s-x)^{-\beta} (s-y)^{-\beta'} ds. \quad . \quad (15)$$

Here I (15) has been assumed again and x, y, $-x$, $-y$, $x-y$, $y-x$ are to have their principal values. Also in (13) s is to have its principal value and

$x \neq 0, y \neq 0,\ \ \arg x - \pi \leq \arg (s-x) \leq \arg x + \pi,\ \ \arg y - \pi \leq \arg (s-y) \leq \arg y + \pi;$

in (14) $s-x$ is to have its principal value and

$$x \neq 0,\ x \neq y,\quad \arg(-x)-\pi \leq \arg s \leq \arg(-x)+\pi,$$
$$\arg(y-x)-\pi \leq \arg(s-y) \leq \arg(y-x)+\pi;$$

finally in (15) $s-y$ is to have its principal value and it is assumed that

$$y \neq 0,\ x \neq y,\quad \arg(-y)-\pi \leq \arg s \leq \arg(-y)+\pi,$$
$$\arg(x-y)-\pi \leq \arg(s-x) \leq \arg(x-y)+\pi.$$

The contour of integration is a loop starting and ending at one of the zeros of the integrand and encircling in the positive direction the line connecting the other two. From I, section 5, it immediately follows that the above integrals represent solutions of (1). Now it will be seen that these solutions have expansions into convergent hypergeometric series of Horn's type (Horn, 1931, p. 384)

$$\Gamma_1(\alpha, \beta, \beta';\ x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_m(\beta)_{n-m}(\beta')_{m-n}}{m!\,n!} x^m y^n \qquad (|x| < 1). \quad (16)$$

Let us use in (13) the expansion

$$e^s = e^x \sum_{n=0}^{\infty} \frac{(s-x)^n}{n!},$$

uniformly convergent in the range of integration. If $|x-y| < |x|$, then the contour of integration can always be taken to lie entirely outside of the circle with centre x and radius $|x-y|$. On such a contour the series

$$(s-y)^{-\beta'} = (s-x)^{-\beta'} \sum_{m=0}^{\infty} \frac{(\beta')_m}{m!}\left(\frac{y-x}{s-x}\right)^m$$

converges uniformly. Hence

$$z_{10} = e^x \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\beta')_m}{m!\,n!}(y-x)^m \cdot \frac{\Gamma(\beta+\beta')\Gamma(2-\gamma)}{\Gamma(\beta+\beta'-\gamma+1)2\pi i}\int_0^{(x+)} s^{\beta+\beta'-\gamma}(s-x)^{n-m-\beta-\beta'}ds.$$

The last integrand having no singularity at $s=y$, the contour of integration has been replaced by a loop starting and ending at zero and encircling $s=x$ only. Hence

$$z_{10} = e^x \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\beta')_m}{m!\,n!}(y-x)^m \frac{x^{1-\gamma-m+n}}{(\beta+\beta')_{m-n}(2-\gamma)_{n-m}}$$
$$= x^{1-\gamma}e^x\Gamma_1\left(\beta', 1-\beta-\beta', \gamma-1;\ \frac{y}{x}-1, x\right),$$

and from the symmetry of z_{10} in x, β and y, β' we have, if only $|x-y| < |y|$,

$$z_{10} = y^{1-\gamma}e^y\Gamma_1\left(\beta, 1-\beta-\beta', \gamma-1;\ \frac{x}{y}-1, y\right).$$

Similarly there are two expansions, convergent in different domains, for each of the three new solutions. Thus we obtain

$$z_{10} = x^{1-\gamma} e^{x} \Gamma_1\left(\beta', 1-\beta-\beta', \gamma-1; \frac{y}{x}-1, x\right) \qquad (|x| > |x-y|)$$

$$= y^{1-\gamma} e^{y} \Gamma_1\left(\beta, 1-\beta-\beta', \gamma-1; \frac{x}{y}-1, y\right) \qquad (|y| > |x-y|), \quad (17)$$

$$z_{11} = (-x)^{1-\gamma} \Gamma_1\left(\beta', \beta-\gamma+1, \gamma-1; -\frac{y}{x}, -x\right) \qquad (|y| < |x|)$$

$$= (y-x)^{1-\gamma} e^{y} \Gamma_1\left(\gamma-\beta-\beta', \beta-\gamma+1, \gamma-1; \frac{y}{x-y}, y-x\right) \quad (|y| < |x-y|), \quad (18)$$

$$z_{12} = (-y)^{1-\gamma} \Gamma_1\left(\beta, \beta'-\gamma+1, \gamma-1; -\frac{x}{y}, -y\right) \qquad (|x| < |y|)$$

$$= (x-y)^{1-\gamma} e^{x} \Gamma_1\left(\gamma-\beta-\beta', \beta'-\gamma+1, \gamma-1; \frac{x}{y-x}, x-y\right)$$

$$(|x| < |x-y|). \quad (19)$$

The three solutions obtained thus are not linearly independent. For all three loops in (13)–(15) can be replaced, with suitable phases of s, $s-x$ and $s-y$, by the triangle with vertices 0, x and y; hence z_{10}, z_{11} and z_{12} are expressible as linear combinations of z_1, z_2, z_3, and it has been seen (I, p. 230) that these solutions are not linearly independent of each other. Any two of the three solutions z_{10}, z_{11}, z_{12} are linearly independent, and hence any two taken together with z_0 form a fundamental system of solutions.

5. In the preceding section two expansions of each of the solutions z_{10}, z_{11}, z_{12} have been obtained. Each of these solutions possesses, however, like z_1, z_2 and z_3, twelve different expansions. This depends on certain transformations of the series Γ_1 which we are now going to investigate.

Let us start with the integral representation

$$\Gamma_1(\alpha, \beta, \beta'; x, y) = \frac{\Gamma(1-\beta)\Gamma(1-\beta')}{\Gamma(1-\beta-\beta')2\pi i}\int_{-1}^{(0+)} e^{y/t} t^{\beta'-1}(1+t)^{-\beta-\beta'}(1-xt)^{-\alpha}\,dt \quad (20)$$

and expand the exponential in three different ways, corresponding to (10),

$$e^{y/t} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{y}{t}\right)^n = e^{-y}\sum_{n=0}^{\infty}\frac{1}{n!}\left(y\frac{1+t}{t}\right)^n = e^{xy}\sum_{n=0}^{\infty}\frac{1}{n!}\left(y\frac{1-xt}{t}\right)^n. \quad . \quad (21)$$

Let us combine each of these three expansions with each of the four expansions (in the second case of (21) β being replaced by $\beta-n$ and in the third case α by $\alpha-n$)

$$
\begin{aligned}
(1-xt)^{-\alpha} &= \sum_{m=0}^{\infty}\frac{(\alpha)_m}{m!}(xt)^m = (1+x)^{-\alpha}\sum_{m=0}^{\infty}\frac{(\alpha)_m}{m!}\left(x\frac{t+1}{x+1}\right)^m \\
&= (1-xt)^{\beta+n-1}\sum_{m=0}^{\infty}\frac{(1-\alpha-\beta-n)_m}{m!}\left(\frac{-xt}{1-xt}\right)^m \\
&= (1+x)^{1-\alpha-\beta-n}(1-xt)^{\beta+n-1}\sum_{m=0}^{\infty}\frac{(1-\alpha-\beta-n)_m}{m!}\left(-x\frac{1+t}{1-xt}\right)^m \qquad (22)
\end{aligned}
$$

corresponding to (11). Thus, integrating term-by-term, the following twelve expansions of the right-hand side of (20) are obtained:—

$$
\begin{aligned}
&\Gamma_1(\alpha, \beta, \beta';\ x, y) \\
&= e^{-y}\mathrm{H}_8(\beta', \alpha, 1-\beta-\beta', 1-\beta;\ -x, -y) \\
&= (1+x)^{1-\alpha-\beta-\beta'}e^{xy}\mathrm{H}_{11}(\beta, 1-\alpha-\beta, 1-\beta-\beta', 1-\beta';\ -(1+x)y, x) \\
&= (1+x)^{1-\alpha-\beta-\beta'}e^{-y}\Psi_1(1-\beta-\beta', 1-\alpha-\beta, 1-\beta, 1-\beta';\ -x, (1+x)y) \\
&= e^{xy}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m-n}(\beta')_{m-n}(1-\alpha)_n(\beta)_{n-m}}{m!\,n!}x^m(-y)^n \\
&= (1+x)^{1-\alpha-\beta-\beta'}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1-\beta-\beta')_m(\alpha+\beta)_n(\beta)_{n-m}(1-\alpha-\beta)_{m-n}}{(1-\beta')_n m!\,n!}x^m y^n \\
&= (1+x)^{-\beta'}e^{xy}\Gamma_1\left(1-\alpha-\beta, \beta, \beta';\ \frac{-x}{x+1}, (1+x)y\right) \\
&= (1+x)^{-\beta'}e^{-y}\mathrm{H}_8\left(\beta', 1-\alpha-\beta, 1-\beta-\beta', 1-\beta;\ \frac{x}{x+1}, -(1+x)y\right) \\
&= (1+x)^{-\alpha}\mathrm{H}_{11}\left(\beta, \alpha, 1-\beta-\beta', 1-\beta';\ -y, \frac{-x}{x+1}\right) \\
&= (1+x)^{-\alpha}e^{-y}\Psi_1\left(1-\beta-\beta', \alpha, 1-\beta, 1-\beta';\ \frac{x}{x+1}, y\right) \\
&= (1+x)^{-\beta'}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\beta)_{n-m}(\beta')_{m-n}(1-\alpha-\beta)_{m-n}(\alpha+\beta)_n}{m!\,n!}\left(\frac{-x}{x+1}\right)^m\{-y(1+x)\}^n \\
&= (1+x)^{-\alpha}e^{xy}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m-n}(1-\alpha)_n(1-\beta-\beta')_m(\beta)_{n-m}}{(1-\beta')_n m!\,n!}\left(\frac{-x}{x+1}\right)^m\{(1+x)y\}^n. \qquad (23)
\end{aligned}
$$

The first six and the other six series are convergent in the domains $|x| < 1$ and $\mathbf{R}(x) > -\frac{1}{2}$ respectively for finite values of x and y.

Applying (23) to the solutions z_{10}, z_{11} and z_{12}, thirty-six series are obtained, six of which are convergent in each of the domains of convergence of (17)–(19). These series represent three different (though not linearly independent) solutions. For the sake of brevity a table of these thirty-six series has been omitted.

TRANSFORMATION THEORY OF THE SOLUTIONS.

6. Any of the thirteen solutions of (1) can be represented as a linear combination of any three linearly independent solutions. All the formulæ of this kind together constitute the transformation theory of the solutions. We shall express all solutions in terms of z_4, z_5 and z_6 which in I, section 10, have been proved to be linearly independent.

To fix the ideas we shall assume that $0 < \arg x < \arg y < \pi$ and $\arg x < \arg (y-x) < \pi$. Hence for the principal values of the arguments of $-x$, $-y$ and $x-y$ we have

$$\arg(-x) = \arg x - \pi, \quad \arg(-y) = \arg y - \pi, \quad \arg(x-y) = \arg(y-x) - \pi.$$

Now, to express z_0 as a linear combination of z_4, z_5, z_6, let us replace the contour $-\infty \ldots (0+, x+, y+) \ldots -\infty$ in I (13) by three loops $-\infty \ldots (0+) \ldots -\infty$, $-\infty \ldots (x+) \ldots -\infty$, $-\infty \ldots (y+) \ldots -\infty$, each of them enclosing only one singularity of the integrand. In consequence of I (34), (35), (36), the integral along a single loop can be expressed in terms of z_4, z_5 or z_6, whichever may be in question. Taking all phases of s, $s-x$ and $s-y$ into due consideration, we arrive at the transformation formula

$$z_0 = e^{(\beta+\beta')\pi i}\frac{\Gamma(\gamma)}{\Gamma(\gamma-\beta-\beta')}z_4 + e^{(\beta+2\beta'-\gamma)\pi i}\frac{\Gamma(\gamma)}{\Gamma(\beta)}z_5 + e^{(\beta'-\gamma)\pi i}\frac{\Gamma(\gamma)}{\Gamma(\beta')}z_6. \tag{24}$$

This transformation formula shows the asymptotic behaviour of z_0 when $x \rightarrow \infty$, $y \rightarrow \infty$ and $y - x \rightarrow \infty$.

7. Next z_1, z_2, z_3 will be expressed in terms of z_4, z_5, z_6. The notation of matrix calculus will be used.

Now, $\int_0^x \ldots = \int_0^{-\infty} \ldots + \int_{-\infty}^x \ldots$, and by I (31) and (32) the first integral can be expressed in terms of z_4, the second one in terms of z_5. Hence, by I (16), z_1 has been expressed as a linear combination of z_4 and z_5. Transforming similarly z_2 and z_3, always taking into consideration the phases of s, $s-x$ and $s-y$ as fixed in I, we arrive at a set of three transformation formulæ which may be condensed into the following matrix equation:—

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = e^{(\beta+\beta'-\gamma)\pi i}\begin{bmatrix} -\frac{\Gamma(2+\beta'-\gamma)}{\Gamma(1-\beta)}, & \frac{\Gamma(2+\beta'-\gamma)}{\Gamma(\beta+\beta'-\gamma+1)}, & 0 \\ -\frac{\Gamma(2+\beta-\gamma)}{\Gamma(1-\beta')}, & 0, & \frac{\Gamma(2+\beta-\gamma)}{\Gamma(\beta+\beta'-\gamma+1)} \\ 0, & -e^{-\beta\pi i}\frac{\Gamma(2-\beta-\beta')}{\Gamma(1-\beta')}, & e^{-\beta\pi i}\frac{\Gamma(2-\beta-\beta')}{\Gamma(1-\beta)} \end{bmatrix}\begin{bmatrix} z_4 \\ z_5 \\ z_6 \end{bmatrix}. \tag{25}$$

The matrix on the right-hand side is a singular one, *i.e.* its determinant vanishes, thus showing that $\{z_1, z_2, z_3\}$ is *not* a linearly independent set of solutions.

8. When transforming z_7, we have to put in I (42)

$$\int_{-\infty}^{(0+,\,x+)} \ldots = \int_{-\infty}^{(0+)} \ldots \int_{-\infty}^{(x+)} \ldots$$

and then use I (34) and (35). Dealing similarly with z_8 and z_9, we arrive at

$$\begin{bmatrix} z_7 \\ z_8 \\ z_9 \end{bmatrix} = \begin{bmatrix} e^{\beta\pi i}\dfrac{\Gamma(\gamma-\beta')}{\Gamma(\gamma-\beta-\beta')}, & e^{(\beta+\beta'-\gamma)\pi i}\dfrac{\Gamma(\gamma-\beta')}{\Gamma(\beta)}, & 0 \\ e^{\beta'\pi i}\dfrac{\Gamma(\gamma-\beta)}{\Gamma(\gamma-\beta-\beta')}, & 0 \quad , & e^{(\beta+\beta'-\gamma)\pi i}\dfrac{\Gamma(\gamma-\beta)}{\Gamma(\beta')} \\ 0 \quad , & e^{\beta'\pi i}\dfrac{\Gamma(\beta+\beta')}{\Gamma(\beta)} \quad , & e^{-\beta\pi i}\dfrac{\Gamma(\beta+\beta')}{\Gamma(\beta')} \end{bmatrix} \begin{bmatrix} z_4 \\ z_5 \\ z_6 \end{bmatrix}. \qquad (26)$$

The matrix appearing herein is a non-singular one (except for integer values of β). Hence $\{z_7, z_8, z_9\}$ is, in general, a linearly independent set of solutions.

9. We complete our transformation formulæ by expressing z_{10}, z_{11} and z_{12} in terms of z_4, z_5, z_6. In order to do so we have to decompose the loop of (13) according to the scheme

$$\int_0^{(x+,\,y+)} \ldots = \int_0^{-\infty} \ldots + \int_{-\infty}^{(x+)} \ldots + \int_{-\infty}^{(y+)} \ldots + \int_{-\infty}^{0} \ldots,$$

and in a like manner for (14) and (15). $\int_0^{-\infty}$ and $\int_{-\infty}^{0}$ do not cancel because of the different phases of $s-x$ and $s-y$ along them. For the first and fourth integral we use I (31), for the two middle ones I (35) and (36) respectively. So we obtain

$$\begin{bmatrix} z_{10} \\ z_{11} \\ z_{12} \end{bmatrix} = \Gamma(2-\gamma) \begin{bmatrix} -\dfrac{e^{(\beta+\beta'-\gamma)\pi i}}{\Gamma(1-\beta-\beta')}, & \dfrac{e^{(\beta+2\beta'-\gamma)\pi i}\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta+\beta'-\gamma+1)}, & \dfrac{e^{(\beta'-\gamma)\pi i}\Gamma(\beta+\beta')}{\Gamma(\beta')\Gamma(\beta+\beta'-\gamma+1)} \\ \dfrac{e^{(\beta-\beta')\pi i}\Gamma(\gamma-\beta)}{\Gamma(1-\beta)\Gamma(\gamma-\beta-\beta')}, & -\dfrac{e^{\beta\pi i}}{\Gamma(1+\beta-\gamma)}, & \dfrac{e^{(\gamma-\beta')\pi i}\Gamma(\gamma-\beta)}{\Gamma(\beta')\Gamma(1-\beta)} \\ \dfrac{e^{(\beta+\beta')\pi i}\Gamma(\gamma-\beta')}{\Gamma(1-\beta')\Gamma(\gamma-\beta-\beta')}, & \dfrac{e^{(\beta+2\beta'-\gamma)\pi i}\Gamma(\gamma-\beta')}{\Gamma(\beta)\Gamma(1-\beta')}, & -\dfrac{e^{\beta'\pi i}}{\Gamma(1+\beta'-\gamma)} \end{bmatrix} \begin{bmatrix} z_4 \\ z_5 \\ z_6 \end{bmatrix}. \qquad (27)$$

Again the matrix is a singular one, showing that a linear relation between z_{10}, z_{11} and z_{12} is bound to exist (see also section 4).

10. From equations (24)–(27) the whole transformation theory of the solutions follows. First let us collect our results into one single matrix equation

$$\{z_0, \ldots, z_{12}\} = A_{4\;5\;6}^{0\ldots12}\{z_4, z_5, z_6\},$$

the big matrix $A_{4\;5\;6}^{0\ldots12}$ having thirteen rows and three columns. The 5th, 6th and 7th rows are occupied by the elements of the unit matrix of order three.

Now, in order to express our solutions in terms of any three linearly independent ones, z_j, z_k, z_l say, let us consider first the matrix $A_{4\,5\,6}^{j\,k\,l}$ consisting of the $(j+1)^{th}$, $(k+1)^{th}$ and $(l+1)^{th}$ rows of the big matrix. z_j, z_k, z_l being assumed to be linearly independent, $A_{4\,5\,6}^{j\,k\,l}$ will be non-singular. Hence it possesses a reciprocal matrix which will be denoted by $A_{j\,k\,l}^{4\,5\,6}$. Then, according to the elementary rules of matrix algebra,

$$\{z_4, z_5, z_6\} = A_{j\,k\,l}^{4\,5\,6}\{z_j, z_k, z_l\}$$

and

$$\{z_0, \ldots, z_{12}\} = A_{j\;k\;l}^{0\ldots12}\{z_j, z_k, z_l\},$$

where

$$A_{j\;k\;l}^{0\ldots12} = A_{4\;5\;6}^{0\ldots12}A_{j\,k\,l}^{4\,5\,6}$$

is again a matrix of thirteen rows and three columns.

The transformation theory also solves the question of the analytical continuation of the solutions. For any point (x, y) in the finite part of the four-dimensional space of the complex variables x, y there are three linearly independent solutions, z_j, z_k and z_l say, expressible by hypergeometric series convergent in a certain domain containing this point and extending to the next singularity. Then the matrix $A_{j\;k\;l}^{0\ldots12}$ determines the analytical continuation of all the solutions into the above-mentioned domain.

THE MONODROMIC GROUP

11. The monodromic group of the system (1) is the group of all substitutions which a fixed system of three linearly independent solutions of (1) undergoes when the point (x, y) of the four-dimensional space of the complex variables x and y describes any closed circuit. The transformations of the monodromic group can be composed of the simple transformations which the selected set of three solutions undergoes when one variable is fixed and the other encircles *one* of the singular manifolds of (1) in the positive direction. Only singularities in the finite part of the four-dimensional space of the x, y need to be considered.

On the other hand, having the transformation theory of the previous sections, the substitution undergone by the *selected* system can be determined as soon as the substitution which *any* linearly independent set of three solutions undergoes is known. To fix ideas, let us select $\{z_4, z_5, z_6\}$ again as the fixed set. Now assume that there is a set $\{z_j, z_k, z_l\}$ of three linearly independent solutions of (1) and the substitution, characterised by the 3×3 matrix S, is known which $\{z_j, z_k, z_l\}$ undergoes when (x, y) describes a certain closed circuit. Then clearly the substitution of $\{z_4, z_5, z_6\}$ for the same circuit is given by the matrix

$$\mathrm{A}^{4\,5\,6}_{j\,k\,l}\,\mathrm{S}\,\mathrm{A}^{j\,k\,l}_{4\,5\,6}.$$

Hence we shall attach to each of the singular manifolds a fundamental system of three linearly independent solutions, and find the substitution S which this system undergoes when (x, y) encircles the corresponding singular manifold. Then all the simple substitutions of the monodromic group can be obtained in the manner described in the last paragraph, the matrices A being known or easily computable from the results of the preceding sections.

12. Let us begin by assuming y to be constant and different from 0 and ∞, and only x variable. Then there will be three singularities in the x-plane corresponding to the three singular manifolds $x=0$, $x=y$ and $x=\infty$ of (1).

To $x=0$ we can attach, for instance, the fundamental system $\{z_0, z_1, z_{12}\}$, to $x=y$ the set $\{z_3, z_9, z_{10}\}$, and to $x=\infty$, *e.g.*, the set $\{z_2, z_4, z_8\}$. This is by no means the only possible choice of the fundamental systems. To the singular manifolds in the finite part of the plane we naturally attach solutions which can be expanded into series convergent in the neighbourhood of these singular manifolds. This is not possible with the singularities at infinity, these being "irregular" ones. Yet there is at least *one* solution in the case of $x=\infty$, namely z_2, which is expressible as a series convergent in the vicinity of $x=\infty$.

The linear independence of the fundamental systems chosen can be tested, *e.g.*, by evaluating the determinants of the corresponding matrices $\mathrm{A}^{j\,k\,l}_{4\,5\,6}$.

Next let x be kept constant and different from 0 and ∞, while y is variable. Again there are three singularities, $y=0$, $y=x$, $y=\infty$, and the corresponding fundamental systems may be taken, for instance, to be $\{z_0, z_2, z_{11}\}$, $\{z_3, z_9, z_{10}\}$ and $\{z_1, z_4, z_7\}$ respectively.

13. Now when x encircles the singularity $x=0$ in the positive direction, and no other singularity, $\{z_0, z_1, z_{12}\}$ changes to a system

$$\{z_0^*, z_1^*, z_{12}^*\} = \mathrm{S}_0 \cdot \{z_0, z_1, z_{12}\}.$$

The linear transformation of $\{z_0, z_1, z_{12}\}$ into $\{z_0^*, z_1^*, z_{12}^*\}$ is characterised by the diagonal matrix

$$S_{0\,\bullet} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2(\beta'-\gamma)\pi i} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This can be seen easily from the expansions I (29), I (24), and (19) of the present paper.

Similarly if x encircles the singularity $x=y$ in the positive direction and no other singularity, the corresponding fundamental system $\{z_8, z_9, z_{10}\}$ undergoes a substitution characterised by the diagonal matrix

$$S_{y\,\bullet} = \begin{pmatrix} e^{-2(\beta+\beta')\pi i} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

derived from I (26), I (50), and (17) of the present paper. $\{z_8, z_9, z_{10}\}$ undergoes the same transformation when y encircles $y=x$, hence $S_{\bullet\,x} = S_{y\,\bullet}$.

Finally if y encircles the origin in the positive direction and no other singularity, the corresponding transformation of $\{z_0, z_2, z_{11}\}$ is characterised by the diagonal matrix

$$S_{\bullet\,0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2(\beta-\gamma)\pi i} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now the monodromic group can be defined by the aggregate of all (in general non-commutative) products of positive and negative powers of the matrices

$$T_1 = A^{4\ 5\ 6}_{0\ 1\ 12}\ S_{0\,\bullet}\ A^{0\ 1\ 12}_{4\ 5\ 6},$$

$$T_2 = A^{4\ 5\ 6}_{0\ 2\ 11}\ S_{\bullet\,0}\ A^{0\ 2\ 11}_{4\ 5\ 6},$$

and

$$T_3 = A^{4\ 5\ 6}_{8\ 9\ 10}\ S_{y\,\bullet}\ A^{8\ 9\ 10}_{4\ 5\ 6}.$$

COMPARISON WITH HORN'S RESULTS.

14. Beside the papers cited in I there is another paper by Professor Horn (Horn, 1939) dealing with the integration of the system (1), which came to my hands after I had been written.

The most interesting of Horn's results on the system of linear partial differential equations connected with Φ_2 are contained in his two last papers (Horn, 1938 and 1939), so only the results of these two papers will be discussed and compared with those of I and the present lines.

This comparison seems to indicate that the method of integral representations is superior to the other methods enumerated in section 1 of I.

Beside the well-known solution z_0, Horn (1938) finds z_4, z_5 and z_6 as divergent series formally satisfying (1). The existence of solutions represented asymptotically by these divergent series, *i.e.* the existence of the solutions z_4, z_5, z_6, is proved by putting $x=at$, $y=bt$ and investigating the differential equation satisfied by z *qua* function of t. Horn finds also an integral representation of z_4 but it is less simple than I (31). (See also the next section.)

In the second paper (Horn, 1939) three more formal series of the "mixed" type (being convergent in one variable and divergent in the other) are found and converted into Laplace integrals. Here the superiority of the method of integrating (1) directly by Laplace integrals is even more convincing.

Horn finds first our z_8 (Horn, 1939, equation (1′)), then (*ibidem*, equation (2′)) the series

$$z=\sum_{n\geq m}\frac{(\beta)_m(\beta+\beta'-\gamma+1)_n}{(n-m)!\,m!\,(\beta-\gamma+2)_n}x^{-\beta-m}y^{\beta-\gamma+n+1},$$

which is easily seen to be identical with our solution z_2. There is also the solution (*ibidem*, equation (3′))

$$z=e^x\sum_{m\geq n}\frac{(\gamma-\beta)_m(\beta')_n}{n!\,(\gamma-\beta)_n(m-n)!\,(\beta)_{n-m}}\left(-\frac{1}{x}\right)^{\gamma-\beta+m}(-y)^n, \qquad . \quad (28)$$

but this is merely another asymptotic representation of z_5 and hence identical with Horn's solution (2). Yet from the methods used by Horn this is not immediately evident.

To prove that (28) is equal to z_5, let us use in I (35)

$$\left(1-\frac{s-x}{y-x}\right)^{-\beta'}=(y-x)^{\beta'}(-x)^{-\beta'}\left(1-\frac{s-x}{-x}\right)^{-\beta'}\left(1-\frac{y/(-x)}{1-(s-x)/(-x)}\right)^{-\beta'}$$

$$=(y-x)^{\beta'}(-x)^{-\beta'}\sum_{n=0}^{\infty}\frac{(\beta')_n}{n!}\left(-\frac{y}{x}\right)^n\left(1-\frac{s-x}{-x}\right)^{-\beta'-n},$$

thus obtaining

$$z_5=\sum_{n=0}^{\infty}\frac{(\beta')_n}{n!}\left(-\frac{y}{x}\right)^n(-x)^{\beta-\gamma}\frac{\Gamma(\beta)}{2\pi i}\int_{-\infty}^{(x+)}e^s(s-x)^{-\beta}\left(1-\frac{s-x}{-x}\right)^{\beta-\gamma-n}ds,$$

and this series is convergent for any $|\arg(-x)|<\pi$ in some neighbourhood of $y=0$. The last equation can be written also

$$z_5=(-x)^{\beta-\gamma}e^x\sum_{n=0}^{\infty}\frac{(\beta')_n}{n!}\left(-\frac{y}{x}\right)^n {}_2F_0\left(1-\beta,\gamma-\beta+n;\ \frac{1}{x}\right), \qquad . \quad (29)$$

and this is identical with (28).

15. It has been mentioned already in I, section 10, that Horn's integral representation of z_4 is different from and more elaborate than our integrals. There are certain integral representations of the solutions of (1) which contain as special cases our simple integral representations as well as those of Horn.

These integral representations are of the type

$$z=c\int_a^b e^s s^{-\alpha} \mathrm{F}_1\left(\alpha, \beta, \beta', \gamma; \frac{x}{s}, \frac{y}{s}\right) ds, \quad . \quad . \quad . \quad (30)$$

where α is an arbitrary parameter and F_1 is Appell's function (Appell-Kampé de Fériet, 1926, p. 14)

$$\mathrm{F}_1(\alpha, \beta, \beta', \gamma; x, y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}m!n!}x^m y^n. \quad . \quad . \quad (31)$$

This integral representation can be obtained in many ways. In the case of the solutions z_4, z_5, z_6 the simplest way is to use fractional integration by parts. Also in the case of z_0 it is very easy to verify (30) with $c=\Gamma(\alpha)/(2\pi i)$, the path of integration being the loop $-\infty \ldots (0+, x+, y+) \ldots -\infty$. Further, putting (30) into (1) and using the system of differential equations of F_1 this integral representation can be proved easily.

Plainly $\alpha=\gamma$ yields the integral representations of the type I (14). In this case F_1 degenerates. In the general case as well as in the case of Horn's integral representations, F_1 is not expressible as a product of elementary functions.

16. Beside the advantages of the method exhibited in the preceding sections there is one more which is of great weight, namely, that the integral representations bring out clearly the *relations between different systems of partial differential equations*. Thus the results of I, section 7, suggest that (1) is equivalent to, *i.e.* transformable into, the system of linear partial differential equations attached to Φ_1.

In fact, let us put, for instance,

$$A=\beta+\beta'-\gamma+1, \quad B=\beta', \quad C=\beta'-\gamma+2, \quad X=\frac{x}{y}, \quad Y=x, \quad Z=x^{\gamma-\beta'-1}y^{\beta'}z. \quad (32)$$

Then (1) transforms into

$$\left.\begin{aligned} X\frac{\partial^2 Z}{\partial X\partial Y}+Y\frac{\partial^2 Z}{\partial Y^2}-X\frac{\partial Z}{\partial X}+(C-Y)\frac{\partial Z}{\partial Y}-AZ=0,\\ X\frac{\partial^2 Z}{\partial X\partial Y}-\frac{\partial Z}{\partial X}+B\frac{\partial Z}{\partial Y}=0. \end{aligned}\right\} \quad . \quad . \quad (33)$$

The first equation (33) is identical with the second equation of the system of Φ_1 as given by Humbert (1920–21, p. 77, bottom; see also Appell and Kampé de Fériet, 1926, p. 128). The second equation (33) is also well known to be satisfied by Φ_1 (*ibidem*). Both together are the most concise way of representing the system of differential equations satisfied by Φ_1.

17. The results of section 4 suggest that (1)—or (33)—is equivalent to the system of differential equations satisfied by Γ_1. More precisely, (19), for instance, suggests putting

$$A=\beta, \quad B=\beta'-\gamma+1, \quad B'=\gamma-1, \quad X=-\frac{x}{y}, \quad Y=-y, \quad Z=y^{\gamma-1}z. \quad (34)$$

Substituting this in (1), we obtain

$$\left.\begin{aligned} \frac{\partial^2 Z}{\partial X \partial Y}-X\frac{\partial Z}{\partial X}-AZ=0, \\ X\frac{\partial^2 Z}{\partial X \partial Y}-Y\frac{\partial^2 Z}{\partial Y^2}+X\frac{\partial Z}{\partial X}+(B'-1-Y)\frac{\partial Z}{\partial Y}-BZ=0. \end{aligned}\right\} \quad (35)$$

The last equation is the second one of the system as given by Horn (1935, p. 644). The first equation of (35) is also known to be satisfied by Γ_1 provided that $B+B' \neq 0$, *i.e.* $\beta' \neq 0$ (*ibidem*). Apart from this case which has to be excluded throughout, (35) is the shortest representation of the system of differential equations satisfied by Γ_1.

Horn has also dealt shortly with the integration of this system (*ibidem*, p. 660 and *seq.*), but his results here are less complete than those regarding (1).

Our results also indicate the connection between (1) and particular cases of the systems of Ψ_1, Ξ_1, H_2 and H_{11}.

Not only does the integral representation reveal the relation between different systems, but it also enables us to choose, between infinitely many equivalent forms, a fairly simple one. If we start, for instance, with the system of Φ_1 and arrive at the integral representation I (22), it is rather obvious that a substitution $uy=t$, $y/x=x'$, $y^{\alpha}z=z'$, $\alpha-\gamma+1=\beta'$ will render a greater symmetry. By this one is automatically led to a system nearly related to (1).

(1) is the most symmetrical among the known systems of partial hypergeometric differential equations which are equivalent to it; therefore it is advisable to choose this system for investigation. Nevertheless our results give implicitly also a complete theory of the solutions of (33) or (35) or of any equivalent system.

SUMMARY.

18. Continuing the integration of the system of two linear partial differential equations of second order of which Humbert's confluent hypergeometric series Φ_2 of two variables is one solution, a great number of convergent expansions of the solutions introduced in a preceding paper is given. Also three new solutions are established and for each of them twelve different convergent expansions obtained.

Thus a total of seven different solutions expressible by ninety-three convergent hypergeometric series of two variables has been investigated. There are two more sets each of which contains three linearly independent solutions, represented by divergent or mixed type hypergeometric series of two variables respectively.

The full transformation theory of the solutions is outlined and the monodromic group of the system determined.

The equivalence of the system discussed with the systems of partial differential equations, of which Humbert's series Φ_1 or Horn's series Γ_1 are solutions respectively, has been established and the results compared with those arrived at by Horn.

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XXVII.—The Design and Interpretation of Experiments based on a Four-fold Table: the Statistical Assessment of the Effect of Treatment.

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THE tabulation of results in the form of a four-fold table is frequently of great convenience, especially in the biological sciences. By a four-fold table is meant a table of the following type in which, for example, the two columns represent survivors and dead, and the two rows represent untreated and treated.

	Survived.	Died.	Totals.
Untreated	a	b	$a+b=n_1$
Treated	c	d	$c+d=n_2$
Totals . . .	$a+c=n_3$	$b+d=n_4$	N

It will be seen that there are four separate compartments containing respectively the numbers of those who were untreated and survived, who were untreated and died, who were treated and survived, and who were treated and died. There are also four marginal totals giving respectively the total numbers of untreated and of treated, of survivors and of dead. In planning an experiment to determine some particular point it is often found advantageous to adopt this four-fold form. For example, in the many instances when it is a question of deciding whether a treatment is or is not effective, or which of two methods of treatment is the better, an experiment of the above type naturally suggests itself.

The purpose of this paper is to present a number of considerations which may guide the investigator in planning and in interpreting such an experiment.

1. THE χ^2 TEST.

It may be convenient first of all to give a brief account of the important χ^2 test as it is commonly employed in relation to a four-fold table. We calculate the value of the quantity

$$\chi^2=\frac{\mathrm{N}(ad-bc)^2}{n_1 . n_2 . n_3 . n_4}. \qquad (1)$$

It is easy to see that, in a general way, the greater the effect of the treatment, the greater will be the value of χ^2. However, even if the treatment were really without effect, χ^2 would not necessarily be zero; when the animals are distributed quite at random as between treated and untreated, it may happen as the result of mere chance that a larger proportion of highly resistant animals are included in the one group than in the other. One now finds the value of P (from a table such as is given by Udny Yule and Kendall, 1937, pp. 534–535), which corresponds to the calculated value of χ^2. P is the chance that in an experiment in which the treatment had no effect at all we would obtain a value of χ^2 equal to or greater than that calculated from the actual experimental figures. It is not difficult to see that the result of an effective treatment would be to render large values of χ^2 likely. Hence it is natural to regard a large value of χ^2—unlikely to have occurred in the absence of effective treatment, and likely to occur when the treatment is effective—as evidence of the effectiveness of the treatment. The value of χ^2 to be regarded as critical is determined by reference to the corresponding value of P. In practice it is usual to regard values of $P < 0{\cdot}05$ as significant, that is values of $\chi^2 > 3{\cdot}85$. In other words, we reject as not significant (that is to say not demonstrating that the treatment is effective) all experiments which give a χ^2 of less than 3·85. If the treatment were in reality quite ineffective, a value of χ^2 as great or greater than this would be obtained as the result of mere chance in only 5 per cent. of many repeated experiments.

It is perhaps desirable to refer here to Yates's "correction for continuity," which should be made in the formula for χ^2 as given above when relatively small numbers are found in any of the compartments. On the assumption that the product ad is less than bc, then half ought to be added to each of a and d, and subtracted from each of b and c, before carrying out the calculation. It will be observed that this leaves n_1, n_2, n_3, n_4 and N unaffected. For extreme cases still more complicated corrections are in strictness required, but these need not usually be applied.

2. DESIGN OF A FOUR-FOLD EXPERIMENT.

(*a*) In the design of an experiment based on a four-fold table the two most important questions which arise are the number of animals to be used and the degree of mortality to be aimed at in the controls. The first question will be discussed in a subsequent section. With regard to the second, it may perhaps suffice to remark that from the statistical point of view it is always desirable to avoid zero or very small entries in any of the four compartments. This is because when the expectations are small the deviations become disproportionately large, and the entries which must of necessity be integers do not follow a normal distribution; furthermore, if all the controls die, the "test dose" (*i.e.* the quantity of bacteria or of toxic substance injected into all the animals, both treated and untreated) may have been so great that the effectiveness of the treatment had no chance of declaring itself. To give the experiment then the best chance of demonstrating the effectiveness of the treatment, when the number of animals available is limited, it should be so arranged that in the control groups most, but not all, of the animals will die. An exception to this rule arises when the treatment is so effective that it can convert 100 per cent. mortality into 0 per cent. In these circumstances of course it is even an advantage to start with a dose which will kill all of the controls, but it is not in relation to such extreme cases that the problem of careful design arises.

(*b*) *Number of Test Animals required.*—It is next necessary to consider how many animals must be employed if the experiment is to have a reasonable chance of giving a definite result. The larger the numbers in the control and in the treated groups, the greater is this chance likely to be, but, on the other hand, the experiment becomes more costly and laborious to carry out. Frequently in practice it is of great importance to reduce the numbers to a minimum consistent with a reasonable expectation of obtaining a definite result. It is at once clear that unless we have some rough idea not only of the expected death-rate p_1 amongst the controls, but also of the expected efficiency of the treatment, it is impossible to do other than proceed by a process of trial and error. If, however, as often happens, we have some rough idea of the order of the efficiency which we expect a treatment to possess (or if, alternatively, we decide that we are interested in positive results only if these show an efficiency greater than a certain stated minimum), then it is possible to calculate approximately the minimum number of animals which must be employed, in order to obtain a significant result with reasonable certainty. Usually it is the total number of animals N in the two groups—treated and untreated—taken together that must for the sake of economy be kept as low as possible. In the absence of special previous knowledge, the most efficient plan is to make the numbers n_1 and n_2 in the untreated and treated groups equal, and to select the animals at random. If it were known from previous experiments that the untreated animals had a certain average mortality, this knowledge might make it reasonable to increase the number of treated animals at the expense of the number of controls; but usually it is best to employ groups of equal size n.

We shall assume that the death-rate of the untreated controls is p_1, and that of the treated p_2. Let us denote the fraction $\frac{p_1 - p_2}{p_1}$ by y. This fraction may be called the efficiency of the treatment, and as a first approximation may be regarded as a constant independent of p_1. Thus if the experiment were carried out with very large numbers of animals in each of the two groups, the death-rates would be p_1 and $(1-y)p_1$ respectively. If smaller numbers were used, deviations from these values would occur as the result of random sampling. On the average, however, the numbers of dead amongst the untreated would be $p_1 n$, and amongst the treated $(1-y)p_1 n$, where n is the number of animals in the experimental and the control groups, these being assumed to be of equal size.

If we now in theory apply the χ^2 test as described above to these average values, we obtain χ^2 in terms of p_1, n, and y. By putting $\chi^2 = \chi^2_P$ (where χ^2_P is the value of χ^2 corresponding to the value of P regarded as significant) we obtain the condition that the average result, or one showing even greater bias, would have a chance P of occurring in populations in which the treated and the untreated really had equal death-rates. We thus obtain an equation involving n, y, and p_1 and from it we can express n in terms of y and p_1. Thus

$$n = \chi^2_P \frac{(2-y)(2-2p_1+yp_1)}{2y^2 p_1}, \qquad . \quad . \quad . \quad . \quad . \quad (2)$$

from which we have, when p_1 is small compared with unity,

$$n = \chi^2_P \frac{(2-y)}{y^2 p_1}, \qquad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and when y is small compared with unity,

$$n = 2\chi^2_P \frac{(1-p_1)}{y^2 p_1}. \qquad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

So far we have assumed that the totals of treated and untreated in the experiment are equal. As remarked above, this is usually the best arrangement, but sometimes this equality cannot be realised in practice. For instance, statistics of antirabic treatment are collected year by year, giving the numbers of persons subjected to different methods of treatment at different institutes, and the results may be examined with the object of ascertaining whether they exhibit any significant differences in mortality according to the methods of treatment employed. In order to obtain sufficiently large numbers in each category, the figures must be collected over a number of years. The numbers subjected to any two types of treatment are not equal, but the ratio from year to year remains approximately constant. If then we want to arrive at a rough estimate of the total number necessary for the detection of some significant difference between any two methods of treatment, we require modifications of the above formulæ applicable to the case in which the numbers of the untreated and the treated—or of those treated by method A and those treated by method B—are n and βn respectively.

The required formulæ, deduced by the same method as in the case where the numbers are equal, are as follows:—

$$n = \chi_{\mathrm{P}}^2 \frac{\{1+\beta(1-y)\}\{1-p_1+\beta-\beta(1-y)p_1\}}{(1+\beta)\beta y^2 p_1}, \quad . \quad . \quad . \quad (5)$$

or when p_1 is small compared with unity,

$$n = \chi_{\mathrm{P}}^2 \frac{\left(\frac{1}{\beta}+1-y\right)}{y^2 p_1}, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and when y is small compared with unity,

$$n = \frac{1+\beta}{\beta} \chi_{\mathrm{P}}^2 \frac{(1-p_1)}{y^2 p_1}. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

It is to be emphasised that these values of n are based on the assumption that the experiment will give its average or most likely result. In practice in about half of all experiments the value of χ^2 actually found will exceed and in the other half will fall short of this average. Thus if we adopt for our experiment the value of n given by the formulæ, a result which we would judge significant (using as criterion a value of χ greater than χ_{P}) would be obtained in 50 per cent. of all experiments carried out with this number of test animals.

Obviously there would be very little use in carrying out an experiment in which the size of the samples was less than that given by equations (2) to (7), and usually it is desirable that the samples should be even greater than this, especially where it is intended to carry out only a single experiment. In those circumstances it might be reasonable to employ two to four times the number of animals given by the above formulæ. This will ensure that the experiment will indicate the existence of a significant difference between the two populations of treated and untreated, in substantially more than 50 per cent. of all cases.

The following method of approach, whilst more limited in some respects than the previous, allows of a more definite solution of this aspect of the problem being given.

Let us regard the groups, treated and controls, as constituting two independent experiments, from the one of which we may form an estimate p_1' of p_1, and from the other p_2' of p_2. The best values of p_1 and p_2 may be shown to be $p_1' = \frac{a}{a+b}$, and $p_2' = \frac{c}{c+d}$ respectively, and the standard deviations of p_1' and p_2' are

$$\sigma_{p_1'} = \sqrt{\frac{p_1 q_1}{n_1}}$$

and

$$\sigma_{p_2'} = \sqrt{\frac{p_2 q_2}{n_2}},$$

where

$$q_1 = 1 - p_1$$

and

$$q_2 = 1 - p_2.$$

An estimate of $\epsilon = p_1 - p_2$ is given by $\epsilon' = p_1' - p_2'$, with a standard deviation of $\sigma_{\epsilon'} = \sqrt{(\sigma^2_{p_1'} + \sigma^2_{p_2'})}$. We wish to ensure that ϵ' is significantly greater than zero, in, let us say, a proportion P′ of all experiments, where P′ is some fraction arbitrarily decided upon. ϵ' will be deemed to be significantly greater than zero provided that $\frac{\epsilon'}{\sigma_{\epsilon'}}$ is greater than χ_P, where $\sigma_{\epsilon'} = \sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}$, and χ_P is the value of χ corresponding to some arbitrarily chosen standard of significance. But ϵ' has a distribution approximately normal, with mean at ϵ, and standard deviation $\sigma_{\epsilon'}$, so that the value $\chi_P \sigma_{\epsilon'}$ will be exceeded in the fraction P′ of all experiments provided that $\epsilon - \chi_P \sigma_{\epsilon'} = \chi_1 \sigma_{\epsilon'}$, where χ_1 is the value of χ corresponding to α, where $\frac{1+\alpha}{2} = P'$, or $\alpha = 2P' - 1$.

As p_1 and p_2 are not known, $\sigma_{\epsilon'}$ cannot be calculated from the data If, however, $p_1 - p_2$ is not great, we may assume that $p_1 = p_2 = \frac{n_3}{N}$ and $q_1 = q_2 = \frac{n_4}{N}$. Hence $\sigma_{\epsilon'}$ is approximately equal to

$$\sqrt{\frac{n_3 n_4}{N^2}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} = \sqrt{\frac{n_3 n_4}{N n_1 n_2}},$$

whence

$$\epsilon = (\chi_P + \chi_1)\sqrt{\frac{n_3 n_4}{N n_1 n_2}}.$$

But

$$\epsilon = p_1 - p_2 = p_1 y,$$

so that

$$p_1 y = (\chi_P + \chi_1)\sqrt{\frac{n_3 n_4}{N n_1 n_2}}.$$

Now

$$n_2 = \beta n, \quad \text{and} \quad N = (1 + \beta)n,$$

so we find

$$n = (\chi_P + \chi_1)^2 \frac{1+\beta}{\beta}\frac{q_1}{y^2 p_1} = \left(1 + \frac{\chi_1}{\chi_P}\right)^2 n_0, \quad . \quad . \quad . \quad (8)$$

where

$$n_0 = \chi_P^2 \frac{1+\beta}{\beta}\frac{q_1}{y^2 p_1}, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

the number required for significance in 50 per cent. of all experiments.

When P' is taken as 0·5, so that $\alpha = 0$, $\chi_1 = 0$, and $n = n_0$.

When $P' = 0{\cdot}975$, $\alpha = 0{\cdot}95$, and $\chi_1 = 1{\cdot}96$; when χ_P is also taken as 1·96 it follows that $n = 4n_0$.

It is easily shown that when $n = 2n_0$ and $\chi_P = 1{\cdot}96$, $\chi_1 = (\sqrt{2} - 1)\chi_P = 0{\cdot}81$, and $P' = 0{\cdot}79$.

Thus doubling the number increases the chance of obtaining a significant result from 50 per cent. to 79 per cent., whilst quadrupling it increases the chances to 97 per cent. When only one experiment is feasible, it therefore seems advisable that if possible at least four times n_0 should be employed. For convenience values of n_0 for varying values of p_1 and y are given in Table I.

TABLE I.

	$p_1 = 0{\cdot}1$	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
$y = 0{\cdot}1$	6930	3080	1797	1155	770	513	330	193	86
0·2	1733	770	449	289	193	128	83	48	21
0·3	770	342	200	128	86	57	37	21	10
0·4	433	193	112	72	48	32	21	12	5
0·5	277	123	72	46	31	21	13	8	3

Values of n_0 for varying values of p_1 and y. $\beta = 1$ (groups equal), $\chi_P = 1{\cdot}96$, $P = 0{\cdot}05$.

Experiments carried out with n_0 animals in both experimental and control groups have a 50 per cent. chance of showing significance; with $2n_0$ and $4n_0$ animals the chances are 79 and 97 per cent. respectively.

3. INTERPRETATION OF A FOUR-FOLD EXPERIMENT.

(a) *General Remarks.*—In the application of the χ^2 method to four-fold tables examples are frequently encountered where the same value of χ^2 is obtained from experiments of different sizes. In the ordinary routine application of the method, a value of P is deduced from χ^2, and the result of the experiment is considered to be significant or non-significant according as P is less or greater than 0·05. It is obvious, however, that this procedure does not give us all the information contained in the data. For example, it is clear that a value $P=0{\cdot}2$ obtained from an experiment with about 10 million in each category is much more conclusive evidence that the population is really homogeneous, or at least very nearly so, than the same value of P obtained from an experiment with 10 in each category.

The problem may be treated by the method of fiducial probability proposed by Fisher, or the very closely related method of confidence-ranges developed by Neyman. The general problem in statistics is to make statements about a population from our knowledge of a sample of that population, the sample being supposed to have been drawn at random or according to some specified rule. The method of inverse probability originally supported by famous mathematicians, such as Laplace, has in recent years been subjected to very serious criticism, and is probably untenable at least in its general form. Its aim was to calculate the chance that a particular hypothesis about the general population was true. It is clearly necessary in the first place to give a definite meaning to this phrase. The essence of Fisher's proposal is that we should consider the question of chance or probability not directly in relation to the general populations themselves, but in relation to the truth of statements made about these general populations. What we set ourselves to do is to make statements which will have, say, a 95 per cent. chance of being true. More precisely this means that we set out to find a rule such that, if we follow it, then on the average 95 out of every 100 statements we make according to this rule will be true ones (or more generally at least 95 per cent. will be true ones). It is probable that some idea of this nature was really in the minds of the exponents of the theory of inverse probability, although it is not explicitly stated in this form in their writings.

All that the scientist can hope to do in drawing his conclusions is to reduce the number of his erroneous statements to a minimum. This can always be done by reducing the content of the statements which he makes. For instance, he can with certainty say that the distance between Paris and London is between 100 and 1000 miles, and if he restricts himself to vague statements of this kind every one of them will be correct. The more precise and unconditional his statements the more difficult is it for him to ensure that some definite percentage of them will be correct.

The chief object of the theoretical scientist who is presented with certain experimental facts is to make statements which will have the maximum of content compatible with a definite degree of truth or, what is almost the same thing, the maximum of truth compatible with a certain assigned degree of content. Somewhat loosely expressed, the object is to extract the largest possible amount both of truth and of content from the data. The science of statistics attempts to solve this problem in the case where the experimental observations are subject to variations which cannot be directly controlled.

(b) *Illustrations of General Theory.*—Before going on to apply these principles to the four-fold table, we shall, in the first place, consider one or two rather simpler examples. These refer to the interpretation of the so-called standard error of the mean, and amongst the various cases which may occur are the following:—

(1) Let us suppose that we have a normally distributed population with an unknown mean μ and a known standard deviation σ. Let us suppose that we draw a single sample and find that it has the value x. Then it emerges from the theory that the statement that the real mean lies between $x+\phi_P\sigma$ and $x-\phi_P\sigma$ (where ϕ_P is the abscissa of the normal curve with standard deviation unity and origin at the mean, such that the proportion P of the total area lies between ϕ_P and $-\phi_P$) has the chance P of being true. If we chose $P=95$ per cent. then $\phi_P=1{\cdot}96$, and the statements so made will be correct on the average 95 times, and wrong 5 times, in 100. If $P=0{\cdot}5$ then $\phi_P=0{\cdot}6745$, and the statements are then correct in only 50 per cent. of all cases.

(2) Let us next consider the case where we take a sample n from the same population, and let the mean of the sample be $\bar{x}$, the fiducial statement is that the real value of the mean μ lies within the range $\bar{x} \pm \phi_P\frac{\sigma}{\sqrt{n}}$, where as before if we require a 95 per cent. reliability $\phi_P=1{\cdot}96$, and for a 50 per cent. reliability $\phi_P=0{\cdot}6745$. It may be pointed out that the ordinary probable error, $\pm 0{\cdot}6745\frac{\sigma}{\sqrt{n}}$, evidently gives the range for a 50 per cent. reliability, and a range of three times the probable error corresponds roughly to a 95 per cent. reliability.

(3) As a third example let us suppose that we have a normal population of which we do not know either the mean μ or the standard deviation σ, and that we draw from it a sample of size n. Let $t_{(n,\,P)}$ be the value of t which in the well-known Student's t distribution is such that P per cent. of the area of the curve (lying to the right of the origin) is contained between the ordinate at t and the y axis. The fiducial statement is that

μ lies within the range

$$\bar{x} \pm t(n, \mathrm{P})\frac{s}{\sqrt{n}},$$

where

$$\bar{x} = \frac{\mathrm{S}x_r}{n},$$

and

$$s^2 = \frac{\mathrm{S}(x_r - \bar{x})^2}{n - 1}.$$

Such statements will then have a reliability of P. If n be very large it is well known that $t(n, \mathrm{P}) = \phi_\mathrm{P}$, so that for large values of n, the statement is that μ lies within the range $\bar{x} \pm \phi_\mathrm{P}\frac{s}{\sqrt{n}}$. As, for large values of n, s very rarely differs appreciably from σ, of which it is a good estimate, it is clear that this formula is in agreement with that obtained in (*b*) (2) above. The new formula is, however, applicable in the absence of any *a priori* knowledge of σ.

(c) *Application to Four-fold Table.*—We shall now apply these principles to the four-fold table. Let us assume that we have two populations, each a mixture of black and white balls, the probability of a ball being black being p_1 in the first population and p_2 in the second. Let us draw a sample of n_1 balls from the first population and n_2 from the second. We have now an aggregate sample containing two sub-samples which give estimates p_1' of p_1 and p_2' of p_2 respectively. Considering as before the whole population of samples, we have a population of sub-samples of p_1 each associated with a sub-sample of p_2. Each population of sub-samples forms a normal distribution with means at p_1 and p_2, and standard deviations $\sqrt{\frac{p_1q_1}{n_1}}$ and $\sqrt{\frac{p_2q_2}{n_2}}$ respectively. Thus the population consisting of the differences $p_1' - p_2'$ has a normal distribution with mean $p_1 - p_2$, and standard deviation

$$\sigma = \left(\frac{p_1q_1}{n_1} + \frac{p_2q_2}{n_2}\right)^{\frac{1}{2}},$$

then, applying the results of (*b*) (1) above, it follows that the real value of $p_1 - p_2$ lies within the range $p_1' - p_2' \pm \phi_\mathrm{P}\sigma$.

We do not know the real value of σ, but if n_1 and n_2 are large it will very rarely differ appreciably from

$$\left(\frac{p_1'q_1'}{n_1} + \frac{p_2'q_2'}{n_2}\right)^{\frac{1}{2}}.$$

Hence our statement is practically equivalent to saying that $p_1 - p_2$ lies within the range

$$p_1' - p_2' \pm \phi_P\left(\frac{p_1'q_1'}{n_1} + \frac{p_2'q_2'}{n_2}\right)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad (10)$$

provided that n_1 and n_2 are large.

If $p_1' - p_2'$ may be taken as very small compared with p_1' and p_2', so that $p_1' = p_2' =$ approximately $\frac{n_3}{N}$, then the above formula is very nearly equivalent to

$$p_1' - p_2' \pm \phi_P\left(\frac{n_3 n_4}{N n_1 n_2}\right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This formula is rather easier to calculate arithmetically than (10).

It may be mentioned that a formula very similar to (11) can be obtained by a different method in which the real value of $p_1 - p_2$ is assumed to be ϵ, and the value of χ is then calculated as a function of ϵ. From the distribution of χ in terms of ϵ, fiducial limits for ϵ are deduced in terms of χ. The limits so found are

$$\frac{p_1' - p_2' \pm \phi_P\left(\dfrac{n_3 n_4}{N n_1 n_2}\right)^{\frac{1}{2}}}{1 + \phi_P F}, \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where

$$F = \frac{(n_1 - n_2)(n_3 - n_4)}{2(N n_1 \,.\, n_2 \,.\, n_3 \,.\, n_4)^{\frac{1}{2}}}.$$

This formula approximates to that already found when $\phi_P F$ is small compared with unity. This is always true when n_1 is not very different from n_2, and n_3 is not very different from n_4. If either or both of the two dichotomies is very unequal, it can be shown that the whole expression is small compared with unity provided that the product of n_1 and n_3 (the two smaller groups) is much larger than N. (When $\phi_P = 2$, say 100 times as large as N.) In many cases which occur in practice this condition is satisfied, and in our subsequent discussion we shall use the simple formula (11) in which $\phi_P F$ is neglected.

It is of interest to note that (11) may be written in the form $(p_1' - p_2')\left(1 \pm \frac{\phi_P}{\chi}\right)$, where χ has the value given in equation (1), for it may be readily shown that

$$\chi \equiv \frac{p_1' - p_2'}{\left(\dfrac{n_3 n_4}{N n_1 n_2}\right)^{\frac{1}{2}}}.$$

When $\chi = \phi_P$ (*i.e.* $\chi^2 = 3{\cdot}85$ for a 95 per cent. reliability) one end of the range is at the zero point. This means that it is just possible that the two populations are really identical. This is likewise the conclusion that would be drawn by the application of the ordinary χ^2 method; the latter method, however, tells us only that a homogeneous population might reasonably be expected to produce the given result, whereas the present method makes a statement about the range within which $p_1 - p_2$ must lie.

Table II presents an example taken from a review of the results of anti-rabies treatment (M'Kendrick, 1940. Ninth review).

TABLE II.

		Number Treated.	Deaths.	Mortality, %	Upper Limit.	Lower Limit.
Europeans:	Cords	65,025	95	0·15	+0·0214	−0·0602
	Killed phenol	81,555	135	0·17		
		146,580	230			
Non-Europeans:	Cords	80,952	423	0·52	+0·0078	−0·1056
	Killed phenol	383,955	2194	0·57		
		464,907	2617			

During the last few years, formula (11), and the extension of it to be defined in the next section of this paper, have been applied in the analysis of these rabies statistics, but the derivation of the formulæ has not previously been published.

(d) *Application to* $2 \times r$ *Table.*—Sometimes the experiment refers not to two treatments only but to several. In this case the results are conveniently represented in a $2 \times r$ table.

	Dead.	Survivors.	Totals.
Treatment 1	a_1	b_1	n_1
,, 2	a_2	b_2	n_2
,, 3	a_3	b_3	n_3
: :	:	:	:
,, r	a_r	b_r	n_r
Totals	m_1	m_2	N

Clearly the figures available in such a table enable us to apply tests for the existence of a significant difference between any two treatments. We may also examine whether any one treatment is different from the aggregate of all the others. We may also, by the method described under (c) in the previous section, obtain an estimate of the difference $p_s - p_t$, where p_s and p_t are the death-rates for any two treatments s and t respectively, and likewise for the value of $p_s - p_\rho$, where p_ρ refers to the remaining treatments (excluding treatment s). However, both these methods of calculation are to some extent unsatisfactory, because we obtain a series of ranges which are not directly comparable with each other. For example, in the series p_1-p_ρ, p_2-p_ρ, . . ., etc., the p_ρ's in each expression are obviously different from each other.

This difficulty is avoided in the following way. We note that if $p_1, p_2, p_3, \ldots p_r$, be the various death-rates, and $p_{m_1}=\frac{m_1}{\mathrm{N}}$ the death-rate of the whole aggregate, then it may be shown that

$$p_1-p_{m_1}=\frac{\mathrm{N}-n_1}{\mathrm{N}}(p_1-p_\rho)=\frac{n_\rho}{\mathrm{N}}(p_1-p_\rho). \qquad . \qquad . \qquad . \quad (13)$$

Now by condensing the above table into a four-fold table of which the rows refer to treatment 1, and to the remainder respectively, we obtain as the fiducial limits of p_1-p_ρ

$$p_1'-p_\rho' \pm \phi_{\mathrm{P}}\left(\frac{m_1m_2}{\mathrm{N}n_1n_\rho}\right)^{\frac{1}{2}},$$

whence the fiducial limits of $\frac{\mathrm{N}-n_1}{\mathrm{N}}(p_1-p_\rho)$ are

$$\frac{\mathrm{N}-n_1}{\mathrm{N}}(p_1'-p_\rho') \pm \frac{\mathrm{N}-n_1}{\mathrm{N}}\phi_{\mathrm{P}}\left(\frac{m_1m_2}{\mathrm{N}n_1n_\rho}\right)^{\frac{1}{2}}. \qquad . \qquad . \qquad . \quad (14)$$

Clearly as equation (13) is a purely algebraical identity it must hold

TABLE III.

Europeans.	Number Treated.	Deaths. Observed.	Deaths. Calculated.	Percentage Mortality.	Upper Limit.	Lower Limit.
1. Cords . .	65,025	95	100	0·15	+0·0201	−0·0365
2. Dilutions .	73,364	125	113	0·17	+0·0425	−0·0103
3. Killed phenol .	81,555	135	126	0·17	+0·0425	−0·0202
4. Live phenol .	7,006	6	11	0·09	+0·0228	−0·1600
5. Fermis vacc. .	17,723	23	27	0·13	+0·0324	−0·0814
6. Fermis S.V. .	390	0	1	0	+0·2351	−0·5437
7. Heated . .	140,959	258	217	0·18	+0·0464	+0·0110
8. Killed ether .	90,919	112	140	0·12	−0·0078	−0·0544
9. Mixed (*a*) .	57,227	65	88	0·11	−0·0102	−0·0712
(*b*) .	7,307	14	11	0·19	+0·1267	−0·0521
(*c*) .	6,646	8	10	0·12	+0·0599	−0·1277
10. Yatren . .	5,384	13	8	0·24	+0·1915	−0·0171
	553,505	854	852	0·15		

$\chi^2=29·8$. $\mathrm{P}=0·001$.

In the last two columns the limits of the superiority or the inferiority of the particular treatment as compared with the aggregate of all treatments are shown. In three instances the range does not overlap the zero point, and within the limits of reliability chosen (95 per cent.) we may state that two of these treatments have a superiority over the average of between 0·0078 and 0·0544 per cent. and 0·0102 and 0·0712 per cent. respectively, and the other an inferiority of between 0·0464 per cent. and 0·0110 per cent. Attention may be drawn to the extremely large range of treatment 6. This emphasizes the fact that when small numbers are involved, the conclusions which may be drawn are necessarily vague.

both for the real and the observed values of the p's. Whence by (14) the limits of $p_1 - p_m$ are

$$p'_1 - p'_m \pm \frac{\phi_P}{N}\left(\frac{m_1 m_2 n_p}{n_1 N}\right)^{\frac{1}{2}}. \qquad (15)$$

These are the limits within which we may assert that the true value of $p_1 - p_m$ lies.

In Table III this formula has been applied to certain rabies statistics from the Ninth League of Nations Review (M'Kendrick, 1940).

SUMMARY.

1. Formulæ are given in a convenient form for calculating the numbers of animals likely to be required in a controlled experiment for testing the effect of a treatment, when a rough knowledge of the possible maximum efficiency of the treatment is available, and the level of mortality amongst the controls is approximately known. For most purposes formula (8) is sufficiently accurate.

2. A formula (11) is deduced for calculating the limits between which the real advantage of the treated over the control is likely to lie.

3. The method has been extended to the case where various treatments are compared amongst themselves. A formula (15) is deduced for the calculation of the limits within which the real superiority (or inferiority) of any one treatment over the aggregate of all the treatments is likely to lie.

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OBITUARY NOTICES.

Charles Eugène Barrois, For. Mem. R.S., Hon. F.R.S.E.

CHARLES BARROIS was born on April 21, 1851, and died November 5, 1939. No man since Murchison has won such renown for high-class stratigraphical research, within and without the bounds of his own country. Belonging to a great industrial family of the Nord, he was free from the first to follow his natural bent for Science. In 1871 he entered at Lille what was then a constituent college of the University of France. Immediately he fell under the influence of Professor Jules Gosselet, to whose enduring geological researches one may trace much indeed of our present-day knowledge concerning the Ardennes and the associated Franco-Belgian coalfield—including tectonic principles at once transmitted, through Marcel Bertrand's intuition, to the far-distant Alps. Thus Barrois enrolled as the first definitely geological student of Gosselet's school, still in its birth-struggles; and before the year closed he had assisted his master in founding the Société géologique du Nord, which under his constant care has grown to be one of the most distinguished geological societies of the world. There was nothing evanescent in Barrois' enthusiasm. When the age-limit set Gosselet on one side in 1902, it was Barrois who took his place as Professor of Geology in the University of Lille; and, when Gosselet died in 1916, it was Barrois again who succeeded as Director of the Société du Nord. From active occupation of the former post he retired in 1926, though retaining the honorary title to the end.

Barrois first attracted general attention through a six-week tour of the Chalk of England and Antrim, where he established (1876) the continuation of fossil zones already elaborated by Hébert and others in the north of France. The thoroughness and precision of his onslaught took the breath from contemporary workers in England, and to this day prove an unfailing source of amazement, admiration, and inspiration to successive generations of students making first acquaintance with the subject.

Barrois is also gratefully remembered for friendly invasions into northern Spain, where we owe to him most of our knowledge of the local development of the Hercynian Chain (1882), and into the Sierra Nevada farther south (1884). He travelled widely and fruitfully in other lands, but his greatest contribution to Geology has been his overwhelming share in the survey of Armorica, by which is understood the whole of Brittany along with a border zone of neighbouring provinces, particularly Normandy. If one emphasises Barrois' contributions as a stratigrapher, one perhaps runs a risk of underrating his general accomplishment. Barrois was a complete geologist, distinguished not only in the field, but also in the laboratory, as a palæontologist well versed in many classes of the animal kingdom, and as a petrologist equally at home among igneous and metamorphic problems. He was main author of more than 20 sheets of Armorica in the 1 : 80,000 map of France. In fact a geologist may well say: Armorica, *c'est* Barrois. Armorica is comparable in size with Scotland, and, though more accessible, and favoured with more convenient weather, its geological problems present much the same range and intricacy as do those of the Highlands and Southern Uplands of our country. To his colleagues in Scotland it has seemed very wonderful that one man could cover the whole of this region, everywhere showing manifest delight in, and mastery of, both local detail and wide-reaching principle. The achievement, though in less heroic mould, reminds one of Törnebohm's contribution to Scandinavian research.

As Gosselet's strength failed Barrois seems increasingly to have felt the call of his home coalfield, tectonically the most complicated in the world. His foundation of the Musée Houiller de l'Université de Lille in 1908, and his close personal underground investigation of structural perplexities, proved very valuable to the industry. Versailles later widened his opportunities, and he extended his researches into the relatively simple coalfield of the Saar and its concealed prolongation in Lorraine.

Meanwhile, the last war had burst in 1914. It found Gosselet old, but living; and he and Barrois considered that the highest service that they could offer to their country was to remain at Lille in charge of the University Collections, with the German soldiery in occupation of the town. Gosselet succumbed in 1916, following an attempt to cope with an explosion. Barrois alone in 1918 had the sad happiness of seeing a riotous welcome given to relieving French and British troops: sad because it brought home to him with fresh poignancy the many friends whom he would see no more.

Now the grand old man has died, listening to distant guns of another war. His town, to its own honour be it said, did not forget realities in the midst of such distractions, and assembled to give a civic funeral presided over by a Cardinal Archbishop. An eloquent oration was delivered by Barrois' pupil and successor, Professor P. Pruvost; while Britain was represented by Major (Professor) W. B. R. King, official geologist on the Western Front.

Unassuming and accessible, ever the friend of the young beginner, Barrois did not escape world-wide recognition. He was honoured by the Governments of France, Belgium, Sweden, Spain, and Austria. France created him Chevalier of the Légion d'Honneur when only thirty-seven, and gave him the Cravatte de Commandeur in 1923—Teall used to tell how helpful the little red button of the Legion proved in finding accommodation even in the holiday season in crowded Brittany. The Académie des Sciences de Paris elected him to its membership in 1904, and sat under his presidency in 1927. The main scientific societies of London, Edinburgh, New York, Brussels, Madrid, and the Papacy placed him on their honorary roll, as also the geological societies of Austria, Germany, Belgium, Great Britain, and the United States. Medals are too numerous to recall, but it is significant of his early promise and mature fulfilment that he was awarded both the Bigsby (1881) and the Wollaston (1901) medals of the London Geological Society. Among honorary degrees may be mentioned the doctorates of St Andrews and Oxford.

Though better known personally in England than in Scotland, Barrois is remembered as one of the distinguished foreigners who attended the ever-memorable excursion to the North-West Highlands led by Peach and Horne after the Dundee Meeting of the British Association in 1912.

E. B. B.

Sir Thomas Hudson Beare, B.A., B.Sc., LL.D., M.Inst.C.E., M.I.Mech.E.

Sir Thomas Hudson Beare, Regius Professor of Engineering at the University of Edinburgh, died on June 10, 1940.

He was born at Adelaide, South Australia, in 1859, and was educated at Prince Alfred College and the University of Adelaide. After taking first place in the Civil Service examination of South Australia, he obtained an appointment in the Public Works Department, where he was engaged on railway work for three years. After resigning from this post, he completed the degree of B.A., was awarded the Fife (Australia) Scholarship, and proceeded to London, where he studied at University College and obtained the B.Sc. degree of London University.

In 1884, after further practical experience in this country, he was invited to join the staff at University College, London, where he assisted Professor A. B. W. Kennedy in teaching and in private professional work.

His connection with Edinburgh began in 1887, when he became the first occupant of the Chair of Mechanics and Engineering at the Heriot-Watt College where, in two years, he built up a most successful department and designed and equipped the first Engineering Laboratory in Scotland for teaching and research work.

On the resignation of Professor Kennedy, in 1889, Hudson Beare was appointed to the Chair of Mechanical Engineering at University College, London, and was largely responsible for the planning and equipment of the new Engineering Department which was built in 1895. During this period he was able to carry out a considerable amount of research work, and was awarded the Telford Premium by the Institution of Civil Engineers for a valuable paper on "The Building Stones of Great Britain."

He was a member of the Committee, of the Institution of Civil Engineers, on Standards of Thermal Efficiency for Steam Engines and of the Marine Engine Trials Committee, of the Institution of Mechanical Engineers.

In 1901, Hudson Beare returned to Edinburgh on his appointment to the Chair of Engineering at the University and at once began the reorganisation of what was then a small and poorly equipped department. In 1905, new laboratories were occupied at High School Yards. Under his influence the Edinburgh School attained a very high status, students being attracted to it from all parts of the Empire, and their number increased very rapidly.

The accommodation at High School Yards was taxed to the utmost, and, although the equipment had been recently increased and the laboratories reorganised, Sir Thomas eagerly seized the opportunity for further expansion, and in 1931 the department was moved to the Sanderson Laboratories at King's Buildings.

In 1913, Sir Thomas was elected Dean of the Faculty of Science, which post he held continuously for the period of twenty-eight years, and his expert knowledge and great administrative ability enabled him to render invaluable service to the University.

For many years he was Convener of the Works Committee of the University Court and carried the enormous responsibility of supervising the design and erection of many new departments and of the many alterations required to adapt the older buildings to modern requirements.

Appointed Convener of the Military Education Committee in 1904, he continued in that office until his death, and his keen personal interest was largely instrumental in raising the Edinburgh University Contingent of the Officers' Training Corps to its very high state of efficiency. He also represented the University as a member of the City of Edinburgh Territorial and Air Force Association, and for a period was Chairman of that body.

He was Vice-President of the Board of Governors of the Edinburgh and East of Scotland College of Agriculture from 1932, Chairman of the Board of Governors of the Royal (Dick) Veterinary College from 1923, and for many years a Governor of the Heriot-Watt College.

In 1921, he was appointed, by the Secretary of State for Scotland, an Assessor on the Central Miners' Welfare Committee. and served on the Commission until the time of his death.

Knighthood was conferred upon him in 1926, and ten years later he received the Honorary Degree of LL.D. from the University of Edinburgh.

He was a member of the Institution of Civil Engineers, a member, and later an Honorary Life Member, of the Institution of Mechanical Engineers, and a member and Vice-President of the Institution of Structural Engineers. He was a Justice of the Peace and Deputy Lieutenant of the County of the City of Edinburgh.

Hudson Beare was a man of outstanding ability and of tremendous energy, and in the midst of his many responsibilities he maintained his keen interest in his profession and in the work of his department. Keeping closely in touch with the latest developments in Engineering, his power as a teacher enabled him to do much for the advancement of Engineering

Education, and he was happy in being able to carry on this work right up to the end of his life.

His friendly interest in students and in all their activities inspired in them feelings of regard and confidence which were reflected in the close associations they maintained with him throughout their subsequent careers.

He travelled widely and also found recreation in an intensive study of the systematics of the Coleoptera, on which subject he was responsible for numerous publications and was accepted as an authoritative specialist.

Elected a Fellow in 1888, he served on the Council from 1907 to 1909 and acted as Vice-President from 1909 to 1915, and again from 1923 to 1926.

J. B. T.

Sir George Andreas Berry, LL.D., F.R.C.S.E.

SIR GEORGE BERRY died at his residence, "King's Knoll," North Berwick, on June 18, 1940, in his eighty-seventh year.

George Andreas Berry was the eldest son of Walter Berry of Glenstriven, Argyllshire. Born in 1853, he was educated at Marlborough and at Edinburgh. He studied Mathematics under Professor P. G. Tait, and became one of the six members of Tait's Senior Class, where he developed his remarkable knowledge of this subject. He graduated M.B., C.M., at Edinburgh in 1876. In 1881 he became a Fellow of the Royal College of Surgeons of Edinburgh.

Before beginning practice, George Berry prosecuted his special studies in Ophthalmology for some years in various universities and hospitals. In Copenhagen he worked with his distinguished uncle, Professor Hansen Grut; and he also studied in France, Austria, Germany, and Holland at this period of his career.

While he was Resident House Surgeon at Moorfields Eye Hospital, London, in 1878–79, Berry was one of the moving spirits in the formation of the Ophthalmological Society of the United Kingdom, which was founded in 1880. Throughout his lifetime he took the greatest interest in this Society, of which he was an Original Member.

While still a comparatively young man he was recognised as an authority in Ophthalmology, and enjoyed a high reputation on the Continent and in America, as well as at home. He was the author of a text-book entitled: *Diseases of the Eye—A Practical Treatise for Students of Ophthalmology.* This text-book was referred to by many of the leading Continental professors of Ophthalmology as being not only an exhaustive treatise, but also distinguished by many original observations and ideas. It was widely used as a standard work for many years after its publication. The terms of the dedication of this book to Edmund Hansen Grut, M.D., show the keen appreciation, the enthusiasm, and the loyalty which were essentially characteristic of Berry. All his assistants, and many others who had the privilege to know him, recognised the magnetism of his personality, the strength of his intellect, and his independent and original type of mind.

Two monographs from his pen were also much appreciated and widely used: *Subjective Symptoms in Eye Disease* and *Ophthalmoscopic Diagnosis.* That on subjective symptoms, published in 1886, was especially interesting and characteristic. His interest in, and remarkable knowledge of, the significance of subjective symptoms in the early diagnosis of disease proved of great value in his clinical work and teaching, which was not only instructive to all, but stimulating and inspiring to his senior students and assistants.

He was a well-known figure at the meetings of the Ophthalmological Society of the United Kingdom, and was President from 1909 to 1911. In 1917 the Society paid him the highest honour by electing him to be Bowman Lecturer. His important contributions in discussion and his able and numerous papers which appeared in the Society's *Transactions*, and also elsewhere, give evidence alike of his ability, and of the lively interest which he always took in the advance of Ophthalmology.

He was a member of the British Medical Association for many years. He was awarded the Middlemore Prize in 1886, and was Vice-president of the Section of Ophthalmology in 1889 and 1898, and President in 1905.

Berry served on the staff of the Eye Department of the Royal Infirmary, Edinburgh, for twenty-three years, and retired from the position of Senior Surgeon of the Department in 1905. As Lecturer on Diseases of the Eye in the University of Edinburgh he was responsible for the training of many students, not a few of whom have obtained distinction as oculists at home and in Overseas Dominions.

He became a Fellow of the Royal Society of Edinburgh in 1893, in which year he contributed "A Note on the Focus of Concavo-Convex Lenses the surfaces of which are of Equal Curvature" (*Proc. Roy. Soc. Edin.*, vol. xx, 1893, pp. 192–195). He was a Member of Council of the Society from 1916 to 1919, and Vice-President from 1919 to 1922.

Berry was an Honorary Fellow of the Royal Academy of Medicine of Ireland; President of the Royal College of Surgeons of Edinburgh from 1910 to 1912; a Member of the Edinburgh University Court from 1911 to 1923; and a Manager of the Royal Infirmary. During the last war he established and conducted the Ophthalmic Department of the Second Scottish General Hospital, Edinburgh. He was appointed Honorary Surgeon Oculist in Scotland to H.M. King Edward VII and to H.M. King George V, and in 1916 the honour of knighthood was conferred upon him. From 1922 to 1931 he was Member of Parliament (Conservative) for the Scottish Universities, and at the end of this period he received the Honorary LL.D. of Edinburgh University.

Sir George Berry was a man of remarkable ability. Physically vigorous, he enjoyed outdoor sports—especially golf and angling. He was much interested in music and played the 'cello. He took a prominent part in relation to the Foundation of the Reid Chair of Music in the

University of Edinburgh. His great interest in mathematics was abundantly shown in his text-book and in his contributions to this subject. He possessed a remarkable acquaintance with modern languages and especially with the less generally known Norse tongues, with which he had become familiar in early life. His brightness of mind, his scientific imagination, and high and varied attainments were fully recognised and highly valued by his colleagues, and he was much beloved by a wide circle of friends.

When he retired from a long life of high attainment and great activity he resided at North Berwick, where he had built a house and laid out an excellent garden, in which he took great pleasure. He married, in 1883, Agnes Jean, daughter of the late Sir William Muir, K.C.S.I., by whom he was predeceased; and is survived by three daughters.

A. H. H. S.

Waldemar Christofer Brøgger, For. Mem. R.S., Hon. F.R.S.E.

Dr W. C. Brøgger, Emeritus Professor of Mineralogy and Geology at the University of Oslo, died on February 17, 1940, at the age of eighty-eight. Scottish geologists share in the bereavement of their Norwegian colleagues, for in matters geological Scotland stands very near to Norway, nearer even than it does to England. The subjects which Brøgger illuminated in the course of his long life—Early Palæozoic faunas, the mineralogy and petrology of crystalline rocks, glaciation and raised beaches—are all such as commonly attract Scottish investigators. Brøgger, it is true, was particularly fortunate in the wealth of material that lay within easy reach; but this material was even more fortunate in the amazing skill and industry of the master who handled it.

Brøgger was won for geology by Theodor Kjerulf, his predecessor in the Chair at Oslo, at that time known as Christiania. It is fitting therefore that one of his most dramatic triumphs should be connected with a trilobite that carries the specific name *kjerulfi*. As early as 1875 Brøgger pointed out the resemblance of *Paradoxides kjerulfi* Linnarson to an American form *Paradoxides thompsoni* (Hall), both of them later transferred to Olenellid genera away from *Paradoxides*. In 1886 these fossils furnished Brøgger with the basis for a paper "Om alderen af Olenelluszonen i Nordamerika," in which American geologists were asked to revise their stratigraphy and to place their Olenellus fauna below, not above, their Paradoxides fauna, so as to agree with the findings of Scandinavia. The story is well told by C. D. Walcott in *The Fauna of the Lower Cambrian or Olenellus Zone* (1890, p. 545).

On his way to this intercontinental correlation Brøgger, in 1882, produced a notable memoir entitled *Die silurischen Etagen 2 und 3 im Kristianiagebiet und auf Eker*. This dealt in admirable fashion with the stratigraphy and palæontology of the local Upper Cambrian and Lower Ordovician; and it also included a preliminary survey of the considerably younger igneous rocks of the district, which presently were to take first place in the author's affections. These igneous rocks, long thought to be Devonian, have recently been proved to be Permian in age by O. Holtedahl through a discovery of associated fossils.

In 1890 Brøgger succeeded Kjerulf at Oslo, and the same year published his first great work on the Permian igneous assemblage, entitled *Die Mineralien der Syenitpegmatitgänge der südnorwegnischen Augit- und Nephelinsyenite*. This has proved a veritable mine of mineralogical and petrological interest; but far from being an end in itself it was followed between 1894 and 1933 by *Die Eruptivgesteine des Kristianiagebietes* in seven volumes. The whole of this series is not directly concerned with the local Permian, for Vol. II introduces a comparative account of the Triassic igneous rocks of Predazzo in Italy, while Vol. IV is given up to an exceedingly interesting set of Precambrian igneous rocks rich in calcite occurring in the Fenn district not far from Oslo. The special alkali character of the Oslo Permian eruptives necessitated much definition and naming of new rock-types, supported by abundant chemical analyses. Brøgger excelled in such systematic work, but he never let himself become its slave. He readily recognised a community in the diversity of the Oslo assemblage, and applied to it the conception of differentiation from a parental stock. He saw too that the chemistry of the intrusions was in complete contrast to that of the country rock, and he strongly opposed any idea of assimilation. He postulated, perhaps too freely, a laccolithic mechanism of emplacement, drawing analogies from the laccoliths described by Gilbert in the Henry Mountains. Brilliant in so many directions, Brøgger failed somewhat in tectonic perception. Thus he overlooked the presence of quite a number of cauldron subsidences and related structures. Their discovery was reserved for his pupil and successor J. Schetelig, about the same time as Scottish geologists were interpreting like phenomena in Glencoe. Schetelig had the great satisfaction of converting his master to the new interpretation, but their joint descriptions never saw the light of day. In spite of hopes expressed, when in 1911 he received the Wollaston Medal from the Geological Society of London, Brøgger never quite completed his "main lifework: 'The History of the Eruptive Province of the Kristiania Region,' the finishing of which, by the force of circumstances, has been interrupted and postponed by official duties for several years."

The official duties to which Brøgger referred were indeed onerous, but they were not alone responsible for his leaving to others some igneous work still remaining to be done in the Oslo district. His scientific production in other directions was enough in itself to occupy any ordinary man. Here we can only refer to *Om de senglaciale og postglaciale nivåforandringer* (1900–1901, with English summary) and *Strandliniens beliggenhed under Stenalderen* (1905). The latter provides the first full treatment of the time relation connecting successive Stone Age cultures with migrations of strand line in the Oslo Fjord.

He was elected an Honorary Fellow in 1905.

E. B. B.

Edmund W. Wace Carlier, B. ès Sc., M.Sc., M.D., F.R.E.S.

PROFESSOR E. W. WACE CARLIER, who died on September 2, 1940, at his home, Morningside, Dorridge, Warwickshire, was the son of Antoine Guillaume Carlier, Officier d'Académie. Born at Norwich in 1861, he was educated at King Edward VI School, Norwich, the Lycée de Valenciennes, France, and the University of Edinburgh, where he graduated M.B., C.M., with honours in 1886. Five years later he obtained the M.D. degree with the award of a gold medal, and became an assistant to Professor Rutherford in the Department of Physiology. In 1895 he became Senior Lecturer.

On the death of Professor Rutherford, in 1899, Dr Carlier was appointed to conduct the work of the Physiological Chair in Edinburgh, and in the same year he was elected Professor of Physiology in Mason College in Birmingham, which in 1900 became the University of Birmingham. He held this post until his retirement twenty-eight years later, when he became Emeritus Professor.

From 1909 to 1914 he was examiner in Physiology for the first fellowship examination of the Royal College of Surgeons, England. He was Vice-President of the Physiological Section of the British Medical Association at Ipswich in 1900, and at Birmingham in 1911, and a former Honorary Secretary and Vice-President of the Scottish Microscopical Society.

Professor Carlier was a tall, imposing figure, with bearded face and piercing gaze. A dramatic and lucid lecturer, he held the attention of his students, whose welfare he had greatly at heart. His main interest lay in Histology, and his collection of slides was unique as regards both quality and quantity. His researches were chiefly directed towards the correlation of cellular changes with function, and included investigations on the functions of the nucleolus in cellular fatigue, gastric secreting cell changes during digestion, and the histology of the so-called hibernating gland of the hedgehog.

He had many outside interests—natural history, gardening, and music—and was always active. His retirement was largely spent in carrying on his researches in entomology and he could often be seen in the University Library in Birmingham, or paying a visit to his old friends, with whom he never lost contact.

In 1898 Professor Carlier married Hannah, daughter of Mr G. Culver of Hughenden, Buckinghamshire. She died in 1929. Professor Carlier leaves a daughter and two sons.

He was elected a Fellow in 1899.

I. DE B. D.

Alexander Charles Cumming, O.B.E., D.Sc., F.I.C.

ALEXANDER CHARLES CUMMING was born in Melbourne, Victoria, in 1880. From Caulfield Grammar School he proceeded to the University of Melbourne, where he gained several scholarships and graduated as B.Sc., with First Class Honours in Chemistry, in 1902. During the two years following he held the Kernot and Dixon Research Scholarships and engaged in chemical research under Professor (later Sir) David Orme Masson and acted as demonstrator. In 1904 he was awarded an 1851 Exhibition Research Scholarship and spent the first year with Professor (later Sir) James Walker in Dundee and the second with Professor Abegg in Breslau. At this period (1905) he received the degree of D.Sc. of Melbourne University. The scholarship having been extended for a third year, he worked with Sir William Ramsay at University College, London, until appointed Lecturer at Birkbeck College in 1907. When James Walker came to Edinburgh as Professor of Chemistry, in 1908, he appointed Cumming to a Lecturership in the subject, a post that he held for nine years. During this period he was in charge of the Medical Practical Class and of the Senior Laboratories for Final Science students. In addition to the duties of the Lecturership, he engaged in research work upon various subjects of chemical, physico-chemical, and technical interest, and contributed papers to the Royal Societies of London and of Edinburgh, the Chemical Society and the Faraday Society, which were printed in their respective journals (see notice in *Journ. Chem. Soc.*).

He also entered upon extensive literary work, which took up much of his attention both during and subsequent to his connection with Edinburgh University. His *Practical Chemistry for Medical Students* appeared in 1911 and passed through three editions. In collaboration with the late Dr Sydney A. Kay he produced a *Text-book of Quantitative Chemical Analysis*, which was highly successful from the first, and in the revised seventh edition (1939) remains a standard work on the subject. A valuable series of Chemical Monographs written by specialists and edited by Cumming appeared during the period 1913–22. The editing of a new issue of the six large volumes of Lunge's classical *Sulphuric Acid and Alkali*—the most extensive of Cumming's literary efforts—was carried out between 1923 and 1925. He also revised in 1929 the third edition of *The Technical Chemist's Handbook* by the same author.

The outbreak of the Great War in 1914 formed a turning-point in Cumming's life. Filled with a desire to devote his energies to the assistance of the country, he offered his services to the Government as a chemist, with a view to the establishment of an explosives factory in the neighbourhood of Edinburgh. After negotiations and approval, in 1915 he proceeded, in conjunction with Professor James Walker and Mr John W. Romanes, to the fitting up in Broughton Road, Edinburgh, of the Lothian Chemical Company's Factory, of which he became manager, for the manufacture of trinitrotoluene (T.N.T.). In this enterprise, the initiative and also the working details of the manufacturing process were his own. The story of the factory is told in an interesting pamphlet—*Starting a War-time Factory*—which he circulated to friends after the war. The factory was a distinct success, and for about two years it turned out "T.N.T." in large quantity and of the highest quality. Later the Government asked the company to build a larger factory in a safer area, and H.M. Factory, Craigleith, was built, and the production was continued there for some time. In recognition of his services, Cumming received the appointment of O.B.E. in 1918.

At the end of the war, Cumming returned to Edinburgh University for a short period as Lecturer on Technical Chemistry. In 1921 he went to Liverpool as consultant to Macfie & Sons, Ltd., Sugar Refiners, of which company he became Managing Director three years later. He reorganised the factory, and remained there during fourteen strenuous years and until its closing in 1938.

In recent years Cumming lived at Caldy, Cheshire, or at "Craigness," Glenisla, Perthshire. A world tour to Singapore, the Dutch Indies, Australia, New Zealand, and Canada was undertaken in 1938–39. He died suddenly on September 28, 1940, while staying in Wales.

Gifted with much force of character, Cumming was outstanding as a capable and confident teacher, a wise counsellor, a hard worker, and, to those who knew him well, a firm, constant, and sympathetic friend. His immediate circle have to mourn the passing of a devoted spouse and parent.

He was elected a Fellow in 1914.

L. D.
J. E. M.

Carstairs Cumming Douglas, M.D., D.Sc.

THE sudden death on September 28, 1940, of Dr Carstairs Cumming Douglas must inevitably be felt as a withdrawal of weight from extramural teaching in Glasgow and as a loss to the medical profession. One of a West of Scotland family, he was born in Fife and educated at George Watson's College. His Alma Mater was the University of Edinburgh, where he was a distinguished student and won many scholarships and prizes. Graduating as Bachelor of Medicine and Master of Surgery, with First Class Honours, in 1890, he became a Doctor of Medicine, with Honours, of that University six years later. In 1906 he added to his academic distinctions by graduating as a Doctor of Science, and in 1901 became a Fellow of the Royal Society of Edinburgh. Dr Carstairs Douglas also spent a period of study in Berlin. His long and valuable association with the Anderson College of Medicine in Glasgow commenced in 1899, when he received the appointment of Professor of Medical Jurisprudence and Public Health, which office he held with distinction until his death. During the long period of forty years he held numerous additional appointments, including the Deanship of Anderson College, the Chairmanship of the Council of the Medical and Dental Defence Union of Scotland, the Principal Lecturership in Hygiene, Glasgow Provincial Committee for the Training of Teachers, and the Examinerships for the Diploma in Public Health of the three Royal Scottish Medical Corporations and in Medical Jurisprudence for the Scottish Conjoint Board. During his long period as a teacher he gave of his best to the instruction of many generations of students, making his teaching a joy and inspiration, and he won the respect and esteem of all sections of a wide community. He proved himself a valued colleague and his unfailing courtesy and fair-mindedness endeared him to all. As an author, he wrote *Chemical and Microscopical Aids to Diagnosis*, *The Laws of Health*, *A Manual of School Hygiene*, in addition to numerous scientific papers. He is survived by his wife, also a son and daughter.

J. G.

Thomas William Drinkwater, Ph.D., L.R.C.P.E., L.R.C.S.E., F.I.C.

DR T. W. DRINKWATER died on January 25, 1940, in his eighty-seventh year, almost realising his desire to finish his course "in harness." Born in Ireland in 1852, he was educated at King's Lynn Grammar School, studied under Francis Sutton, Public Analyst, Norwich, with Professor A. J. Bernays at St Thomas's Hospital, London, and for a time in Germany. He came to Edinburgh as assistant to Dr (later Sir) Henry Littlejohn, Professor of Medical Jurisprudence, and for a time held the appointment of Chief Chemist with Messrs Hope & Company, Leith Chemical Works. He qualified L.R.C.P.E., L.R.C.S.E., in 1877, and from the year following until nearly the end of 1939 was Lecturer on Chemistry in the School of Medicine of the Royal Colleges of Physicians and Surgeons of Edinburgh. In 1928 he was the guest of honour at a dinner in Surgeons' Hall, given by the Fellows of the two Royal Colleges in celebration of his Jubilee as a Lecturer in the Edinburgh Medical School. He practised as an Analyst and Consultant in the City, and held appointments as Public Analyst for the counties of Ross and Cromarty, the Royal Burgh of Inverness, and the Burgh of Fortrose. He also acted as examiner in Chemistry for the Royal College of Physicians, the Royal College of Surgeons, and other bodies.

The foregoing presents a selection of facts, but does little to depict the man—familiarly and affectionately known as "Drinky" to students and colleagues alike. He was short and sturdy, with a smooth clean-shaven face which remained almost boyish despite advancing years; indeed a stranger recently observing his active movements and alert air in classroom or laboratory would have been astonished to learn that he had long passed "three score and ten."

The Doctor did not fail to keep in touch with modern developments, a fact well illustrated by the eagerness with which, a year or two ago, he sought the writer's advice regarding the installation of an Ultra-Violet Ray Lamp which he employed to demonstrate and apply some of the newest methods of fluorescence analysis.

Dr Drinkwater's interests outside his professional activities were many and varied, and included sport, photography, literature, and music. His after-dinner musical contributions were familiar, and many of us can recall him in this connection as author, vocalist, composer, and accompanist.

His fellow-lecturers miss their friend and colleague, who, though he had held office before most of them were born, leaves a memory of youthfulness both in outlook and appearance. In him were blended the intangible qualities which go to the making of a remarkable personality.

He was elected a Fellow in 1901. C. N. K.

Percy Hall Grimshaw, I.S.O., F.R.E.S.

Percy Hall Grimshaw was born in Leeds in 1869, and after a conventional education there entered the service of one of the city's banks. His interests, however, were in natural history, so that when opportunity offered in 1895 he followed a fellow-townsman, Dr Eagle Clarke, to the Edinburgh Museum of Science and Art. There, under its newer name of The Royal Scottish Museum, he occupied increasingly important posts in the Natural History Department, until when he retired in 1935 he had been its Keeper for five years.

Although Grimshaw's earlier interests had been botanical, the needs of the Museum demanded that he should turn to the lower animals, and particularly Insects. He took up the study with zest, and one result became evident in the galleries, where for the first time typical collections of Insects, British and foreign, neatly arranged and properly labelled, made their appearance. He entered the Museum service under a regime which regarded exhibits as devised for the serious student and looked upon the inquisitive public as a body to which no special encouragement should be held out. Latin names demarked the specimens; English names, even of British birds, were frowned upon as "unscientific." He lived through a period when the general cultural value of museum collections has been increasingly recognised, and as a consequence he saw many changes in the aim and methods of museum display. It is evidence of his movement with the times that one of the developments of his keepership was the creation of a "Children's Gallery" in which natural history truths are suggested by simple series of exhibits.

As well as revising and regenerating both the exhibited and cabinet collections of Insects, Grimshaw made many contributions to our knowledge of the class. He contributed to the *Transactions* of this Society in 1898 an account of certain type-specimens of Lepidoptera and Coleoptera in the famous collection of Du Fresne of Paris, which had been purchased by the University of Edinburgh in 1819 for £3000 and had with much heartburning been handed over to the Government at the foundation of its Museum some forty years later. But his investigations mostly related to the Diptera and their distribution in Scotland, and in pursuit of that study he made many collecting trips to parts of the mainland and to the outer islands. The result has been a great addition to the knowledge of the

Dipterous fauna of this country, mainly contained in the pages of *The Annals of Scottish Natural History* and its successor *The Scottish Naturalist*, with the editing of which he was for long closely associated.

He shared in the production of Bartholomew's great *Atlas of Zoogeography*, and the records he collected for that outstanding work were, like all his accomplishment, characterised by close attention to detail and minute accuracy. His services were publicly recognised when, in 1933, the Imperial Service Order was bestowed upon him.

Still full of vigour, he died suddenly in Edinburgh on November 14, 1939, at the age of sixty-nine years.

He was elected a Fellow in 1910.

J. R.

Robert Kerr Hannay, M.A., LL.D., His Majesty's Historiographer for Scotland and Sir William Fraser Professor of Ancient History and Palæography in the University of Edinburgh.

ROBERT KERR HANNAY (born 1867), the third and youngest of a distinguished group with Andrew Lang and Hay Fleming of Scottish historians for whom St Andrews was a perpetual inspiration, partook of the gaiety and ingenuity of the one and the severe exactitude of the other. He had a special claim to be celebrated in the *Proceedings of the Royal Society of Edinburgh.* For in his historical labours he required from himself and from others the disinterested spirit of scientific inquiry and verification with as little bias as is humanly possible and with as little anticipation of the result as is consistent with purposive research. The field of that inquiry and research was the abundant store of Scottish archives, particularly from the late fifteenth and the sixteenth centuries, with which first his collaboration with Sir John Herkless in *The Archbishops of St Andrews*, then his curatorship of the Historical Department in the Register House, and lastly and chiefly his professorship in Edinburgh University and his appointment as Historiographer Royal had made him more familiar than any other historian of Scotland. He generously paid tribute to his forerunners in disentangling the complex web of the Scottish past, not least to his predecessor in his professorship, Hume Brown, for the "remarkable achievement" of his *History of Scotland.* But Hannay clearly saw that what was needed, if the record, especially for the earlier periods, was to be established with something like the completeness and substantial reliability of English or French history, was an exploration of the uncharted, and a re-charting of the fairly familiar, manuscript sources, by which alone picturesque fiction could be discredited, doubts dispelled, gaps filled, and facts proved. He not only pleaded for such a scrutiny of the *terra incognita* in his inaugural lecture on "Scottish History and the National Records," but himself did yeoman service; and many were the accepted traditions and oft-repeated statements from which he cut away the ground by proving the documentary truths. In a long series of papers in the *Scottish Historical* and the *Juridical Reviews* and other periodicals, in his introductions to *The Register of the Privy Council of Scotland*, *The Acts of Council, 1501–1554*, and the *Rentale Dunkeldense* and the *Rentale Sancti Andree*, and in his *Scottish Crown and the Papacy, 1424–1560*, and

The College of Justice, he illuminated for every future researcher and for the ideal historian of Scotland who is yet in the womb of time our national institutions, civil, ecclesiastical, educational, legislative, executive, and juridical.

Not only did he labour with unflagging energy in these regions, but he fired with his own enthusiasm a numerous group of students who have branched out in many directions from him and have carried out some of his fruitful suggestions for investigation. As a teacher of history he realised the importance, particularly for his own speciality, of taking even beginners back to the raw material—the untouched documents. Into these dry bones he had the gift of breathing life because he was never a mere antiquary or pedant in the things of the past. Rather the records, faded and dusty, were by him opened out as revelations of the business and activities of men and women who were as alive in their day as we in ours, and whose individual and communal life, though necessarily viewed by him mainly for its political importance, was as real and as vital as our own.

But fascinating as Hannay was as a teacher and oral expositor of Scottish history, he was always the scrupulous scholar, never the populariser or propagandist. He belonged to the school of Stubbs and Tout. He was, as someone has said, "an historian's historian," one to whose foundational work succeeding teachers and students will turn for guidance, knowledge, and proof.

That work will remain for the Scottish historians of the future. But the genial, tolerant, and warm-hearted man and friend has gone. His was a mercurial and sociable nature, gay, easy of approach, generous in praise, ever ready to help and advise, modest, humorous, and humane. He loved his fellow-men, none the less for their absurdities; and that love of multi-faceted human nature was manifested by an Herodotean liveliness of anecdote of which the supply was as inexhaustible as the choice was apt.

He was elected a Fellow in 1922, and died on March 19, 1940.

A. M. C.

Forrest Hay Lightbody.

Mr Lightbody entered George Watson's College in 1870, the first year that the College was open as a day school. In 1884–85, he attended the Edinburgh University Class of Engineering under Professor Fleeming Jenkin, gained first prize and was medallist for the year. Throughout his life his interest in the Watsonian Club and in Rugby and cricket never flagged. It was his proud boast that he had not missed a single Rugby international in Edinburgh since 1875.

He was the prime mover in the formation of the Society of Ordained Surveyors, which was inaugurated in 1899. The Society prepared and issued Modes of Measurement for measuring building works and set up an examination board for the qualification of its students in the art of measuring artificers' work, which was the first great step taken to place the profession on a sound basis. Mr Lightbody realised that the Society of Ordained Surveyors, being an Edinburgh body, was not on a big enough scale, so he worked hard for the formation of a Scottish body of surveyors. In 1913 he was the leader in the formation of the Faculty of Surveyors of Scotland, which was then granted a Royal Charter, and Mr Lightbody was elected its first President.

On the passing of the Finance (1909–10) Act, Mr Lightbody published a book on *The Valuation of Undeveloped Land*. His services were widely used as an arbiter and valuer.

Mr Lightbody was a well-known figure in connection with building contracts in Scotland, and a man of eminence in his profession. He worked hard indeed to raise the status of the quantity surveyor, and his passing was a great loss to the profession in Scotland.

He was elected a Fellow in 1905, and died on July 8, 1940.

W. M. B.

Sir George Macdonald, K.C.B., LL.D., Hon. F.R.S.E.

Sir George Macdonald was born in Elgin in 1862. He received his schooling at Ayr Academy, where his father, from whom he inherited his lifelong interest in archæology, was Rector. Then he went on to the University of Edinburgh, and the completion of his undergraduate studies there was marked by a First Class in Classics and the Ferguson Classical Scholarship, open to graduates of the four Scottish Universities. Oxford was obviously the next stage, Balliol was no less obviously his college, and Firsts in Classical Moderations and in *Literae Humaniores* followed in due course. He also studied abroad, especially in Germany, where he laid the foundations of an excellent knowledge of German, which he wrote with remarkable facility and accuracy.

In 1887 he became a master at Kelvinside Academy. Five years there, and twelve years as Greek Lecturer in the University of Glasgow till 1904, when Sir Henry Craik and Sir John Struthers were glad to secure his assistance in the Secretariat of the Scottish Education Department. First as Assistant Secretary, then as Second Secretary, and finally as Secretary of the Department, he played a most important part in the great changes in Scottish education that took place in the first thirty years of the century. Though his deepest interests lay in Classics, Classical Archæology and Numismatics, he was entirely free from pedantry in his attitude to the school curriculum and fully realised the fact that a large proportion of our secondary pupils find more profit in subjects that are not linguistic or severely academic.

While he was connected with the Department he was a most active and valuable member of two important Government Committees—on Modern Languages, and the Superannuation of School Teachers. Sir George could touch nothing without going into it with heart and soul, with unerring judgment, and with an industry and painstaking research that amazed his friends and were the despair of those who differed from him. The distinction with which he discharged his duties in the Department was recognised by his election as an Honorary Member of the Edinburgh Merchant Company, and as an Honorary Fellow of the Educational Institute of Scotland, and by the presentation to him of his portrait, painted by Greiffenhagen.

Sir George enjoyed twelve years of retirement, and died in Edinburgh on August 9, 1940. Though he had his full share of illness he possessed remarkable powers of recuperation and was busily occupied with his scholarly pursuits to the very end. For scholarship was his ruling passion, and one is tempted to wish that he had spent his life in an Oxford College or a Scottish University. But that view would be mistaken. It was a real gain to Scotland that his signal gifts were devoted for so many years to Scottish education; for our Scottish schools and central institutions owe a very great deal to him, both educationally and materially.

Sir George, however, was far from thinking that his day was done when his table was clear at the office. He followed the Periclean doctrine that the best relaxation is a change of activity, and to archæology, history, and numismatics he made a contribution that was a life's work in itself. When he was at Glasgow he produced his first great work, the *Catalogue of the Hunterian Collection of Greek Coins in the University*, which was crowned by the French Académie des Inscriptions et Belles Lettres. It revealed his mature classical scholarship, his relentless industry, his remarkable power of handling and marshalling evidence, and his unerring judgment. A list of his publications, compiled by his German friends for a *Festschrift* in honour of his seventieth birthday, contained many scores of papers and articles contributed to learned journals in Britain and abroad, and he continued his scholarly activities to the end of his life. He was one of the greatest European authorities on Roman Antiquities, and his *Roman Wall in Scotland* is, and is likely to remain, a classic.

Honours were showered on him. The King created him C.B. in 1916 and K.C.B. in 1927. He was Hon. LL.D. of Glasgow and Edinburgh, Hon. D.Litt. of Oxford, Hon. Litt.D. of Cambridge, Hon. Fellow of Balliol, and a Fellow of the British Academy. In addition, he was Hon. R.S.A., Hon. Curator of the Hunterian Coin Cabinet, Medallist of the Royal Numismatic and the American Numismatic Societies, and member of the German Archæological Institute.

His combination of learning and administrative ability was of great service to various societies of classical scholars and numismatists, and at one time or another he was President of the Classical Associations of England and Wales, and of Scotland, the Royal Numismatic Society, the Society for the Promotion of Roman Studies, and the Anthropology Section of the British Association.

He was also a member of the Royal Commission on Museums and Galleries and, subsequently, of the Standing Commission on Museums and Galleries. These Commissions dealt with England and Wales as well as with Scotland, and Sir George was one of their most zealous members, as he was of every committee, commission, or society with which he was associated. In his native country he was a pillar of strength to the Royal Commission on Historical Monuments, the Ancient Monuments Board, the Society of Antiquities, and, on the academic side, the Carnegie Universities Trust. For many years, too, he was a most active member of the Treasury Grants Committee. In these multifarious activities it will be difficult, or rather impossible, to fill his place.

A word about the man. To those who did not know him he may have seemed aloof or even formidable, to his friends he was the soul of kindliness and loyalty. He possessed encyclopædic knowledge, he knew his English literature well, including the Bible, and he was a keen student of Dante and Browning. He wrote admirably clear and forceful prose, he had a keen sense of humour, sometimes sardonic, and he was no mean versifier. George Macdonald was without doubt one of the most distinguished intellects of his generation, and he would have attained eminence in any profession he had entered. It was the good fortune of scholarship that it was his main interest throughout.

He was elected an Honorary Fellow in 1933.

W. W. M'K.

The Very Rev. John Harry Miller, C.B.E., T.D., D.D.

In a Society largely representative of the sciences Dr Miller might well be regarded as an expert in the comparatively recent science of Sociology. To that profoundly important and increasingly influential study the Providence that shapes our ends led him by way of a happy ministry in the holiday resort of Elie, where he evinced a brotherly interest in the young folks who thronged its golf course in summer, a varied experience which afterwards bore rich fruit in religious work amongst schoolboys and University students in colleges, camps, and conferences, in England as well as in Scotland. Transferred to a city ministry at Roseburn Church, Edinburgh, he was thence called to an office of which he was the first occupant, the Wardenship of the New College Settlement in the Pleasance, with which ten years later he combined the arduous ministry of the Pleasance Church. At last he had found his true *métier*. Under his guidance the mission became a clinic in the technique of social work for the students of New College. The premises at that time were very inadequate for the work urgently demanding to be done, of which he took the broadest view. Soon he had devised elaborate plans of development such as might have staggered any nature less buoyant and resolute than his. Neighbouring properties were gradually acquired, including a derelict brewery, whose possibilities had been perceived by earlier leaders in the Mission, who lacked, however, the wherewithal to realise them. That was left to the Warden. With his magnetic nature, his gay and genial temperament, his diplomatic and persuasive tongue, his infectious and irrepressible enthusiasm, he gathered round him a devoted band of sympathetic and generous friends, and with their hearty support he gained the satisfaction of achieving in large measure the fulfilment of his haunting dreams. A Trust was formed which raised the funds, amounting to several thousand pounds, necessary for transforming the acquired properties into buildings containing rooms and halls, big and small, suitable for all kinds of clubs and meetings, culminating in the Little Theatre with its up-to-date accessories, a spacious gymnasium with an elaborate equipment, and a Health Centre and Orthopædic Clinic with furnishings for sun-ray treatment.

This splendid hive of varied social activities, consecrated to the spiritual uplift of the demoralised locality, won the compliment of a royal visit, when

King George and Queen Mary expressed an admiration not unmixed with astonishment at the scientific completeness and fitness of its arrangements. The Warden had indeed reason to be proud of his handiwork, for perhaps no one but he, with his peculiar endowments of mind and character, his wide and influential connections, his tireless energies, could have executed an undertaking on such an ambitious scale. In the practical work carried on by the students under his superintendence he was always the leader, who not merely directed but showed the way, constantly engaging in making contacts with the people in their miserable dwellings, trying in his own brotherly and humoursome manner to inspire with new desires and aspirations minds and hearts dulled to very deadness by the gloom and hardship of their starved and dismal lot. Opportunity to pass on the accumulated and pondered results of that intimate experience, and of his eager and wide study of appertaining literature, offered itself when in 1922 he was appointed the first Lecturer on Sociology to the students of New College.

After twenty-seven years of ungrudging labour in this exacting sphere, which at last undermined his health and strength, he passed to the quietness of St Andrews, where in 1935 his career was crowned by his appointment as Principal of St Mary's College. But surely he will most of all be remembered as an outstanding pioneer in social work, who left in the Pleasance Settlement and Mission a monument to his constructive imagination, his fervent Christian idealism, his largeness of heart, which ever sought to make life worth living for the denizens of habitations so destructive of the best in human nature.

He was a late-comer into the list of the Society's membership, having been elected a Fellow only in 1936. But his name, especially in view of his fine achievements as a practical exponent of Social Science, may well adorn its roll. He died on January 3, 1940.

A. M. H.

Robert Cockburn Mossman

Robert Cockburn Mossman was born in Edinburgh in 1870, and was educated at the Royal High School of the city. He was elected a Fellow of the Society in 1891 and died at Buenos Aires on July 19, 1940.

From an early age his main interest in life was the study of meteorology, more especially in its climatological aspects. He began by compiling a weather record for Edinburgh, based on his own observations during the years 1886–1900. This brought him into touch with Dr Alexander Buchan, the well-known meteorologist, who gave him every encouragement in his study of the science. It also led to his frequent employment as assistant at Ben Nevis Observatory, where he made a special study of weather phenomena in Glen Nevis. In 1902, he was appointed meteorological observer on the *Scotia*, the ship in which Dr W. S. Bruce led the Scottish Antarctic Expedition, 1902–5. In 1906 and 1907 he visited Spitzbergen and the Greenland Sea.

While on the *Scotia* expedition, Mossman had visited Laurie Island in the South Orkneys, and had established a temporary weather station on the island. This was afterwards taken over by the Argentine Government, and the relations so established led to his appointment in the Argentine meteorological service. In this he remained, with the exception of a break from 1913 to 1920, until 1938, being chiefly concerned with the climatological records compiled by the department.

Mossman's papers on Climatology are very numerous, reaching close on a hundred. They began with studies of the climates of Edinburgh and London, but his special opportunities led to the compilation of many papers on the climate of the Argentine and of the Antarctic. His papers are almost wholly confined to statistical results, and are well known for the care with which these have been compiled and presented.

In recognition of his work in the Antarctic, Mossman was awarded the Gold Medal of the Royal Scottish Geographical Society in 1905, and the Keith Prize of the Royal Society of Edinburgh in 1918.

A. C. M.

Benjamin Dawson Porritt, M.Sc., F.I.C., F.Inst.P., F.I.R.

BENJAMIN DAWSON PORRITT died at Croydon on January 28, 1940, at the age of fifty-six years. The only son of the late Herbert Thomas Porritt of Armley, Yorkshire, he was born in Canada and educated at Whitgift Grammar School, Croydon.

He studied at University College, London, from 1903 to 1906, graduating B.Sc. in 1906, and subsequently proceeded to M.Sc. After a year on research with Sir William Ramsay and Professor Norman Collie, he was appointed chemist to the North British Rubber Company, Edinburgh, with whom he remained until 1920, being appointed chief chemist in 1912 and research superintendent in 1916. During this period he carried out valuable work for the fighting services in the War of 1914–18.

When, in 1920, the British rubber manufacturers decided to set up a co-operative industrial research association under the scheme introduced by the Department of Scientific and Industrial Research, Porritt became Director of the Research Association of the British Rubber and Tyre Manufacturers, a position which he occupied until his death. The building up of the new research organisation and its guidance during a period beset with difficulties was a task for which Porritt was well fitted, and he devoted himself to it with unsparing energy and perseverance.

He was the author of many scientific and technical papers relating to rubber and of a monograph on *The Chemistry of Rubber*.

Porritt was elected a Fellow of this Society in 1919. He was a Fellow of the Institute of Chemistry, a Fellow and Vice-President of the Institution of the Rubber Industry, and a Fellow of the Institute of Physics. He served on the councils of the Institute of Chemistry, the Institution of the Rubber Industry, and the Chemical Society. In 1938 he was awarded the Colwyn Medal.

A keen member of the Rugby Fifteen at University College, he retained his interest in Rugby all his life, but his main recreation was to get out with one or two friends in a small yacht and potter round the coast.

A man of genial disposition and invariable good nature, Porritt had a multitude of friends. He leaves a widow and two daughters.

A. C. C.

William Alexander Robertson, *P.*F.A., F.C.I.I.

THE sudden death of William Alexander Robertson, President of the Faculty of Actuaries, on April 4, 1940, whilst on a business visit to London, came as a shock not only to the insurance world but also to a much wider circle.

Educated at George Watson's College, which he left in 1894 to enter the insurance profession, he commenced his career at the Head Office of the Scottish Metropolitan Insurance Company. A few years later he joined the Scottish Union and National Insurance Company. He entered the service of the Century Insurance Company Limited in 1908 as Actuary. Later he became Assistant Manager and Actuary and afterwards Manager and Actuary. This position he held at the date of his death.

He was a dynamic personality. He endeared himself to all with whom he came in contact by his enthusiasm about everything he took in hand. Much of his time he devoted to assisting in their studies the young men in the insurance business. For many years he was an outstanding tutor of Actuarial Science and for a time acted as Lecturer on this subject at the University of Edinburgh. He contributed many papers to insurance societies all over the country. *Actuarial Theory*, of which he was joint author, has been of inestimable value to actuarial students throughout the world. He was a leading authority on sickness and disability insurance and his contributions on these subjects have enhanced insurance literature. He was a former President of the Insurance Society of Edinburgh and a former President of the Watsonian Club. He did much for the senior boys of George Watson's College.

He was an enthusiast for Rugby football. He strongly advocated its value in developing character, sportsmanship and what he called the "team spirit." During thirty years of active refereeing he handled many important games, including the International matches between England and France at Twickenham in 1919 and between Ireland and England at Dublin in 1920. Not infrequently he refused an important senior match so that he might referee school games and afterwards lecture to the boys on the intricacies and spirit of the game. He gave valuable service in organising the annual seven-a-side tournaments in aid of the Edinburgh Royal Infirmary.

William Alexander Robertson by his enthusiasm, energy and thoroughness set a very high example, and he had that rare quality of bringing out the best in all with whom he came in contact.

He was elected a Fellow in 1919.

A. G. R. B.

Ralph Allen Sampson, M.A., D.Sc., LL.D., F.R.S.

RALPH ALLEN SAMPSON was born at Skull, County Cork, on June 25, 1866. When he was five the family moved to Liverpool, and it was at the Liverpool Institute that he received his early education. He eventually entered St John's College, Cambridge, as a sizar, and graduated as third Wrangler in the Mathematical Tripos of 1888. In the following year he gained the first Smith's Prize and was elected a Fellow of St John's. At this time he held a Lectureship in Mathematics at King's College, London, and was engaged in hydrodynamical research.

In 1890 he returned to Cambridge as the first holder of the newly founded Isaac Newton Studentship in Astronomy and Physical Optics. He worked in astronomical spectroscopy with H. F. Newall, and in 1893 he published a paper of 62 pages entitled "On the Rotation and Mechanical State of the Sun." This paper is historically interesting as it advanced for the first time the hypothesis of radiative equilibrium, a hypothesis which has since dominated the theory of the interior of a star. On several occasions during Sampson's career he put forward suggestions which subsequently were followed up by others and proved to be of first-class importance. He possessed the adventuring spirit of the pioneer, combined with a tremendous capacity for work. It naturally happened that on occasions he embarked on a line of research which eventually proved fruitless, and his remarkable patience and persistence would lead him to continue work in a particular direction long after a point had been reached at which other investigators would have given up the struggle. It was this same patience and capacity for work which saw him through the research on the theory of Jupiter's inner satellites, for which he was awarded the Gold Medal of the Royal Astronomical Society in 1928.

In 1893 Sampson was elected to the Chair of Mathematics in the Durham College of Science at Newcastle-on-Tyne, and in 1895 he removed to Durham itself as Professor of Mathematics. It was during his Durham days that he carried out the heavy but congenial task of editing the unpublished manuscripts of John Couch Adams, the co-discoverer of Neptune, who had been Sampson's teacher. And it was at Durham that the work on the theory of the Four Great Satellites of Jupiter was commenced. He formed the project of basing the elements of the orbits entirely on the Harvard photometric observations of the eclipses of the satellites, and from a careful discussion of these observations he deduced fresh elements by comparison with existing theory. He then found that there were substantial discrepancies when the individual observations were compared with the theory using the new elements, and Sampson undertook the heavy task of working out the dynamical theory afresh. This work was completed by the time he left Durham in 1910 and tables based on the new theory were published. It was not, however, until 1920 that the new theory itself was printed, and in the intervening period the whole work was revised and a few omissions were found and tabulated. But in spite of the refinements introduced, substantial discrepancies still remained when comparison was made with observation, and it has been since shown by De Sitter that, contrary to expectation, the Harvard photometric observations of the eclipses were affected by rather serious systematic errors, and that consequently it is unadvisable to derive elements from eclipse observations alone. But Sampson's research remains as a necessary stage in our efforts to obtain a really satisfactory theory of Jupiter's Galilean Satellites. These satellites form a very stable system which is of special dynamical interest on account of the commensurabilities of the periods of revolution of the three inner ones.

In December 1910 Sampson was appointed Astronomer Royal for Scotland and Professor of Astronomy in the University of Edinburgh in succession to F. W. Dyson. During his tenure of office he took a great interest in problems connected with the accurate measurement of time, and it was largely due to his co-operation in testing the first free pendulum clock devised by W. H. Shortt that this type of clock, in its developed form, is now part of the standard equipment of so many observatories. In 1921 Sampson published an important paper in the *Monthly Notices of the Royal Astronomical Society* on clock errors and wireless time-signals, in which he called attention to the serious discrepancies between time observations at different observatories. There can be no doubt that this paper greatly stimulated the interest of astronomers in the problem of time determination, which is of fundamental importance in certain aspects of astronomical and geophysical research. In consequence of the attention which has, as a result of Sampson's stimulation, been lavished on the problem, and of the appearance of the free pendulum type of clock, the discordances have now been greatly reduced.

A very striking example of Sampson's pioneering instinct is to be found in the work carried out at Edinburgh on the distribution of energy in the continuous spectra of stars. Sampson was interested in photometric research, and during the war years of 1914–1918 he instructed E. A. Baker to embark on photo-electric experiments. At this time the idea was to fit a cell on a telescope and obtain direct readings of stellar magnitudes (as has indeed been done by other workers), but the aim of the research changed and the result was the construction of a photo-electric photometer designed to measure the densities of star images on photographic plates. Then came the idea of applying the instrument to the measurement of images of stellar spectra. More research by Baker followed on the intensity-density relationship of photographic emulsions, and finally a spectrophotometric programme was commenced which yielded what have since become known as colour temperatures of stars, although Sampson used the term "effective temperature," which is now reserved to denote a different concept. The stress was now laid on the methods of photographic spectrophotometry as worked out by Sampson and Baker, and the photo-electric photometer took its place as an accessory instrument for measuring the densities of the spectra. The methods employed have since been copied and extended by other workers, and modifications in Sampson's original pioneer scheme were only to be expected. But any subsequent modifications and improvements do not alter the fact that other workers in this field were inspired by Sampson's researches, and they know that their work is the logical continuation of the lead given at Edinburgh.

In the Edinburgh colour temperature work the spectra were photographed with a 6-inch photo-visual to which an objective prism was attached. The exposures were necessarily long and the programme was limited to stars brighter than the third magnitude. It had been obvious for some time that the observatory required extensive re-equipment, and in 1927 Sampson was able to announce that plans had been sanctioned for the erection of a 36-inch reflecting telescope and the acquisition of a new camera employing a 10-inch Cooke Triplet lens. Provision was also made for a modern spectrograph to be attached to the 36-inch reflector and for other accessories. The new equipment was finally completed and installed in 1932.

Sampson's active career was now approaching its close. He continued in the direction of the observatory until 1937, when ill-health hastened his retirement. For some months he travelled abroad, but eventually settled in the south of England, and on November 7, 1939, he died at Bath, being survived by his widow, a son and four daughters.

The Gold Medal of the Royal Astronomical Society was awarded to Sampson in 1928 for his work on Jupiter's Satellites. He was awarded the Keith Prize of the Royal Society of Edinburgh for the period 1919–1921. He served on the Council of the Royal Society of Edinburgh during the periods 1912–1915 and 1919–1921, was Vice-President 1915–1918 and 1933–1936, Secretary to Ordinary Meetings 1922–1923, and General Secretary of the Society 1923–1933.

See also *Obituary Notices of Fellows of the Royal Society*, vol. iii, No. 8, 1940, pp. 221–226

W. M. H. G.

John James McIntosh Shaw, M.C., M.A., M.D., F.R.C.S.E.

"AND he said to me as we walked across the fields, 'To-night we shall be at war, to-morrow we begin to pay the price which war demands, and it will be in the dark currency of death.'" This quotation came to my mind when I heard that J. J. M. Shaw was dead. The Edinburgh Medical School has been called upon at an early stage to pay an item in the grim tally, and it has been in the loss of one who could ill be spared, a man who was accomplishing a great service in enhancing the reputation of his school and in fashioning the direction of its future. Shaw laid down his task while in his fifty-fourth year, an age when, as some would say, life is at its best—an age which looks back on years full of happy memories, of friends, of difficulties overcome, of things accomplished, of work well done; an age which looks forward to the days which lie ahead with hope that the future holds further achievements ere the sun sets. In our friend's case the curtain has fallen in life's midday; the future and its promised bright visions have been denied to him, but we who knew him can look back upon the part he played, and, as we record it, we pay tribute to its sincerity and its worth. Were I asked to choose the motto which best describes the impulse of Shaw's being my choice would fall upon the single word "*Service*," and surely there could be no finer testimony to his memory. From his student days, when first I knew him, until the hour when his spirit passed into the Egyptian night, his life was one of service to others, of devotion to such things as are worthy and of good report. There are few aspects of our social, academic, and professional activities which have not benefited from his interest, his wisdom, and his genius. As an undergraduate he played a leading part in the life and development of the University Union; he occupied the honourable position of President, an indication of his popularity and of the esteem with which his fellow-students regarded him. Athletics claimed his interest, and again the recognition of his merits placed him in the Presidentship of the University Athletic Club. He entered the "Battery," and within the space of a few years the Bombardier had become a Lieutenant. So far as I am aware the trilogy is unique, and it surely has its significance, for it bespeaks the faculty of leadership, the ability to rise to the top wherever his lot might be cast. It was no jealous sense of power which placed him in these high places, it was his own urge to serve, and the recognition by his fellows that here was one upon whose word and judgment they might rely.

In the War of 1914–18 there arrived an opportunity which brought out some of his finest qualities. His devotion to duty was unswerving, his reputation as an indefatigable worker placed him in centres where pressure was most heavy, and amidst these supreme demands he pursued his task with a courage and a cheerfulness which made light of danger and difficulty. It was a fitting recognition of his merits when he was decorated with the Military Cross and the Croix de Guerre.

When peace came it was characteristic that Shaw should turn his thoughts and energies to reconstruction—the act of service in its best and highest form. The spheres in which he laboured are too numerous to mention individually, but one of them stands out supreme—the grim yet pathetic field of plastic surgery. He worked here with diligence and success, and his reward was to bring happiness, confidence, and self-respect to men whose forms and features had been disfigured by the ravages of war.

In later years he became one of our most successful medical administrators. He was responsible for the creation of the Cancer Control Organisation in south-eastern Scotland, he occupied positions of high responsibility in the National Radium Trust, the Radium Commission, and the Scottish Cancer Advisory Committee. He took the keenest interest in all that concerned the treatment of malignant disease, and had he lived he would have seen in all probability the fulfilment of his highest ambition, the establishment of a Cancer Hospital and Research Institute in this city.

Amidst his many professional activities he was not forgetful of his former interests in University social life. The Union, the Athletic Club, the Settlement, and the Graduates' Association—with each of these he kept in touch, indeed he did more, he gave unstintingly of his time and energy in order that they might flourish.

Claims such as these might seem sufficient, and yet they were not all. Shaw was a surgeon of the first rank, a teacher of recognised distinction, a tireless investigator in the search after knowledge.

Is it surprising that we mourn his passing? We shall not easily look upon his like again, and the most fitting monument to his memory is what men are saying about him—"He spent his life in service to others."

He was elected a Fellow in 1931, and died "On Active Service" in Egypt in September 1940.

J. F

Sir Joseph John Thomson, O.M., D.Sc., LL.D., F.R.S.

JOSEPH JOHN THOMSON * was the son of a Manchester bookseller, and was born in that city on December 18, 1856. At the age of fourteen he entered Owens College (now the University of Manchester), and four years later won an entrance scholarship in mathematics at Trinity College, Cambridge, where he read for the Mathematical Tripos. In 1880 he graduated as Second Wrangler, and at once set to work on the preparation of a thesis for the Fellowship competition of the College.

His starting-point was the known theorem of dynamics that if two bodies, A and B, influence each other's motion, say by attraction, and if we observe the motion of B which results when A moves in a definite way, then by dynamical theory we can deduce the motion of A when that of B is specified, even though we do not know the precise nature of the connection between them. Believing that all physical phenomena could be explained by dynamical principles, Thomson tried to extend this theorem to the more general case, which occurs in Physics and Chemistry, when instead of two bodies attracting each other we have two phenomena, A and B, which mutually influence each other. We may be ignorant of the details of the molecular mechanism connecting the phenomena, but, as Thomson showed, we may by general dynamical theory be able to deduce the effect of A on B when we have observed the effect of B on A. Thus, if we observe that the coefficient of magnetisation of a piece of soft iron decreases as the temperature increases, then we can infer that a magnet will become heated when it is moved from weak to strong parts of the magnetic field. The dissertation containing these results won a Fellowship at the first attempt, and the dominant idea in it has had a great influence on the subsequent development of physical chemistry.

It was on the border-line between chemistry and physics that Thomson found the subject which he next attacked, and which by a series of brilliant discoveries he ultimately solved, namely, the conduction of electricity through gases. His starting-point here was Lord Kelvin's representation of the atoms of matter by vortex-rings in a perfect fluid. In 1883 Thomson constructed a model of the electric discharge through gases by representing the atoms of a gas as vortex-rings: the combination of two atoms to form

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* This obituary notice is extracted from the President's Address delivered at the Annual Statutory Meeting of the Society on October 28, 1940.

a molecule would then correspond to the union or pairing of two vortex-rings, a phenomenon which was well known experimentally and which he had investigated mathematically. Now suppose that there is a quantity of gas in an electric field. The electric field he represented by a distribution of velocity in the medium whose vortex-motion constitutes the atoms of the gas: the disturbance due to this distribution of velocity will cause some of the paired vortex-rings to dissociate, and so a mathematical theory of the phenomenon of conduction in gases becomes possible. He worked out the conditions for the disruptive discharge, and showed that the intensity of the field necessary to produce discharge should diminish when the gas is rarefied.

To the question of conduction in rarefied gases he now devoted his attention, being moved by the consideration that the properties and laws of gases are simpler than those of solids and liquids, and that the kinetic theory enables us to form clearer mental pictures of what goes on in a highly rarefied gas. It had been shown in 1869 by Hittorf that the phosphorescent light in the neighbourhood of the cathode in a discharge-tube consists of a disturbance of some kind which is propagated in straight lines from the cathode: for when a solid body is placed in the tube it casts a shadow. This disturbance thus came to be known as the *cathode rays*: and a lively discussion went on for over twenty years as to whether these rays were, like rays of light, a disturbance in the æther, or whether they consisted of a torrent of small charged particles torn off the cathode. Most of the German physicists, especially Hertz and Lenard, held to the æther hypothesis, while the British, particularly Varley, Crookes, Fitz-Gerald, and Thomson, advocated the projectile theory. Hertz supported his thesis by showing experimentally that the rays did not appear to produce any external electric or magnetic field, or to be deflected by an electrostatic field, and, above all, that they were capable of passing through films of metal which were so thick as to be quite opaque to ordinary light: it seemed inconceivable that particles of matter should not be stopped by even the thinnest gold-leaf. (At that time the alpha rays of radium, which are undoubtedly streams of particles and have a remarkable penetrating power, were not as yet discovered.) On the other hand, the upholders of the particle theory showed that the cathode rays are deflected by a magnetic field: and in 1894 Thomson, by means of a rotating mirror, succeeded in measuring their velocity, obtaining a value ($1{\cdot}9 \times 10^7$ cm. per sec.) so much smaller than the velocity of light that it seemed scarcely possible to conceive of the rays as vibrations of the æther. A further blow was dealt at the æther hypothesis in the following year, when Perrin, having received the rays in a metallic cylinder placed in front of the cathode, found that the cylinder became charged with negative electricity. Thomson then extended this experiment by showing that when the rays were deviated by a magnet they still communicated a charge to any vessel placed to receive them. This appeared to demonstrate conclusively that they transported negative electricity, and therefore were, presumably, particles of matter.

It seemed probable that if the charged-particle theory were true, the charges of all the particles would be equal, and would in fact each be equal to an elementary quantum or atom of electricity, which would not be capable of further subdivision. Thus the question as to whether electricity was continuous, or existed only in discrete and identical units, was involved in the discussion. The theory of electric displacement, which had been put forward by Maxwell, favoured the concept of electricity as a continuum: and Maxwell's theory had by this time won general acceptance, chiefly as a result of Hertz's experimental demonstration of electric waves. On the other hand, the facts of electrolysis seemed to show that an absolute unit of electric charge exists, namely, that amount of it which attends each chemical bond or valency: and in fact the magnitude of the charge carried by the hydrogen atom in electrolysis was well known, the charges carried by other atoms being either equal to this or (positive or negative) multiples of it.

In 1896 and 1897 Thomson, by a brilliant series of experiments, succeeded in removing all the objections to the charged-particle theory of the cathode rays, and finally determined their actual nature. He first dealt with Hertz's failure to deflect the rays by an electrostatic field; by carrying the exhaustion to a much higher degree, he removed the effects of the ionisation of the residual gas in the tube, which, as he showed, had been the cause of Hertz's negative result, and thus he obtained the expected deflection. He now subjected the rays to an electric and a magnetic field simultaneously, and adjusted the fields until the deflections in opposite directions balanced each other: this experiment, as he showed, provided data for determining the velocity of the particles. Then, by measurements of the deflection by a magnetic field alone, he obtained the value of $\frac{mv}{e}$, where m denotes the mass of one of the particles, v its velocity, and e its electric charge. Since v was already known, he was able to calculate (m/e), the ratio of the mass of a cathode-ray particle to the charge which it carries. This ratio was found to be independent of the nature of the rarefied gas or of the cathode: and, to Thomson's great surprise, it proved to be less than the thousandth part of the value of (m/e) for the hydrogen ion in electrolysis. If the charge were the same as that of the hydrogen ion—and by use of a brilliant discovery of C. T. R. Wilson he proved this to be the case—then there was no escape from the conclusion that the particle whose mass was thus measured was very much smaller than the smallest atom.

The first public announcement of this great discovery was made by Thomson in an Evening Discourse at the Royal Institution on April 30, 1897. "The assumption of a state of matter more finely subdivided than the atom of an element," he said, "is a somewhat startling one: but a hypothesis that would involve somewhat similar consequences—viz. that the so-called elements are compounds of some primordial element—has been put forward from time to time by various chemists."

In order to explain this last remark, it must be recalled that from the very beginnings of physics, amongst the Milesian philosophers in the sixth century B.C., the doctrine had been maintained that all things are but different forms of one single primary and universal substance. Thales conjectured that it might be water, Anaximenes that it might be air, and Anaximander that it was a boundless indefinite something—τὸ ἄπειρον—out of which everything is generated and to which everything returns. Modern science has entertained the same idea, in many different forms: as we have seen, Lord Kelvin suggested that all atoms might be vortex-rings in an infinite liquid, which would be precisely Anaximander's τὸ ἄπειρον: but when J. J. Thomson discovered that the cathode-ray particles were the same, whatever atoms they were torn out of, his mind recurred chiefly to the hypothesis put forward in 1815 by Prout, namely, that the atomic weights of the elements are whole-number multiples of the atomic weight of hydrogen, and that, in fact, hydrogen is the universal primordial substance. Thomson now conjectured that Prout had been wrong in selecting hydrogen, and that the fundamental place in nature belonged rightly to the newly discovered cathode-ray particles, or "corpuscles" as he called them, which were very much smaller than the hydrogen atom and were now proved to be contained in the atoms of all elements. This idea seems to have fascinated him at the time, as well it might.

Later researches—most of them carried out or inspired by Thomson himself—have shown the necessity of modifying this opinion as to the place of the cathode-ray corpuscles in the scheme of the universe. Since they carry charges of negative electricity, while the atom as a whole is neutral in respect of charge, it follows that some part of the atom must be positively charged. To investigate this, Thomson examined another set of rays obtained in the vacuum discharge. More than a decade previously, Goldstein had shown that when the cathode of a discharge-tube is perforated, radiation of a certain type passes through the perforations into the part of the tube behind the cathode, travelling in the direction opposite to the cathode rays: to this radiation he had given the name *canal rays*. Shortly after Thomson's work on the cathode rays, it was shown that the canal rays are torrents of positively charged particles, the value of (m/e) being immensely larger than that obtained for the cathode rays, and indeed of the same order of magnitude as the corresponding ratio in electrolysis. In 1907 Thomson made an extensive experimental study of these canal rays, or *positive rays* as he now called them, determining the velocity and the ratio (m/e) by shooting the rays through a narrow tube, so as to obtain a small spot on a phosphorescent screen or a photographic plate, and subjecting them between the tube and the screen to an electric field and also a magnetic field, so as to deflect the beam of particles. He showed that all particles having the same value for (m/e) would be spread out by the two fields so as to strike the screen in points lying on a parabola; thus, particles of different mass would give different parabolas. Parabolas were found corresponding to the atoms and molecules of various gases in the discharge-tube: and the atomic weights of the particles could be at once inferred from measures of the parabolas. By applying this method of positive-ray analysis to the gas neon (since used in the familiar advertisement lights) he found, in addition to a parabola belonging to atomic weight 20, another corresponding to atomic weight 22. These proved to be, both of them, atoms of neon, but of different masses: Thomson had in fact discovered the first example of two atoms having the same chemical behaviour but different physical characteristics: as we now say, two *isotopes* of the same chemical element.

The innumerable investigations that have been carried out in atomic physics since the beginning of the present century have not so far led up to any universal primordial substance such as Thomson thought he had found in 1897. We have progressed somewhat in its direction, for we now know that the atoms of the 92 chemical elements are all compounded of a much smaller number of more fundamental particles. Of these, the first is Thomson's cathode-ray corpuscle, now always known as the electron: it is not, however, the only elementary particle: there are others—the proton, the neutron, the positron, the light-quant, the meson, and perhaps the neutrino. Some of these can certainly be generated from the others: and it seems probable that every one of them can be created or annihilated, so that the doctrine of permanent indestructible ultimate particles, as taught by Leukippus and Democritus, is definitely false: but more than this we cannot, in the present state of knowledge, affirm.

The discovery of the electron established Thomson's reputation as the first of living experimental physicists: and it was in this aspect that he was generally regarded during the latter half of his life. But his contributions to theoretical or mathematical physics were scarcely less important than the fruits of his researches in the laboratory. Perhaps his greatest achievement in pure theory was the discovery (in 1893) of electromagnetic momentum; namely, that in an electromagnetic field there is stored in every unit volume an amount of mechanical momentum, proportional to the vector-product of the electric and magnetic vectors. This principle is necessary for the construction of the energy-tensor; and without it, neither relativity-theory nor quantum-electrodynamics could have developed.

During the whole of his tenure of the Cavendish chair Thomson was the active head and inspirer of a great research school. The institution of the status of "advanced student" at Cambridge in 1895 led to a great increase in the numbers of young graduates of other universities who came to work in his laboratory, with the consequence that to-day nearly all the important chairs of Physics in the Empire are filled by his disciples.

In 1918 he was elected Master of the great foundation of which he had been a member uninterruptedly since 1875.

"How fortunate I have been throughout my life!" he wrote, near the end of it. "I have had good parents, good teachers, good colleagues, good pupils, good friends, great opportunities, good luck, and good health." He lived to be eighty-three, dying at Trinity Lodge on August 30, 1940, and was buried on September 4 in Westminster Abbey.

He was elected an Honorary Fellow in 1905.

E. T. W.

Sir Donald Francis Tovey, Kt., B.A. (Oxon.), M.Mus. (Hon.), Birmingham.

PROFESSOR TOVEY came to Edinburgh in the autumn of 1914, and he died in Edinburgh on July 10, 1940, at the regrettably early age of sixty-four, after completing twenty-five years as Reid Professor of Music. His tenure of office thus opened and closed in the dark days of war, when the arts must inevitably suffer severely. This fact tends to throw into sharper relief the brilliance of his professorial career. He came with a formidable reputation as a scholar; Joachim had, indeed, said of him, as a young man, that he was the most learned musician in Europe. His amazing scholarship was, moreover, combined in a unique way with great executive powers as a pianist, and with a profoundly poetic imagination; and this phenomenal combination of gifts was allied to the simplicity and true humility of a great mind. The fact that his genius chose to express itself in music was merely a fortunate accident for music. He might just as easily have been a famous classical scholar, a great scientist, or a distinguished philosopher. Music was to him an aspect of ultimate truth and the great powers of his intellect were focussed on the apprehension of this truth.

Two years after his arrival in Edinburgh Sir Donald commenced the most important undertaking of his professorial career—namely, the founding of a local symphony orchestra which should form part of the teaching equipment of the Faculty of Music. His achievement in establishing the Reid Symphony Orchestra is known to many who have never heard this orchestra play, but who have read the programme notes which he wrote for its concerts. These notes have now been collected and published by the Oxford University Press in five volumes of essays, which are universally acknowledged to be the most brilliant, witty, and learned musical analyses ever written. This important tangible record of his work is, however, eclipsed by the fact that the foundation and training of an orchestra, on the lines of his far-sighted policy, has enormously increased the value of the work done in the Faculty of Music, and has perceptibly raised the standard of music throughout Scotland. It has indeed established a great tradition, and has—musically speaking—"put Edinburgh on the map."

It is a tragedy that no permanent record remains of his great pianoforte playing. Valuable and increasingly influential as his writings are, a dozen gramophone records—made when his playing had reached the peak of its eloquence—would have been more valuable still. The loss of this part of his teaching is immeasurably great, and can be assessed only by those who treasure memories of his triple series of concerts—Reid, Historical, and Sunday Evening. His powers of interpretation were without equal, both as a pianist and as a conductor. Through him, music spoke directly to the hearer.

As a composer Professor Tovey was not so widely known as he should have been. His thought expressed itself in classical, rather than in modern, style (though in a distinctive idiom), and his more important works are long—a fact which is against them in an age of speed. Despite this, such fine works as the 'cello concerto and the Gluck variations are already assured of a place in musical repertoire; and *The Bride of Dionysus* will one day be hailed as a masterpiece of English opera.

His greatest contribution to musical æsthetics is an important one: the theory of tonality, which he expounded in the article on "Harmony" in the *Encyclopædia Britannica*, is the only comprehensive and scientific systemisation of the practice of the classical masters. It is based on living music and not on theoretical treatises, and is as yet both inadequately recognised and inadequately understood. A collection of the more important of the forty-odd articles which he wrote for the *Encyclopædia* is envisaged at some future date, and should provide material for at least half a dozen text-books.

Professor Tovey's lectures were as vivid and as varied as his writings, and were illuminated by his lively wit and by voluminous and apt musical illustrations. Even the dullest student could not fail to catch some of the fire of his enthusiasm, and his teaching remains a constant source of inspiration. He never suffered fools gladly, and was intolerant of humbug, "perkiness," or of makeshift in art. He was, however, patient and generous to a degree in his help, encouragement, and appreciation to every *genuine* musician or music-lover, however humble their attainments might be. Professor Tovey was, to sum up, a leader who inspired devotion and affection alike in students, orchestra, fellow-artists, and a host of friends throughout the world: these remember him with gratitude and with gladness.

He was elected a Fellow in 1917.

M. G.

Richard Turner, O.B.E., M.B., C.M.

Richard Turner, son of John Turner of Ratho, Midlothian, was born in 1856. He was educated at Edinburgh University, where he graduated M.B., C.M., in 1880. His active life as a medical practitioner was spent in York, and in 1921 he retired and settled in Peebles. He died in Edinburgh on April 9, 1940.

Dr Turner served for a time as President of the Yorkshire Branch of the British Medical Association, acted as Honorary Consulting Medical Officer of York County Hospital, and during the War of 1914–18 rendered valuable services in military hospitals in York, for which he was awarded the O.B.E. His retiral to Peebles allowed him leisure and scope to follow up his early love for natural history, geology, and archæology. He took a deep interest in the Chambers Institution, rearranging and adding to the exhibited material, and in 1927 published his *Descriptive Catalogue of the Geological Collections.* This work, which occupied him for several years, is much more than its name implies, for Dr Turner aimed at an individual account of the minerals, rocks, and fossils exhibited, and succeeded in presenting the essential information in a simple and attractive manner. He was drawn also to the study of the remains of early settlements in the Peebles district and carried out, for example, excavations at some of the old hillside terraces at Romanno, Dunsyre, etc. For a number of years he kept daily meteorological records.

In a quiet and unobtrusive way Dr Turner did a great deal to foster in the citizens of Peebles an interest in the natural history and geology of their countryside, and in the classified and augmented collections in the geological museum of the Chambers Institution he has left them a legacy of great and abiding value. He was created a burgess of Peebles in 1923, in recognition of his services to the community.

He was elected a Fellow in 1924.

M. M.

Vito Volterra, For. Mem. R.S., Hon. F.R.S.E.

ONE of the outstanding mathematicians of our time, Professor Vito Volterra, died at Rome on October 11, 1940.

Born at Ancona in 1860, Volterra began his university career at Florence in 1878, but soon transferred to the University of Pisa, where he graduated in 1882. At the early age of twenty-three he was elected to the Chair of Mechanics at Pisa, which he held for nine years. In 1892 he became Professor of Mechanics in the University of Turin, and finally, in 1900, he succeeded Beltrami as Professor of Mathematical Physics at Rome. During the War of 1914–18 Volterra volunteered his services and became an officer in the corps of Engineers, conducting experimental and theoretical investigations in aeronautics. From 1921 he was President of the International Committee of Weights and Measures.

The scientific work of Volterra covered many branches of analysis and mathematical physics. His publications, some two hundred in number, include substantial contributions to the theories of partial differential equations, the equilibrium of elastic bodies, optics, and mathematical biology. But the field which is associated primarily with his name is that of the functional calculus. This new development of analysis may be said to have been founded by him in a series of papers of which the first appeared in 1887; it has proved one of the most fruitful adventures in modern mathematics, and Volterra's work on it alone would entitle him to rank in the line of great mathematicians. Starting from functions, defined by partial differential equations, which depend not on a finite number of initial constants but on the infinite set of values given by functions defined initially, he was led to the new conception of *functionals*—or *functions of lines*—quantities whose value depends on the infinite set of values assumed by a given function. For these new functions, Volterra built up a new calculus—the functional calculus.

The methods of Volterra were rapidly diffused, and opened up new fields of research. The calculus of variations appeared now as a branch of the functional calculus, and was developed along new lines by Tonelli from this point of view. Integral equations arise in the theory of functionals, and Volterra's solution of the integral equations which now bear his name was the first general treatment of a class of integral equations. It thus paved the way for the whole theory of integral and integro-differential equations, to which Volterra made many further fundamental contributions. The theory of permutable functions, the work of Fantappiè on analytic functionals, and the work of Moore and Fréchet on general analysis, all derive more or less directly from Volterra's ideas.

His first papers on mathematical biology appeared in 1926. They dealt with highly simplified mathematical models of systems of competing species. In numerous subsequent papers, and in his book *Leçons sur la Théorie Mathématique de la Lutte pour la Vie* (1931), he generalised the simple treatment, introducing many complications which would make the mathematical model a closer representation of the struggle for life in nature. The biological deductions have been strikingly verified in various ways; for example, in statistical observations of fish populations and the changes caused by cessation of fishing; and in *ad hoc* laboratory experiments, such as those of G. F. Gause on cultures of Protozoa. The subject of animal ecology has flowered intensively during the past two decades; and Volterra's mathematical contributions, along with the parallel pioneer work of W. R. Thompson and A. J. Lotka, have been an essential influence in its development.

Volterra lectured in foreign universities on many occasions. We may mention particularly a course on partial differential equations at Stockholm, the Vanuxem Lectures in 1912 at Princeton on permutable functions, and the lectures on functionals at Madrid. These latter formed the basis of his standard book on functionals, the English edition of which was published in 1930.

He was a man of vigorous personality and decided views. He refused in 1931 to take the oath of allegiance to the Fascist régime, required of all Italian professors. He was deposed from his Chair at Rome in consequence, and his compulsory resignation from all Italian scientific societies followed. Many of the most celebrated universities and learned societies of the world honoured Volterra. He visited Scotland on more than one occasion, and was given the Honorary Degree of LL.D. by the University of Edinburgh during the British Association Meeting of 1921. He was elected an Honorary Fellow of the Royal Society of Edinburgh in 1913.

R. P. G.
I. M. H. E.

William Moses Feldman, M.D., B.S., F.R.C.P.(Lond.), F.R.A.S., Senior Physician, St Mary's Hospital for Women and Children, Plaistow, had been Examiner to the General Nursing Council, and Chadwick Lecturer on Ante-Natal and Post-Natal Child Physiology and Hygiene (1924). He was the author of *Biomathematics* (2nd ed. 1935), *Principles of Ante-Natal and Post-Natal Child Hygiene* (1927), and other publications. He was elected a Fellow in 1923, and died in London on July 2, 1939.

Gilbert Macintyre Hunter, M.Inst.C.E., M.Inst.E.S., formerly of Iquique, Chile, where he was resident engineer of the Nitrate Railways, died in Edinburgh on December 7, 1939. He was elected a Fellow in 1911.

Joshua Law Kerr, M.D., J.P., was a graduate of the University of Aberdeen, where, before proceeding to Australia, he was Examiner in Chemistry and Medical Jurisprudence, and was an official of the General Council of the University. He was elected a Fellow in 1891, and died at Stratford, Victoria, Australia, on February 7, 1940.

Nathan Raw, C.M.G., M.D., F.R.C.S.E., was from 1918 to 1922 Coalition Unionist Member of Parliament for the Wavertree Division of Liverpool, and had been Lord Chancellor's Visitor in Lunacy. He was born on August 2, 1866, and educated at Durham University and College of Medicine, where he was awarded Prizes and Scholarships and took honours in several subjects. In 1893 he was appointed Medical Superintendent of the Dundee Royal Infirmary, and, in the course of his career, held several important asylum and hospital appointments. He was a member of the International Commission for the Prevention of Tuberculosis. Colonel Raw served in France during the War of 1914–1918 and was made C.M.G. His publications included numerous works on his special subjects. He was elected a Fellow in 1900, and died in London on August 28, 1940.

Edward Smart, B.A., B.Sc., J.P., was a native of Forfar, and a notable figure in educational work in Perth for over forty years. His teaching career in Perth commenced in 1894 as mathematical master of Sharp's Institution, which became the junior school of the Academy; and in 1899 he was appointed to the mathematical staff of the Academy. He was Rector of the Academy from 1915 to 1930. Mr Smart was elected a Fellow in 1901, and died on December 5, 1939.

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PROCEEDINGS OF THE STATUTORY GENERAL MEETING

Beginning the 157th Session, 1939–1940.

At the Statutory General Meeting, held in the Society's Rooms, 24 George Street, on Monday, October 23, 1939, at 4 P.M.

Sir D'Arcy Wentworth Thompson, President, in the Chair.

The Minutes of the Statutory Meeting held on October 24, 1938, were read, approved, and signed.

The President nominated as Scrutineers, for the election of Office-Bearers and Council, Professor F. W. SHARPLEY and Mr D. HALDANE.

The Ballot was then taken.

The General Secretary submitted the following report:—

GENERAL SECRETARY'S REPORT, OCTOBER 23, 1939.

During the session six Addresses have been given by request of the Council: on October 24, 1938, by Mr JOHN L. BAIRD on "The Development of Television"; on December 5, 1938, by Mr ALEXANDER R. GLEN on "The Oxford University Arctic Expedition to North East Land, 1935–36"; on January 9, 1939, by Mr F. I. G. RAWLINS on "Physical Methods in the Investigation of Painting"; on February 6, 1939, by Professor P. A. M. DIRAC, F.R.S., under the terms of the JAMES SCOTT PRIZE (1938), on "The Relation between Mathematics and Physics"; on May 1, 1939, by Professor P. M. S. BLACKETT, F.R.S., under the terms of the BRUCE-PRELLER LECTURE FUND (1939), on "The Mesotron: the New Unstable Cosmic Ray Particle"; and on July 3, 1939, by Dr MOWBRAY RITCHIE on "Radiation in Chemistry." Professor DIRAC's Lecture was published in *Proc. R.S.E.*, vol. 59, pp. 122–129.

33 papers were read as compared with 37 in the previous session. The papers were in the following subjects: Mathematics and Statistics, 13; Physics, 5; Chemistry, 1; Geology and Palæontology, 5; Zoology, 5; Animal Genetics, 4. 9 papers have been published in the *Transactions* and 22 in the *Proceedings*. 3 papers have been withdrawn, and several papers are still under the consideration of the Council.

The Society has lost by death 18 Fellows and 2 Honorary Fellows. 31 Fellows and 3 Foreign Honorary Fellows were elected. 1 Fellow resigned and 1 Fellow was removed from the Roll of Fellows.

Invitations were received and the Society was, or will be, represented as follows on the occasions mentioned:—

1. Réunion Internationale pour la Commemoration de la Découverte du Radium, Paris, November 26–30, 1938. Letter sent.
2. Maurice Caullery, Scientific Jubilee, Paris, February 25, 1939. Letter sent.
3. Lazzaro Spallanzani Celebration, University of Pavia, April 1939. Professor F. A. E. CREW, F.R.S.
4. Memorial Service for Sir ROBERT PHILIP, St Giles Cathedral, January 29, 1939. Lt.-Col. A. G. MCKENDRICK, Em. Professor R. STOCKMAN, and Lt.-Col. W. F. HARVEY.
5. Lord High Commissioner's Levee, Palace of Holyroodhouse, May 24, 1939. Professor F. A. E. CREW, Dr JAMES WATT, Dr A. C. AITKEN, and Dr. J. E. MACKENZIE.
6. Georgetown University, 150th Anniversary of Georgetown College, May 28 to June 5, 1939. Letter sent.
7. Centenaire de TH. RIBOT, Paris, June 1939. Letter sent.
8. National Association for the Prevention of Tuberculosis, June 29 to July 1, 1939, at Belfast. Letter sent.
9. Seventh International Congress of Genetics, Edinburgh, August 23–30, 1939. The PRESIDENT and Professor F. A. E. CREW.

10. Royal Swedish Academy of Sciences, 200th Anniversary, Stockholm, September 23–25, 1939. Address sent. Owing to the War our representative, Professor JAMES KENDALL, could not attend.
11. First Conference of the International Association for Scientific Tobacco Research, Bremen, September 25–30, 1939. Letter sent.
12. Royal Microscopical Society, London, Centenary Celebration, October 25–26, 1939. Mr WILLIAM WILLIAMSON. This Celebration has been postponed owing to the War.
13. Catholic University of America, 50th Anniversary, Washington, November 11–13, 1939. Professor HUGH S. TAYLOR, F.R.S.
14. Tenth All-India Oriental Conference, Hyderabad-Deccan, December 1939. Letter sent.
15. International Botanical Congress, Stockholm, July 17–25, 1940. Professor Sir W. WRIGHT SMITH, Professor J. WALTON, and Professor R. J. D. GRAHAM.
16. International Congress of Geology, London, July 31 to August 8, 1940. Dr G. W. TYRRELL and Dr J. E. RICHEY.

At the Ordinary Meeting on December 5, 1938, the Dr W. S. BRUCE MEMORIAL PRIZE (1938) was presented to Mr ALEXANDER R. GLEN; and the DAVID ANDERSON-BERRY PRIZE (1938) to Dr MARY A. C. COWELL.

Professor P. M. S. BLACKETT, F.R.S., was the BRUCE-PRELLER Lecturer, and Professor P. A. M. DIRAC, F.R.S., the JAMES SCOTT Lecturer during the session.

The MAKDOUGALL-BRISBANE PRIZE (1934–1938) was awarded to Professor D. M. S. WATSON, F.R.S., for his paper published in the *Transactions* of the Society, within the period, entitled "On *Rhamphodopsis:* a Ptyctodont from the Middle Old Red Sandstone of Scotland," and for his many distinguished contributions to Vertebrate Palæontology.

The JAMES GREGORY TERCENTENARY MEMORIAL VOLUME, edited by Professor H. W. TURNBULL, F.R.S., has been published for the Society by Messrs G. Bell & Sons, Ltd., London. It includes his life and his mathematical work, which contains many advances, remarkable for the time and hitherto unrecorded, in the calculus and theory of numbers; also the important and vivid correspondence (1668–1675) between Gregory and his London friend Collins.

A grant of £200 has been generously awarded by the Carnegie Trust for the Universities of Scotland to Professor TURNBULL to guarantee against loss by the Society on publication of the volume.

The Subscribers to the President's Portrait Fund have presented to the PRESIDENT a smaller portrait of himself by Mr DAVID S. EWART, A.R.S.A. This portrait has been on exhibition in the Galleries of the Royal Scottish Academy during the summer.

Owing to the War and the possibility of damage, the Hume MSS. and two of the Society's pictures—John Robison, by Raeburn, and Sir Walter Scott, by Graham Gilbert—have been placed in safe custody outside our premises. The remainder of the pictures and certain other valuables have been placed in a strong room in the basement of our own building.

In response to a request from the Royal Society of London regarding the preparation of a National Service Central Register, the Council forwarded to the Royal Society a marked list of Fellows, so that the appropriate cards could be sent to them for completion.

Two large new book-cases were erected on the top floor to relieve congestion there.

The sum of £250 was expended on Library binding during the session, and arrears are gradually being overtaken.

A plaster cast of Napier of Merchiston has been presented to the Society by Mrs ROBERTSON, Kirkton, Dumfries, and is now placed in the Lecture Hall.

The cordial thanks of the Society are due to those who have made presentations to the Library.

Mr S. HEDDLE has retired from the post of Housekeeper to the Society, which he, with the late Mrs HEDDLE, so ably filled for a period of thirty years. An allowance has been granted to him. Mr Heddle has been succeeded by Mr WILLIAM BRYCE.

The acknowledgments of the Society are due to the Carnegie Trust for the Universities of Scotland for grants to authors towards the cost of illustrations, tabular matter, etc., of papers published in the *Transactions* and *Proceedings*, amounting to £187, 2s. 7d., to Birkbeck College, University of London, for a grant of £10 towards the cost of publication of Dr VERA FRETTER'S paper in *Transactions* (1939); to the University of St Andrews for £7 towards the cost of Professor PEACOCK and Miss ANN R. SANDERSON'S paper in the *Transactions* (1939); to Mr JAMES WRIGHT for £50 towards the cost of his paper in the *Transactions* (1939); to the London School of Hygiene and Tropical Medicine for £50 towards Dr ALAN MOZLEY'S paper in the *Transactions* (1939); and for £300 from the Government Publication Grant administered by the Royal Society of London, in aid of the cost of the Society's publications for the session 1938–1939.

Treasurer's Report:—

The Treasurer, in submitting the Accounts for the past year, mentioned the leading items of Receipts and Expenditure, comparing them with those of the previous year, and stated that a bequest of £1000 made by Dr James Currie, a former Treasurer, had been received during the session, the income of which would be used in the General Fund of the Society. The Treasurer also said that £460, 15s. 6d., including the £200 set aside in 1937–38, had been paid to the account of the GREGORY TERCENTENARY MEMORIAL VOLUME. The Treasurer indicated that the financial position of the Society was satisfactory.

Colonel J. C. Lamont moved the adoption of the reports, and the reappointment of Messrs Lindsay, Jamieson & Haldane, C.A., as auditors for the ensuing session. The meeting approved.

The Scrutineers reported that the Ballot Papers were in order, and the President declared that the following Office-Bearers and Members of Council had been duly elected:—

Professor E. T. Whittaker, M.A., Hon.Sc.D., D.Sc., LL.D., F.R.S., President.	
Principal J. C. Smail, O.B.E., Companion Inst.E.E. Professor J. Walton, M.A., D.Sc. James Watt, W.S., LL.D. Leonard Dobbin, Ph.D. John Alexander Inglis, K.C., M.A., LL.B. Emeritus Professor Ralph Stockman, M.D., LL.D., F.R.C.P.E.	Vice-Presidents.
Professor James P. Kendall, M.A., D.Sc., F.R.S., General Secretary.	
Alexander C. Aitken, M.A., D.Sc., F.R.S. Professor R. J. D. Graham, M.A., D.Sc.	Secretaries to Ordinary Meetings.
E. Maclagan Wedderburn, O.B.E., LL.D., D.K.S., Treasurer.	
John E. Mackenzie, D.Sc., Curator of Library and Museum.	

MEMBERS OF COUNCIL.

Professor Lancelot T. Hogben, M.A., D.Sc., F.R.S.
Professor James Ritchie, M.A., D.Sc.
G. W. Tyrrell, A.R.C.S., D.Sc., F.G.S.
Emeritus Professor C. T. R. Wilson, C.H., M.A., D.Sc., LL.D., F.R.S.
Professor R. C. Garry, M.B., Ch.B., D.Sc.
Professor D. Murray Lyon, M.D., F.R.C.P.E., D.P.H., D.Sc.
J. E. Richey, M.C., B.A., Sc.D., F.R.S., F.G.S.
The Hon. Lord Robertson.
A. Graham Donald, M.A., F.F.A., F.S.A.Scot.
Alan W. Greenwood, D.Sc., Ph.D.
Emeritus Professor T. H. Milroy, M.D., LL.D.
W. P. D. Wightman, Ph.D., M.Sc.

The Scrutineers were thanked for their services.

The retiring President welcomed the newly elected President of the Society, Professor E. T. Whittaker, F.R.S., and Professor Whittaker paid high tribute to the valuable services rendered to the Society by Sir D'Arcy Wentworth Thompson as President and in other offices.

Professor Whittaker also, in name of the Society, cordially thanked the other retiring Office-Bearers and Members of Council.

Sir D'Arcy Wentworth Thompson introduced Professor D. M. S. WATSON to the meeting, and presented to him the MAKDOUGALL-BRISBANE PRIZE for the period 1936–1938, for his paper published in the *Transactions* of the Society, within the period, entitled "On *Rhamphodopsis*: a Ptyctodont from the Middle Old Red Sandstone of Scotland," and for his many distinguished contributions to the science of Vertebrate Palæontology.

Professor Watson was then called upon to deliver his Address on "The Origin of Frogs."

The Chairman cordially thanked Professor D. M. S. Watson for his Address.

PROCEEDINGS OF THE ORDINARY MEETINGS, Session 1939–40.

FIRST ORDINARY MEETING.

Monday, November 6, 1939.

Professor E. T. Whittaker, M.A., Hon. Sc.D., D.Sc., LL.D., F.R.S., President, in the Chair.

The Minutes of the previous Ordinary Meeting were taken as read, and signed.

The following Communications were submitted:—

1. The Oak Galls in Theophrastus's *Historia Plantarum*. By Professor G. SENN, Basle. Communicated by Sir D'ARCY WENTWORTH THOMPSON, C.B., LL.D., F.R.S. *Trans.*, vol. 60, pp. 343–354.
2. The Geology of the Highland Border from Glenalmond to Glenartney. By D. A. ALLAN, Ph.D., D.Sc. *Trans.*, vol. 60, pp. 171–193.
3. The Maximal Range of Error in Gross Reproduction Rates. By ENID CHARLES, M.A., Ph.D. Communicated by Professor LANCELOT HOGBEN, F.R.S. *Proc.*, vol. 60, pp. 18–32.

Read by Title:—

4. The Normal Œstrous Cycle of the Ferret. The Correlation of the Vaginal Smear and the Histology of the Genital Tract with Notes on the Distribution of Glycogen, the Incidence of Growth and the Reaction to Intra-vitam staining by Trypan Blue. By Professor W. J. HAMILTON, M.D., D.Sc., and J. H. GOULD, B.Sc. *Trans.*, vol. 60, pp. 87–106.
5. Cyclic Variations in the Vascular Architecture of the Uterus of the Guinea Pig. By P. BACSICH, M.D., and G. M. WYBURN, M.B., Ch.B., D.Sc., F.R.F.P.S.G. *Trans.*, vol. 60, pp. 79–86.
6. On a Problem concerning Matrices with Variable Diagonal Cells. By WALTER LEDERMANN. Communicated by Professor GODFREY H. THOMSON. *Proc.*, vol. 60, pp. 1–17.

SECOND ORDINARY MEETING.

Monday, December 4, 1939.

Professor E. T. Whittaker, M.A., Hon. Sc.D., D.Sc., LL.D., F.R.S., President, in the Chair.

The Minutes of the previous Ordinary Meeting were taken as read, and signed.

At the request of the Council Dr DORA ILSE (Institute of Animal Genetics) delivered an address on "Experiments on the Colour Sense of Insects," which was illustrated by a Coloured Cinematographic Film.

THIRD ORDINARY MEETING.

Monday, January 15, 1940.

Professor E. T. Whittaker, M.A., Hon. Sc.D., D.Sc., LL.D., F.R.S., President, in the Chair.

The Minutes of the previous Ordinary Meeting were taken as read, and signed.

At the request of the Council Professor MAX BORN, F.R.S., delivered an Address on "Relativistic Quantum Mechanics and Reciprocity," which was followed by a discussion in which the President and Dr C. A. Coulston (University College, Dundee) and others took part.

Read by Title:—

1. The Periodic Lamé Functions. By E. L. INCE, D.Sc. *Proc.*, vol. 60, pp. 47–63.
2. Harmonic Riemannian Spaces. By Professor E. T. COPSON, M.A., D.Sc., and Professor H. S. RUSE, M.A., D.Sc. *Proc.*, vol. 60, pp. 117–133.

FOURTH ORDINARY MEETING.

Monday, February 12, 1940.

Professor E. T. Whittaker, M.A., Hon. Sc.D., D.Sc., LL.D., F.R.S., President, in the Chair.

The Minutes of the previous Ordinary Meeting were taken as read, and signed.

The following Communications were submitted:—

1. The Life-history and Habits of the Salmon Gill-maggot (*Salmincola salmonea*). By G. F. FRIEND. M.A., B.Sc. Communicated by Professor JAMES RITCHIE, D.Sc.

2. (*a*) The Swimming and Burrowing Habits of *Urothoë marina* (Bate) and a Comparison with those of *Bathyporeia* and *Haustorius*. *Proc.*, vol. 60, pp. 271–280.
(*b*) The Yearly Life Cycle of the Amphipod *Corophium volutator* (Pallas).
By E. EMRYS WATKIN, B.Sc., Ph.D. Communicated by Dr. A. C. STEPHEN.

Read by Title:—

3. Observations on the Œstrous Cycle of the Guinea Pig. By P. BACSICH, M.D., and G. M. WYBURN, M.B., Ch.B. *Proc.*, vol. 60, pp. 33–39.

4. On the Independence of Linear and Quadratic Forms in Samples of Normally Distributed Variates. By A. C. AITKEN, D.Sc., F.R.S. *Proc.*, vol. 60, pp. 40–46.

5. The Estimation of Factor Loadings by the Method of Maximum Likelihood. By D. N. LAWLEY. Communicated by Professor GODFREY H. THOMSON. *Proc.*, vol. 60, pp. 64–82.

FIFTH ORDINARY MEETING.

Monday, March 4, 1940.

Professor E. T. Whittaker, M.A., Hon. Sc.D., D.Sc., LL.D., F.R.S., President, in the Chair.

The Minutes of the previous Ordinary Meeting were taken as read, and signed.

The Ballot for the election of Ordinary Fellows then took place. The President nominated Mr JOHN BARTHOLOMEW and Mr J. MUNNOCH as Scrutineers. The following gentlemen were declared duly elected:—

EDMUND BRUCE BALL, JOHN BOWMAN, BYROM STANLEY BRAMWELL, JAMES BROUGH, ALEXANDER FAIRLEY BUCHAN, JOHN MORRISON CAIE, JOHN CAMERON, JAMES WILFRED COOK, JAMES COUTTS, SATCHIDINANDA DATTA, THOMAS ELDER DICKSON, ARCHIBALD THOMAS JOHN DOLLAR, HAROLD IRVING DREVER, WILLIAM MCCONNACHIE HARROWES, THOMAS HART, CHARLES FREDERICK WILLIAM ILLINGWORTH, JAMES GRAY KYD, PATRICK RAMSAY LAIRD, ROBERT MCADAM, JAMES ALEXANDER MACDONALD, ANGUS ELRICK WILLIAM MCLACHLAN, ANGUS MACNIVEN, GUY FREDERIC MARRIAN, EDWARD ROWLAND ALWORTH MEREWETHER, ROBERT MACFARLANE NEILL, HUGH BRYAN NISBET, JAMES SCOTT CUMBERLAND REID, HARRY RILEY, JAMES THOMSON, HARRY MOSS TRAQUAIR.

The PRESIDENT called upon Professor R. J. D. GRAHAM to intimate the awards of the KEITH and NEILL PRIZES, for the period 1937 to 1939: the KEITH PRIZE to Professor F. A. E. CREW, M.D., D.Sc., Ph.D., F.R.S., for his papers and joint papers in the *Proceedings* of the Society within the period of the award, and in recognition of his valuable contributions to Animal Genetics; The NEILL PRIZE to Mr JAMES WRIGHT, F.G.S., for his important paper on "The Scottish Carboniferous Crinoidea," published in the *Transactions* of the Society within the period of the award.

At the request of the Council Dr G. W. TYRRELL and Dr A. T. J. DOLLAR (Department of Geology, University of Glasgow), delivered addresses on "Recent Scottish Earthquakes" and "The British Earthquake Inquiry" respectively.

Read by Title:—

1. Interpolated Derivatives. By B. SPAIN, Ph.D. Communicated by the PRESIDENT. *Proc.*, vol. 60, pp. 134–140.

2. Further Investigations into the Periodic Lamé Functions. By E. L. INCE, D.Sc. *Proc.*, vol. 60, pp. 83–99.

SIXTH ORDINARY MEETING.

Monday, May 6, 1940.

Professor E. T. Whittaker, M.A., Hon. Sc.D., D.Sc., LL.D., F.R.S., President, in the Chair.

The Minutes of the previous Ordinary Meeting were taken as read, and signed.

The following signed the Roll and were admitted to Fellowship:—Mr EDMUND BRUCE BALL, Mr JOHN MORRISON CAIE, Dr T. ELDER DICKSON, Mr THOMAS HART, Mr PATRICK RAMSAY

Laird, Dr Robert McAdam, Dr James Alexander Macdonald, Dr Angus MacNiven, Dr Hugh Bryan Nisbet, Mr James Scott Cumberland Reid, Mr Harry Riley, and Dr Harry Moss Traquair. Dr Alexander Fairley Buchan and Dr Oswald James Walker also signed the Roll.

Announcement was made of the proposal, approved by the Council, for alteration of Law XVI by the omission of the words underscored below:—

"Fellows who are not in arrear with their Annual Subscriptions and all Honorary Fellows shall be entitled gratis to copies of the Parts of the *Transactions* and the *Proceedings* published subsequently to their admission. Copies of the Parts of the *Proceedings* shall be distributed by post or otherwise, as soon as may be convenient after publication; copies of the *Transactions* or Parts thereof shall be obtainable upon application either personally or by an authorised agent to the Librarian, provided the application is made within five years after the date of publication."

This proposal, with an explanation of its reason, will be placed upon the billet for the June Meeting, and will then come up for discussion and vote. (*See also* General Secretary's Report, p. 433.)

The following Communications were submitted:—

1. The Soils and Vegetation of the Bin and Clashindarroch Forests. By A. Muir, Ph.D., and G. K. Fraser, M.A., D.Sc. Communicated by Dr A. Lauder. With Notes on Afforestation by Professor H. M. Steven, B.Sc., Ph.D. *Trans.*, vol. 60, pp. 233–341.

Read by Title:—

2. Tests of Carcinogenic Substances in relation to the Production of Mutations in *Drosophila melanogaster*. By Charlotte Auerbach, Ph.D. Communicated by Dr A. W. Greenwood. *Proc.*, vol. 60, pp. 164–173.

3. A Theoretical Atomic Distribution Curve for Liquid Argon at 90° K. By G. S. Rushbrooke. Communicated by Professor E. T. Copson, M.A., D.Sc. *Proc.*, vol. 60, pp. 182–191.

4. Reciprocity: Part III: Reciprocal Wave Functions. By Professor Max Born, F.R.S., and K. Fuchs, D.Sc. *Proc.*, vol. 60, pp. 141–146.

5. Reciprocity: Part IV: Spinor Wave Functions. By K. Fuchs, D.Sc. Communicated by Professor Max Born, F.R.S. *Proc.*, vol. 60, pp. 147–163.

6. The Genetical and Mechanical Properties of the Sex Chromosome: VI: *Hexacentrus mundus*. By P. C. Koller, D.Sc., Ph.D. *Proc.*, vol. 60, pp. 174–181.

SEVENTH ORDINARY MEETING.

Monday, June 3, 1940.

Professor E. T. Whittaker, M.A., Hon. Sc.D., D.Sc., LL.D., F.R.S., President, in the Chair.

The Minutes of the previous Ordinary Meeting were taken as read, and signed.

Mr James Brough signed the Roll and was admitted to Fellowship.

Reference Minutes of Ordinary Meeting of May 6, 1940, an amendment of Law XVI was proposed by the President, seconded by Lord Salvesen, and carried unanimously. The amendment was to the effect that the words "of the Parts of the *Proceedings* shall be distributed by post or otherwise, as soon as may be convenient after publication; copies of the *Transactions*, or Parts thereof," should be deleted. The purpose was that the *Proceedings* should henceforth be divided into TWO SECTIONS—"A" (Mathematical and Physical) and "B" (Biological). (*See also* General Secretary's Report, p. 433.)

The following Communications were submitted:—

1. An Analysis of the Influence of Weather upon a Migratory Movement of Birds. By Professor James Ritchie, M.A., D.Sc. *Proc.*, vol. 60, pp. 299–321.

> An extensive migratory movement of Waxwings (*Ampelis garrulus*) from Scandinavia to Great Britain and Denmark in the autumn of 1921 has supplied data which permit analysis of the weather conditions associated with such a movement. The birds apparently arrived in these countries in a series of waves, and the movement as a whole as well as each specific wave was tested with the weather conditions prevailing in S. Scandinavia at the dates of departure. It was found that the start of the movement corresponded with a sudden rise in barometric pressure; that the migration was confined to three weeks when the pressure remained exceptionally high; and that the birds moved from an area of high pressure towards low pressure during anticyclonic conditions. Conditions which seem to favour the setting out of such a migratory wave in autumn are a rising barometer, a steady, relatively high temperature, and overcast skies. Almost invariably the wind was in some degree opposed to the direction of flight, and the appearance of waxwings in Britain takes place only when the wind carries them there.

2. Early Glacial Remains of Reindeer from Glasgow. By MURRAY MACGREGOR, D.Sc., F.G.S., and Professor JAMES RITCHIE, M.A., D.Sc. *Proc.*, vol. 60, pp. 322–332.

During the excavation of a deep sewer in the Mount Florida district of Glasgow reindeer remains were found in a water-rolled gravel 70 feet below the surface. The gravel underlies the thick deposit of boulder clay which forms the Mount Florida drumlin and occupies a hollow in the sub-drift rock-surface. The suggestion is advanced that it represents the deposit of an interglacial period.

The reindeer remains, found 220 feet apart, consist of (*a*) portions of a fore-limb and (*b*) a fragment of antler, and the relative sizes suggest that they may possibly represent two individuals. Both sets of remains bear tooth marks made by carnivorous animals. Observations are made on the significance of the reindeer relics which have previously been recorded from Scottish glacial deposits.

Read by Title:—

3. The Effect of the Inhibition of Respiration and Assimilation on the Diatom *Ditylum brightwelli* (West). By D. BHATIA, Ph.D. Communicated by Professor JAMES RITCHIE, M.A., D.Sc. *Proc.*, vol. 60, pp. 245–259.

4. Random Paths in Two and Three Dimensions. By Professor W. H. MCCREA, M.A., Ph.D., and F. J. W. WHIPPLE, M.A., Sc.D., F.Inst.P. *Proc.*, vol. 60, pp. 281–298.

EIGHTH AND LAST ORDINARY MEETING.

Monday, July 1, 1940.

Professor E. T. Whittaker, M.A., Hon. Sc.D., D.Sc., LL.D., F.R.S., President, in the Chair.

The Minutes of the previous Ordinary Meeting were taken as read, and signed.

The following gentlemen signed the Roll and were admitted to Fellowship:—Dr A. T. DOLLAR and Mr T. T. PATERSON.

The KEITH and NEILL PRIZES were presented.

The following statements were read on presentation of the Prizes:—

Presentation of the KEITH PRIZE to Professor F. A. E. CREW, F.R.S., for the period 1937 to 1939.

Although the problem of inheritance is one that has intrigued scientific minds throughout the centuries it was not until 1866 that Mendel's researches resulted in a clear demonstration of the natural laws of heredity.

Since that time the new science of Genetics has made rapid strides, due largely to the brilliant work of Morgan and his school in America and to Bateson of Cambridge in this country.

No less important have been the activities of the succeeding generation of workers, among the best known of whom in this country may be numbered Professor Crew.

By his genetical writings and original researches he has done a great deal to further his science. The author of more than one hundred scientific papers, he has contributed regularly to this Society since 1923 the results of many important researches.

A man of personality and vigour he has succeeded in building up an important research school in the University of Edinburgh that has been the inspiration of men from all parts of the world.

Presentation of the NEILL PRIZE to Mr JAMES WRIGHT, F.G.S., for the period 1937 to 1939.

Mr Wright, during a long residence at Kirkcaldy, unremittingly devoted his leisure to the collection and study of Carboniferous crinoids which, as he has shown, are developed in great perfection at certain localities in Fife. Since that time he has extended his investigations to other places in Scotland and also in England and Ireland, and the value of his discoveries, with publications dating from the year 1912, was recognised by the Geological Society of London by the award of the Wollaston Fund in 1928. Recently he has contributed to our *Transactions* a memoir dealing with the Scottish Carboniferous Crinoidea, which he has illustrated with a series of beautiful photographs taken by himself. The extent of his scientific share in this subject may be judged from the fact that more than fifty new genera and species from Scotland have been recognised and named by him. He has been not only an apt observer and describer but also has developed an admirable technique for extricating from the enclosing rock the details of these wonderfully shaped fossils.

Mr Wright is recognised as a leading authority of Carboniferous crinoids not only by British but also by American and Continental workers. We understand that he has now begun the study of the much neglected Ordovician forms from Girvan and elsewhere. It is altogether fitting that the Society should recognise his past achievements by this Award, and for his future research we would like to extend our best wishes.

Award of the Dr W. S. BRUCE MEMORIAL PRIZE (1940).

The Joint Committee of representatives of the Royal Society of Edinburgh, the Royal Physical Society, and the Royal Scottish Geographical Society have awarded the above prize to Mr BRIAN ROBERTS, Emmanuel College, Cambridge, for his valuable work in survey, ornithology, and general biology in the Polar regions—north and south.

The Prize will be presented by the President of the Royal Society of Edinburgh at a meeting next session.

The following Communications were submitted:—

1. On the World Correlation of the Pleistocene. By T. T. PATERSON, B.Sc. *Trans.*, vol. 60.

Expeditions to Asia, Africa, and the Arctic have led to the recognition of three great cycles of sedimentation within the Pleistocene, with minor fluctuations due to changes in precipitation. A survey on a world-wide basis shows the correspondence of the first phases of each cycle with Geikie's first three glacial advances, and two minor phases in the third cycle with his fourth and fifth. Three other minor subsidiary phases correspond to interglacial climatic fluctuations. There are appropriate correlations with pluvial periods in the tropics and sub-tropics. At once the system clarifies the question of the time relations of Palæolithic industries and of the remains of fossil man himself. A simplified nomenclature is suggested (retaining Geikie's original classification of glacial stages) with a sub-division into Lower, Middle, and Upper Pleistocene and their sub-stages. The palæontological sequence appears to agree broadly with this system.

2. DEMONSTRATION.—The Preparation and Utility of Lamellar Cellulose Acetate Moulds from Etched Surfaces of Rocks and Minerals. By A. T. DOLLAR, B.Sc., Ph.D., F.G.S.

Read by Title:—

3. The Loss of Heat by Conduction from the Earth's Crust in Britain. By E. M. ANDERSON, M.A., D.Sc. *Proc.*, vol. 60, pp. 192–209.

4. The Design and Interpretation of Experiments based on a Four-fold Table: the Statistical Assessment of the Effect of Treatment. By W. O. KERMACK, M.A., D.Sc., LL.D., and Lt.-Col. A. G. MCKENDRICK, M.B., D.Sc., F.R.C.P.E. *Proc.*, vol. 60, pp. 362–375.

5. The Structure and Behaviour of the Chromosomes of the Sheep during Mitosis and Meiosis. By I. A. AHMED, B.Sc. Communicated by Dr P. C. KOLLER. *Proc.*, vol. 60, pp. 260–270.

6 The Energy Levels of a Rotating Vibrator. By IAN SANDEMAN, M.A., Ph.D., D.Sc. *Proc.*, vol. 60, pp. 210–223.

7. Some Confluent Hypergeometric Functions of Two Variables. By Dr A. ERDÉLYI. Communicated by the PRESIDENT. *Proc.*, vol. 60, pp. 344–361.

8. On the Histogenesis of some Teleost Pituitaries. By T. KERR, M.A. Communicated by Professor F. A. E. CREW, F.R.S. *Proc.*, vol. 60, pp. 224–244.

9. The Effect of Increased Daily Illumination and of Reversed Day and Night on the Œstrus Cycle of the Mouse (*Mus musculus*). By R. A. R. GRESSON, Ph.D. Communicated by Professor J. RITCHIE, M.A., D.Sc. *Proc.*, vol. 60, pp. 333–343.

PROCEEDINGS OF THE STATUTORY GENERAL MEETING

Ending the 157th Session, 1939–1940.

At the Statutory General Meeting, held in the Society's Rooms, 24 George Street, on Monday, October 28, 1940, at 4 P.M.

Professor E. T. Whittaker, M.A., Hon. Sc.D., D.Sc., LL.D., F.R.S., President, in the Chair.

The Minutes of the Statutory Meeting held on October 23, 1939, were approved, and signed.

The President nominated as Scrutineers, for the election of Office-Bearers and Council, Mr DAVID GIBB and Mr W. A. SINCLAIR.

The Ballot was then taken.

The General Secretary submitted the following report:—

GENERAL SECRETARY'S REPORT, OCTOBER 28, 1940.

During the session four Addresses have been given by request of the Council: on October 23, 1939, by Professor D. M. S. WATSON, F.R.S., on "The Origin of Frogs"; on December 4, 1939, by Dr DORA ILSE on "Experiments on the Colour Sense of Insects"; on January 15, 1940, by Professor MAX BORN, F.R.S., on "Relativistic Quantum Mechanics and Reciprocity," followed by a discussion in which the President, Dr C. A. COULSON and others took part; and on March 4, 1940, by Dr G. W. TYRRELL and Dr A. T. J. DOLLAR, on "Recent Scottish Earthquakes" and "The British Earthquake Inquiry" respectively. On July 1, 1940, a Demonstration was given by Dr A. T. J. DOLLAR on "The Preparation and Utility of Lamellar Cellulose Acetate Moulds from Etched Surfaces of Rocks and Minerals." Professor D. M. S. WATSON'S paper was published in *Transactions*, vol. 60, pp. 195–231.

34 papers were read as compared with 33 in the previous session. The papers were in the following subjects: Mathematics and Statistics, 14; Physics, 1; Botany and Physiography, 2; Geology and Palæontology, 4; Zoology and Animal Genetics, 13. 4 papers have been published in the *Transactions* and 27 in the *Proceedings*. 5 papers have been withdrawn, and several papers are still under the consideration of the Council.

The Society has lost by death 23 Fellows and 5 Honorary Fellows. 30 Fellows were elected. 8 Fellows have resigned, and two were removed from the Roll of Fellows.

The undernoted Congresses at which the Society would have been represented have been postponed owing to the war:—

1. International Botanical Congress, Stockholm, July 17–25, 1940.
2. International Congress of Geology, London, 1940.

The Society was represented by Professor J. H. M. WEDDERBURN, F.R.S., at the Bicentennial Celebration of the University of Pennsylvania, held at Philadelphia on September 20, 21, 1940, at which he presented the congratulatory Address of the Society.

At the Ordinary Meeting on July 1, 1940, the KEITH PRIZE for the period 1937–39, was presented to Professor F. A. E. CREW, F.R.S., for his papers and joint papers in the Society's *Proceedings* within the period, and in recognition of his numerous valuable contributions to Animal Genetics; and the NEILL PRIZE for the same period to Mr JAMES WRIGHT, F.G.S., for his important paper on "The Scottish Carboniferous Crinoidea" in the Society's *Transactions*, and published within the period of the award.

A Joint Committee of the Royal Physical Society, the Royal Scottish Geographical Society, and the Royal Society of Edinburgh, has awarded the Dr W. S. BRUCE MEMORIAL PRIZE to Mr BRIAN ROBERTS, Scott Polar Research Laboratory, Cambridge, for his valuable work in Survey, Ornithology, and General Biology, in the Polar Regions—North and South.

Owing to the national necessity for exercising the strictest economy in paper, and in order to reduce the expense of printing and publication, the Society has decided that, as from Vol. LXI, 1940–41, the *Proceedings* shall be published in TWO SECTIONS, viz. "A" (the Mathematical and Physical—including Astronomy, Chemistry, Mathematics, Metallurgy, Meteorology and Physics), and "B" (the Biological—including Anatomy, Anthropology, Botany, Geology, Pathology, Physiology and Zoology). Fellows of the Society, and Institutions with which the Society exchanges publications, will benefit under this arrangement by having, in smaller compass, papers dealing with the subjects in which they specialise.

This scheme involved the amendment of Law XVI, which was approved by the Society at the Ordinary Meeting held on June 3, 1940. Fellows will still have the right under the amended Law to receive BOTH SECTIONS of *Proceedings*, but it has been suggested that, under the circumstances, many Fellows may be willing to forgo this privilege and to ask only for that ONE SECTION in which they are more especially interested; some Fellows may decide to co-operate further by requesting merely occasional separates of particular papers.

No change is proposed in the present form or in the arrangement for the distribution of the Society's *Transactions*.

The Obituary Notices of Fellows, Proceedings of Meetings, List of Fellows, Prizes, etc., formerly published as *Appendices* at the end of each session's volume of *Proceedings* will, under the new scheme, be published separately, and will be sent normally to all Fellows and to those exchanges specially desiring to receive them.

The following were appointed to fill vacancies in the undernoted National Committees, for the periods stated:—

BIOLOGY	Professor EDWARD HINDLE	(Until 1945).
ASTRONOMY	Professor E. T. WHITTAKER	(Until 1945).
"	Sir ARTHUR S. EDDINGTON	(Until 1942).
GEODESY AND GEOPHYSICS	Dr A. CRICHTON MITCHELL	(Until 1945).
PHYSICS	Professor E. TAYLOR JONES	(Until 1945).
SCIENTIFIC RADIO	Professor J. A. CARROLL	(Until 1945).

Dr J. E. MACKENZIE was appointed to succeed Dr A. CRICHTON MITCHELL, who had resigned after long and valuable service, as the Society's representative on the Governing Body of the Heriot-Watt College.

Mr W. H. RUTHERFORD was appointed as junior assistant in the Library of the Society as from March 4, 1940.

During the session a letter was received from H.M. Treasury intimating that the Government had found it imperative to review every branch of State expenditure on services not directly concerned with the war effort, and felt compelled to reduce our present grant from £600 to £500 per annum. The Society has acquiesced for the present in this reduction but had notified H.M. Treasury that a detailed protest would be submitted at the end of the present emergency.

The cordial thanks of the Society are due to the authorities of the New College, Edinburgh, and to Mr W. Williamson, a Fellow of the Society, for the presentation of a number of valuable volumes to the Library, and for gifts from other sources during the session.

A sum of £250 was spent on Library Binding during the session, and arrears of binding are gradually being overtaken.

The acknowledgments of the Society are due to the Carnegie Trust for the Universities of Scotland for grants to authors towards the costs of illustrations, tabular matter, etc., of papers published in the *Transactions* and *Proceedings*, amounting to £163, 5s. 7d.; to Professor W. J. HAMILTON, for a grant of £15 towards the cost of publication of a joint paper in the *Transactions* (1940); to Dr D. A. ALLAN for a grant of £12, 5s. 9d. towards the cost of the coloured map in his paper in *Transactions* (1940); and for £300 from the Government Publication Grant administered by the Royal Society of London, in aid of the cost of the Society's publications for the session 1939–40.

TREASURER'S REPORT:—

The TREASURER in submitting the Accounts for the past year, compared the leading items of Receipts and Expenditure with those of the previous year, and informed the Society that a bequest of £350 made by Dr RICHARD TURNER, a Fellow of the Society, had been received during the session on behoof of the Society's PUBLICATION FUND. The Treasurer indicated that the financial position of the Society was satisfactory.

Dr A. LAUDER moved the adoption of the reports, and the reappointment of Messrs LINDSAY, JAMIESON & HALDANE, C.A., as auditors for the ensuing session. The meeting approved.

The Scrutineers reported that the Ballot Papers were in order, and the President declared that the following Office-Bearers and Members of Council had been duly elected:—

Professor E. T. WHITTAKER, M.A., Hon. Sc.D., D.Sc., LL.D., F.R.S., President.

LEONARD DOBBIN, Ph.D. JOHN ALEXANDER INGLIS, K.C., M.A., LL.B. Emeritus Professor RALPH STOCKMAN, M.D., LL.D., F.R.C.P.E. Professor JAMES RITCHIE, M.A., D.Sc. G. W. TYRRELL, A.R.C.S., D.Sc., F.G.S. Emeritus Professor C. T. R. WILSON, C.H., M.A., D.Sc., LL.D., F.R.S.	Vice-Presidents.

Professor James P. Kendall, M.A., D.Sc., F.R.S., General Secretary.
Professor R. J. D. Graham, M.A., D.Sc.
Professor W. M. H. Greaves, M.A., F.R.A.S. } Secretaries to Ordinary Meetings.
E. Maclagan Wedderburn, O.B.E., LL.D., D.K.S., Treasurer.
John E. Mackenzie, D.Sc., Curator of Library and Museum.

MEMBERS OF COUNCIL.

Professor R. C. Garry, M.B., Ch.B., D.Sc.
Professor D. Murray Lyon, M.D., F.R.C.P.E., D.P.H., D.Sc.
J. E. Richey, M.C., B.A., Sc.D., F.R.S., F.G.S.
The Hon. Lord Robertson.
A. Graham Donald, M.A., F.F.A., F.S.A.Scot.
Alan W. Greenwood, D.Sc., Ph.D.
Emeritus Professor T. H. Milroy, M.D., LL.D.
W. P. D. Wightman, Ph.D., M.Sc.
Professor Edward Hindle, M.A., Sc.D., Ph.D., A.R.C.S.
Professor J. R. Matthews, M.A., F.L.S.
Colonel Sir Arthur Olver, C.B., C.M.G., F.R.C.V.S.
David Russell, LL.D.

The Scrutineers were thanked for their services.

The President cordially thanked the retiring Office-Bearers and Members of Council for their valuable services to the Society, and especially mentioned the name of Dr James Watt, W.S., who had served on the Council since 1924, as Member of Council, Treasurer for eleven years, and Vice-President for the past three years.

The President in his ADDRESS referred to the losses the Society had sustained by death during the past year, mentioning specially Sir Thomas Hudson Beare, Sir George Berry, Professor R. A. Sampson, The Very Rev. J. Harry Miller, Sir Donald Tovey, Professor R. K. Hannay, and Sir J. J. Thomson, O.M., F.R.S., who was an Honorary Fellow of the Society and a contributor to its *Proceedings*, and to whose life and work the greater part of the ADDRESS was devoted. Experimental demonstrations were given by Dr G. A. Carse. (*See* Obituary Notice by the President, p. 410.)

The President was cordially thanked for his ADDRESS.

THE KEITH, MAKDOUGALL-BRISBANE, NEILL, GUNNING VICTORIA JUBILEE, JAMES SCOTT, BRUCE, AND DAVID ANDERSON-BERRY PRIZES, AND THE BRUCE-PRELLER LECTURE FUND.

The above Prizes will be awarded by the Council in the following manner:—

I. KEITH PRIZE.

The KEITH PRIZE, consisting of a Gold Medal and about £30 in Money, will be awarded in the Session 1941–1942 for the "best communication on a scientific subject, communicated,* in the first instance, to the Royal Society of Edinburgh during the Sessions 1939–1940 and 1940–1941." Preference will be given to a Paper containing a discovery. (See also Council's resolutions at the end of these regulations.)

II. MAKDOUGALL-BRISBANE PRIZE.

(*Amended June* 7, 1926.)

This Prize is to be awarded biennially by the Council of the Royal Society of Edinburgh to such person, for such purposes, for such objects, and in such manner as shall appear to them the most conducive to the promotion of the interests of science; with the *proviso* that the Council shall not be compelled to award the Prize unless there shall be some individual engaged in scientific pursuit, or some Paper written on a scientific subject, or some discovery in science made during the biennial period, of sufficient merit or importance in the opinion of the Council to be entitled to the Prize.

1. The Prize, consisting of a Gold Medal and a sum of Money, will be awarded before the close of the Session 1942–1943, for an Essay, Paper, or other work having reference to any branch of scientific inquiry, either material or mental.
2. It is open to all men of science.
3. The specific subjects taken into consideration in the current award are governed by the resolutions of the Council as stated at the end of these regulations.
4. For the current period the Committee is representative of Group *B*.
5. The Committee will consider Papers presented to the Society within the Sessions 1940–1941 and 1941–1942, and will make a recommendation.

It is empowered to recommend either:—

(*a*) An award to the Author of an Essay or Paper considered as above, or
(*b*) That no award be made on the ground that, within its group, no Paper of sufficient merit has been presented, or

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* For the purposes of this award the word "communicated" shall be understood to mean the date on which the manuscript of a Paper is received in its final form for printing, as recorded by the General Secretary or other responsible official.

(*c*) That the Prize be awarded to some distinguished man of learning, who may not have presented a Paper to the Society within the period considered, but who is willing to deliver an address.

III. NEILL PRIZE.

The Council of the Royal Society of Edinburgh having received the bequest of the late Dr PATRICK NEILL of the sum of £500, for the purpose of "the interest thereof being applied in furnishing a Medal or other reward every second or third year to any distinguished Scottish Naturalist, according as such Medal or reward shall be voted by the Council of the said Society," hereby intimate:—

1. The NEILL PRIZE, consisting of a Gold Medal, will be awarded during the Session 1941–1942.

2. The Prize will be given for a Paper of distinguished merit, on a subject of Natural History, by a Scottish Naturalist, which shall have been presented * to the Society during the two years preceding the fourth Monday in October 1941,—or failing presentation of a Paper sufficiently meritorious, it will be awarded for a work or publication by some distinguished Scottish Naturalist, on some branch of Natural History, bearing date within five years of the time of award. (See also resolutions of Council at the end of these regulations.)

IV. GUNNING VICTORIA JUBILEE PRIZE.

This Prize, founded in the year 1887 by Dr R. H. GUNNING, is to be awarded quadrennially by the Council of the Royal Society of Edinburgh, in recognition of original work in Physics, Chemistry, or Pure or Applied Mathematics.

Evidence of such work may be afforded either by a Paper presented * to the Society, or by a Paper on one of the above subjects, or some discovery in them elsewhere communicated or made, which the Council may consider to be deserving of the Prize.

The Prize consists of a sum of money, and is open to men of science resident in or connected with Scotland. The first award was made in the year 1887. The next award will be made in Session 1944–1945.

In accordance with the wish of the Donor, the Council of the Society may on fit occasions award the Prize for work of a definite kind to be undertaken during the three succeeding years by a scientific man of recognised ability.

V. JAMES SCOTT PRIZE.

This Prize, founded in the year 1918 by the Trustees of the JAMES SCOTT Bequest, is to be awarded triennially, or at such intervals as the Council of the Royal Society of Edinburgh may decide, "for a lecture or essay on the fundamental concepts of Natural Philosophy."

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* For the purposes of this award the word "presented" shall be understood to mean the date on which the manuscript of a Paper is received in its final form for printing, as recorded by the General Secretary or other responsible official.

VI. BRUCE PRIZE.

The Society is trustee of a fund, instituted in 1923, to commemorate the work of Dr W. S. BRUCE as an explorer and scientific investigator in polar regions.

The Committee of Award is appointed jointly by the Royal Society of Edinburgh, the Royal Physical Society, and the Royal Scottish Geographical Society.

The Prize consists of a Bronze Medal and sum of Money. It is open to workers of all nationalities, with a preference, *ceteris paribus*, for those of Scottish birth or origin, and is to be awarded biennially for some notable contribution to Natural Sciences, such as Zoology, Botany, Geology, Meteorology, Oceanography, and Geography; the contribution to be in the nature of new knowledge, the outcome of a personal visit to polar regions on the part of the recipient. The recipient shall preferably be at the outset of his career as an investigator.

The next award will be made in 1942. Papers for the consideration of the Committee should be in the hands of the General Secretary of the Royal Society of Edinburgh, 22 George Street, Edinburgh 2, not later than March 31 of that year.

VII. BRUCE-PRELLER LECTURE FUND.

The Council of the Royal Society of Edinburgh having received in 1929 the bequest of the late Dr CHARLES DU RICHE PRELLER of the sum of £500, decided that the income thereof be applied by the Council biennially as an honorarium for a special BRUCE-PRELLER Lecture or Address by an outstanding man of science, its subject to be Geology or Electrical or Physical Science, or in the discretion of the Council some other branch of science. The next award will be made in Session 1942–1943.

VIII. DAVID ANDERSON-BERRY FUND.

The Council of the Royal Society of Edinburgh having received in the year 1930, free of duty, the capital sum of one thousand pounds (£1000), to be used in terms of the will of the late Dr DAVID ANDERSON-BERRY, dated 23rd April 1926, decided that the income thereof be applied triennially, "in the first place in the presentation of a gold medal, and in the second place in the payment of a sum of money to the winner for the year of such gold medal, the winner being the person who, in the opinion of the Society, shall be the producer for the year of the best essay on the nature of X-rays and their therapeutical effect on human diseases."

The third award, which was to be made in July 1941, has been postponed owing to the present international situation.

RESOLUTIONS OF COUNCIL IN REGARD TO THE MODE OF AWARDING PRIZES.

(*See Minutes of Meeting of January* 18, 1915.)

I. With regard to the Keith and Makdougall-Brisbane Prizes, which are open to all Sciences, the mode of award will be as follows:—

1. Papers or essays to be considered shall be arranged in two groups, *A* and *B*—Group *A* to include Astronomy, Chemistry, Mathematics, Metallurgy, Meteorology, and Physics; Group *B* to include Anatomy, Anthropology, Botany, Geology, Pathology, Physiology, and Zoology.
2. These two prizes shall be awarded to each group in alternate biennial periods, provided Papers worthy of recommendation have been communicated to the Society.
3. Prior to the adjudication the Council shall appoint, in the first instance, a Committee composed of representatives of the group of Sciences which did not receive the award in the immediately preceding period. The Committee shall consider the Papers which come within their group of Sciences, and report in due course to the Council.
4. In the event of the aforesaid Committee reporting that within their group of subjects there is, in their opinion, no Paper worthy of being recommended for the award, the Council, on accepting this report, shall appoint a Committee representative of the alternate group to consider Papers coming within their group and to report accordingly.
5. Papers to be considered by the Committees shall fall within the period dating from the last award in Groups *A* and *B* respectively.

II. With regard to the Neill Prize, the term "Naturalist" shall be understood to include any student in the Sciences composing Group *B*, namely, Anatomy, Anthropology, Botany, Geology, Pathology, Physiology, Zoology.

AWARDS OF THE KEITH, MAKDOUGALL-BRISBANE, NEILL, GUNNING, JAMES SCOTT, BRUCE, AND DAVID ANDERSON-BERRY PRIZES, AND THE BRUCE-PRELLER LECTURE FUND.

(*Earlier Awards will be found printed in the preceding volumes of the Proceedings.*)

I. KEITH PRIZE.

49th Biennial Period, 1923–25.—Herbert Westren Turnbull, M.A., for the papers on "Hyper-Algebra," "Invariant Theory," and "Algebraic Geometry," three of which have been published in the Proceedings within the period of award.

50th Biennial Period, 1925–27.—Thomas John Jehu, M.A., M.D., F.G.S., and Robert Meldrum Craig, M.A., B.Sc., F.G.S., for the joint series of papers which have recently appeared in the Transactions of the Society on the "Geology of the Outer Hebrides."

51st Biennial Period, 1927–29.—Christina C. Miller, B.Sc., Ph.D., for her papers on the "Slow Oxidation of Phosphorus Trioxide," published in the Proceedings within the period of the award, and in consideration of subsequent developments on "Slow Oxidation of Phosphorus," published elsewhere.

52nd Biennial Period, 1929–31.—Alan William Greenwood, M.Sc., Ph.D., for his papers on the "Biology of the Fowl," several of which have appeared in the Proceedings within the period of award.

53rd Biennial Period, 1931–33.—A. Crichton Mitchell, D.Sc., for his paper "On the Diurnal Incidence of Disturbance in the Terrestrial Magnetic Field," published in the Transactions within the period of award.

54th Biennial Period, 1933–35.—Lancelot T. Hogben, D.Sc., F.R.S., for his papers on genetical subjects, published alone and in collaboration, which have appeared in the Proceedings within the period of award.

55th Biennial Period, 1935–37.—Harold Stanley Ruse, M.A., for his paper "On the Geometry of Dirac's Equations and their Expression in Tensor Forms," published in the Proceedings within the period, and for his other papers in the Proceedings.

56th Biennial Period, 1937–39.—Professor F. A. E. Crew, M.D., D.Sc., Ph.D., F.R.S., for his papers in the Proceedings within the period of the award, and in recognition of his valuable contributions to Animal Genetics.

II. MAKDOUGALL-BRISBANE PRIZE.

33rd Biennial Period, 1922–24.—Professor H. Stanley Allen, D.Sc., for his papers on the "Quantum and Atomic Theory," published in the Society's Proceedings within the period.

34th Biennial Period, 1924–26.—Charles Morley Wenyon, C.M.G., C.B.E., F.R.S., for his distinguished work in Protozoology extending over a period of twenty-one years.

35th Biennial Period, 1926–28.—W. O. Kermack, M.A., D.Sc., for his contributions to Chemistry, published in the Society's Proceedings and elsewhere.

36th Biennial Period, 1928–30.—Nellie B. Eales, D.Sc., for her papers in the Society's Transactions on "The Anatomy of a Fœtal African Elephant."

37th Biennial Period, 1930–32.—A. C. Aitken, M.A., D.Sc., for various contributions to Mathematics, published in the Society's Proceedings and elsewhere.

38th Biennial Period, 1932–34.—A. E. Cameron, M.A., D.Sc., for his publications in Entomology, including his paper in the Transactions, "The Life-History and Structure of *Hæmatopota pluvialis* Linné (Tabanidæ)."

39th Biennial Period, 1934–36.—E. M. Anderson, M.A., D.Sc., for his paper "The Dynamics of the Formation of Cone-sheets, Ring-dykes, and Caldron-subsidences," published in the Society's Proceedings within the period.

40TH BIENNIAL PERIOD, 1936–38.—Professor D. M. S. WATSON, F.R.S., for his paper published in the Society's Transactions on "*Rhamphodopsis*, a Ptyctodont from the Middle Old Red Sandstone of Scotland" and for his distinguished contributions to Vertebrate Palæontology.

III. THE NEILL PRIZE.

9TH BIENNIAL PERIOD, 1923–25.—FREDERICK ORPEN BOWER, F.R.S., for his recent contributions to Botanical knowledge and in recognition of his published work extending over a period of forty-five years.

10TH BIENNIAL PERIOD, 1925–27.—ARTHUR ROBINSON, M.D., M.R.C.S., for his contributions to Comparative Anatomy and Embryology.

11TH BIENNIAL PERIOD, 1927–29.—EDWARD BATTERSBY BAILEY, M.C., F.R.S., in recognition of his valuable contributions to the Geology of Scotland, two of which have recently appeared in the Transactions of the Society.

12TH BIENNIAL PERIOD, 1929–31.—CHARLES HENRY O'DONOGHUE, D.Sc., for his papers on the "Blood Vascular System," and for his earlier work on the "Morphology of the *corpus luteum*."

13TH BIENNIAL PERIOD. 1931–33.—GEORGE WALTER TYRRELL, A.R.C.S., D.Sc., for his contributions to the Geology and Petrology of Sub-Arctic and Sub-Antarctic Lands.

14TH BIENNIAL PERIOD, 1933–35.—SAMUEL WILLIAMS, Ph.D., for his contributions to the Anatomy and Experimental Morphology of the Pteridophyta.

15TH BIENNIAL PERIOD, 1935–37.—WILLIAM J. HAMILTON, M.D., D.Sc., for his contributions to the Embryology of the Ferret and other work published in the Transactions.

16TH BIENNIAL PERIOD, 1937–39.—JAMES WRIGHT, F.G.S., for his paper on "The Scottish Carboniferous Crinoidea" published in the Transactions.

IV. GUNNING VICTORIA JUBILEE PRIZE.

8TH QUADRENNIAL PERIOD, 1924–28.—Professor E. T. WHITTAKER, F.R.S., in recognition of his distinguished contributions to Mathematical Science, and of his promotion of Mathematical Research in Scotland.

9TH QUADRENNIAL PERIOD, 1928–32.—Emeritus Professor Sir J. WALKER, F.R.S., for numerous contributions to Physical and General Chemistry.

10TH QUADRENNIAL PERIOD, 1932–36.—Professor C. G. DARWIN, F.R.S., for his distinguished contributions in Mathematical Physics.

V. JAMES SCOTT PRIZE.

1ST AWARD, 1918–22.—Professor A. N. WHITEHEAD, F.R.S., for his lecture delivered on June 5, 1922, on "The Relatedness of Nature."

2ND AWARD, 1922–27.—Sir JOSEPH LARMOR, M.A., D.Sc., LL.D., F.R.S., for his lecture delivered on July 4, 1927, on "The Grasp of Mind on Nature."

3RD AWARD, 1927–30.—Professor NIELS BOHR, for his lecture delivered on May 26, 1930, on "Philosophical Aspects of Atomic Theory."

4TH AWARD, 1930–33.—Professor Dr ARNOLD SOMMERFELD, for his lecture delivered on May 1, 1933, on "Ways to the Knowledge of Nature."

5TH AWARD, 1933–38.—Professor P. A. M. DIRAC, F.R.S., for his lecture delivered on February 6, 1939, on "The Relation between Mathematics and Physics."

VI. BRUCE PRIZE.

1ST AWARD, 1926.—JAMES MANN WORDIE, M.A., for his Oceanographical and Geological work in both Polar Regions.

2ND AWARD, 1928.—H. U. SVERDRUP, for his contributions to the knowledge of the Meteorology, Magnetism, and Tides of the Arctic, as an outcome of his travels with the Norwegian Expedition in the "Maud" from 1918 to 1925.

3RD AWARD, 1930.—N. A. MACKINTOSH, M.Sc., A.R.C.S., for his researches into the Biology of Whales in the Waters of the Falkland Islands Dependencies.

4TH AWARD, 1932.—HENRY GINO WATKINS, for important contributions to the topography of Spitsbergen, Labrador, and East Greenland, and investigation of the Ice Cap of Greenland.

5TH AWARD, 1936.—JAMES WILLIAM SLESSER MARR, M.A., B.Sc., for his work in the Southern Ocean and more particularly for his monograph on the South Orkney Islands.

6TH AWARD, 1938.—ALEXANDER R. GLEN, for his work in Spitsbergen, including Survey in New Friesland and the completion of the map of North East Land.

7TH AWARD, 1940.—BRIAN ROBERTS, for his valuable work in survey, ornithology, and general biology in the Polar regions—north and south.

VII. BRUCE-PRELLER LECTURE FUND.

1ST AWARD, 1931.—Professor E. T. WHITTAKER, F.R.S., for his lecture, "James Clerk Maxwell and Mechanical Descriptions of the Universe."

2ND AWARD, 1933.—Professor C. H. LANDER, C.B.E., for his lecture on October 23, 1933, on "The Liquefaction of Coal."

3RD AWARD, 1935.—Professor W. L. BRAGG, O.B.E., F.R.S., for his lecture on February 4, 1935, on "The New Crystallography."

4TH AWARD, 1937.—Professor H. S. TAYLOR, F.R.S., for his lecture on June 7, 1937, on "Heavy Hydrogen in Scientific Research."

5TH AWARD, 1939.—Professor P. M. S. BLACKETT, F.R.S., for his lecture on May 1, 1939, on "The Mesotron: the New Unstable Cosmic Ray Particle."

VIII. DAVID ANDERSON-BERRY FUND.

1ST AWARD, 1935.—CHARLES MELVILLE SCOTT, M.A., M.B., D.Sc., for his essay "On the Action of X- and Gamma-Rays on Living Cells."

2ND AWARD, 1938.—MARY A. C. COWELL, M.B., Holt Radium Institute, Manchester, for her essay "An Investigation into some of the Factors affecting the Response of Human Skin and Human Skin Tumours to Radiation."

ABSTRACT
OF
THE ACCOUNTS
OF
THE ROYAL SOCIETY OF EDINBURGH,

SESSION—1ST OCTOBER 1939 TO 30TH SEPTEMBER 1940.

E. MACLAGAN WEDDERBURN, O.B.E., D.K.S., LL.D.
Treasurer.

I. GENERAL FUND

1. *Funds as at 30th September* 1939, per last Account				£14,235 15 11
2. *Dr Richard Turner Legacy and Interest*				354 5 1
3. *Commutation Fee Fund*— 1 Fellow elected Session 1939–40				52 10 0
				£14,642 11 0
Add Income—				
1. *Contributions for current Session*—				
515 Fellows at £3, 3s. each		£1622 5 0		
Fees of Admission of thirty New Fellows		94 10 0		
First Annual Contribution of twenty-nine New Fellows		91 7 0		
			£1808 2 0	
2. *Extra Contributions for* 1939–40 under Amended Law VI.— Voluntary Contributions			39 17 0	
3. *Interest Received, Untaxed*—				
General Fund—				
On £2100 2½% Consolidated Stock		£52 10 0		
On £11,107, 13s. 2d. 2¾% Guaranteed Stock (1933)		306 18 6		
On Deposit Receipts		8 2 1		
		£367 10 7		
Special Subscription Fund— On £936, 0s. 11d. 2¾% Guaranteed Stock (1933)		25 14 8		
R. M. Smith Legacy— On £554, 13s. 7d. 2¾% Guaranteed Stock (1933)		15 5 0		
Publication Fund— On £2568, 5s. 1d. 2¾% Guaranteed Stock (1933)		70 12 4		
Commutation Fee Fund—				
On £317, 18s. 10d. 2¾% Guaranteed Stock (1933)	8 0 8			
On Deposit Receipts	0 7 9			
		8 8 5		
			487 11 0	
Forward			£2335 10 0	£14,642 11 0

Forward		£2335 10 0	£14,642 11 0
4. *Transactions and Proceedings sold*		37 13 4	
5. *Grants—*			
Annual Grant from Government	£500 0 0		
Grant from Royal Society from Government Publication Grant	300 0 0		
Grants from Trustees of the Carnegie Trust for the Universities of Scotland	163 5 7		
Other Sources	27 5 9		
		990 11 4	
6. *Publication Fund*—Sale of Volumes		1 15 0	
			3365 9 8
			£18,008 0 8
Deduct **Expenditure—**			
1. *Expenses of Transactions and Proceedings—*			
Transactions		£565 10 8	
Proceedings		807 10 0	
		£1373 0 8	
2. *Books, Periodicals, Newspapers, etc.*		190 19 11	
3. *Library Binding*		250 10 10	
4. *Gregory Memorial Volume—*			
Sums paid to Account	£100 0 0		
Less—Sale of Volume	29 6 0		
		70 14 0	
5. *General Upkeep of Society's Rooms—*			
Insurance and Water Rates	£27 14 8		
Repairs and Furnishings	48 16 11		
Heating and Lighting	75 3 11		
Cleaning	18 17 1		
Caretaker's Salary and Uniform	134 14 3		
Charwomen's Wages	72 12 0		
		377 18 10	
6. *Management—*			
Honorarium to General Secretary	£100 0 0		
Salaries	604 6 0		
Fee to Treasurer's Clerk	35 0 0		
Society's Contribution to Staff Pension Fund under Universities Scheme	63 10 0		
State Insurance	13 15 10		
General Printing and Stationery	156 14 0		
Telephone	20 16 0		
Audit Fee	10 10 0		
Tea Expenses at Meetings	28 19 6		
Postages and Petty Outlays	31 15 3		
Miscellaneous	22 16 10		
S. Heddle allowance	50 0 0		
		1138 3 5	
7. *Dr Richard Turner Legacy and Interest—*			
On Deposit Receipt	£354 5 1		
8. *Commutation Fee Fund—*			
Transferred to Special Fund (p. 445)	£52 10 0		
9. *Arrears of Contributions* written off as irrecoverable		25 4 0	
			3426 11 8
Funds as at 30th September 1940			£14,581 9 0

Represented by—			
£11,107, 13s. 2d. 2½% Guaranteed Stock (1933) at cost		£9,298 1 9	
£2100 2½% Consolidated Stock at cost		1,113 0 0	
£554, 13s. 7d. 2½% Guaranteed Stock (1933) at cost—Robert Mackay Smith Legacy		472 8 9	
£2568, 5s. 1d. 2½% Guaranteed Stock (1933) at cost—Publication Fund (comprising Peter Guthrie Tait Memorial Fund and Dr John Aitken Fund)		2,187 9 1	
£936, 0s. 11d. 2½% Guaranteed Stock (1933) at cost—Special Subscription Fund		797 5 2	
Due by Union Bank of Scotland, Ltd., on Deposit Receipt—Dr Richard Turner Legacy and Interest		354 5 1	
£317, 18s. 10d. 2½% Guaranteed Stock (1933) at cost	£243 12 0		
Due by Union Bank of Scotland, Ltd., on Deposit Receipt	52 10 0		
Life Membership Commutation Fee Fund		296 2 0	
Arrears of Contribution at 30th September 1940—			
Present Session	£126 0 0		
Previous Session	28 7 0		
New Member	6 6 0		
		160 13 0	
Due by Union Bank of Scotland Ltd., on Deposit Receipt		21 11 3	
Cash—Imprest Amount		15 0 0	
		£14,715 16 1	
Less—Due to Union Bank of Scotland, Ltd., on Current Account	£116 10 1		
Contributions in Advance for 1940–41	3 3 0		
Part Commutation Fee	14 14 0		
		134 7 1	
			£14,581 9 0

In addition the Society owns the Library, Museum, Pictures, etc., and Furniture in the Rooms 22 George Street, Edinburgh.

Note—

Income for current Session as per preceding Accounts		£3365 9 8
Add—Arrears of Contributions at 30th September 1939		132 6 0
		£3497 15 8
Deduct—*Expenditure* for current Session as per preceding Accounts	£3426 11 8	
Arrears of Contributions at 30th September 1940	160 13 0	
		3587 4 8
Deficit for year to 30th September 1940		£89 9 0

COMMUTATION FEE FUND.

Year to 30th September 1940.

Funds as at 30th September 1939, per last Account		£243 12 0
Fee received for Life Membership—1 Fellow elected 1939–40		52 10 0
Interest received—On Investment	£8 0 8	
On Deposit Receipt	0 7 9	
		8 8 5
		£304 10 5
Deduct—Transferred to General Fund (Interest received)		8 8 5
Total Funds (included in General Funds, above)		£296 2 0
Represented by—		
£317, 18s. 10d. 2½% Guaranteed Stock (1933) at cost	£243 12 0	
Due by Union Bank of Scotland, Ltd., on Deposit Receipt	52 10 0	
		296 2 0

PUBLICATION FUND.

(*Comprising Peter Guthrie Tait Memorial Fund and Dr John Aitken Fund.*)

Year to 30th September 1940.

Funds as at 30th September 1939, per last Account		£2187 9 1
Interest Received: Untaxed—		
Peter Guthrie Tait Memorial Fund—		
On £1929, 17s. 1d. 2½% Guaranteed Stock (1933)	£53 1 4	
Dr John Aitken Fund—		
On £638, 8s. 2½% Guaranteed Stock (1933)	17 11 0	
		70 12 4
Sale of Volumes, Dr John Aitken Fund		1 15 0
		£2259 16 5
Deduct—		
Transferred to General Fund to meet cost of Publication—		
Interest, p. 443	£70 12 4	
Sale of Volumes, p. 444	1 15 0	
		72 7 4
Total Funds (included in General Fund, p. 445)		£2187 9 1
Represented by—		
£2568, 5s. 1d. 2½% Guaranteed Stock (1933) at cost		£2187 9 1

II. PRIZE FUNDS

Year to 30th September 1940.

1. Keith Fund—

Funds as at 30th September 1939, per last Account—		
£801, 13s. 4d. 2½% Guaranteed Stock (1933) at cost		£682 16 1
Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt		60 14 2
		£743 10 3
Interest Received—On Investment	£22 12 8	
On Deposit Receipts	0 7 2	
		22 19 10
		£766 10 1
Deduct—Professor F. A. E. Crew 1937–1939 Award	£48 0 0	
Cost of Medal	2 0 0	
		50 0 0
		£716 10 1

2. Neill Fund—

Funds as at 30th September 1939, per last Account—			
£370 2½% Guaranteed Stock (1933) at cost		£315 2 9	
Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt		28 10 5	
		£343 13 2	
Interest received—On Investment	£10 3 6		
On Deposit Receipts	0 4 9		
		10 8 3	
		£354 1 5	
Deduct—James Wright 1937–1939 Award	£20 0 0		
Cost of Medal	2 0 0		
		22 0 0	
			332 1 5
		Forward	£1048 11 6

Forward			£1048 11 6

3. Makdougall-Brisbane Fund—

Funds as at 30th September 1939, per last Account—			
£493, 6s. 7d. 2½% Guaranteed Stock (1933) at cost .		£420 3 8	
Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt		66 15 1	
		£486 18 9	
Interest received—On Investment . .	£13 11 2		
On Deposit Receipts .	0 6 3		
		13 17 5	
		£500 16 2	
Deduct—Professor D. M. S. Watson 1936–1938 Award	£13 15 0		
Cost of Medal	22 5 0		
		36 0 0	
			464 16 2

4. Makerstoun Magnetic Meteorological Observation Fund—

Funds as at 30th September 1939, per last Account—			
£308, 6s. 9d. 2½% Guaranteed Stock (1933) at cost .		£262 12 5	
Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt		167 0 10	
		£429 13 3	
Interest received—On Investment . .	£13 4 1		
On Deposit Receipts .	0 4 9		
		13 8 10	
			443 2 1

5. Gunning Victoria Jubilee Prize Fund (Instituted by Dr Gunning of Edinburgh and Rio de Janeiro)—

Funds as at 30th September 1939, per last Account—			
£739, 12s. 5d. 2½% Guaranteed Stock (1933) at cost .		£629 19 2	
Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt		121 10 9	
		£751 9 11	
Interest received—On Investment . .	£21 10 4		
On Deposit Receipts .	0 18 5		
		22 8 9	
			773 18 8

6. James Scott Prize Fund—

Funds as at 30th September 1939, per last Account—			
£305, 5s. 2½% Guaranteed Stock (1933) at cost . .		£259 19 10	
Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt		14 5 3	
		£274 5 1	
Interest received—On Investment . .	£8 13 9		
On Deposit Receipts .	0 1 0		
		8 14 9	
			282 19 10

7. Dr W. S. Bruce Memorial Fund—

Funds as at 30th September 1939, per last Account—			
£289, 1s. 5d. 2½% Guaranteed Stock (1933) at cost .		£246 4 2	
Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt		53 5 1	
		£299 9 3	
Interest received—On Investment . .	£8 16 8		
On Deposit Receipts .	0 5 1		
		9 1 9	
			308 11 0
Forward			£3321 19 3

Forward			£3321 19 3
8. Bruce-Preller Lecture Fund—			
Funds as at 30th September 1939, per last Account—			
£140, 9s. Royal Bank of Scotland Stock, taken over at 350%		£491 11 6	
Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt		41 17 9	
		£533 9 3	
Interest received—On Investments (less tax £8, 13s. 1d.)	£16 8 1		
On Deposit Receipts	0 2 5		
		16 10 6	
Repayment of Income Tax for year to December 1939		8 16 9	
			558 16 6
9. Dr David Anderson-Berry Fund—			
Funds as at 30th September 1939, per last Account—			
£1528, 0s. 4d. Local Loans 3% Stock at cost		£1000 0 0	
Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt		52 11 6	
		£1052 11 6	
Interest received—On Investment	£47 6 3		
On Deposit Receipts	0 5 7		
		47 11 10	
			1100 3 4
Total Funds			£4980 19 1
Represented by—			
£3785, 0s. 9d. 2½% Guaranteed Stock (1933) at cost		£3166 18 1	
£1528, 0s. 4d. Local Loans 3% Stock at cost		1000 0 0	
£140, 9s. Royal Bank of Scotland Stock, taken over at 350%		491 11 6	
Due by Union Bank of Scotland, Ltd., on Deposit Receipt		322 9 6	
			£4980 19 1

Note.—Under the Will of the late Professor Charles Piazzi Smyth and his wife, the Society will, on the expiry of certain liferents, become entitled to payment of the residue to be applied as set out in the will.

EDINBURGH, 17*th October* 1940.—We have examined the preceding Accounts of the Treasurer of the Royal Society of Edinburgh for the Session 1939–1940, and have found them to be correct. The Investments held for the General Fund and the Prize Funds as shown in the foregoing Accounts have been verified by us as at 30th September 1940.

LINDSAY, JAMIESON & HALDANE, C.A., *Auditors*.

VOLUNTARY CONTRIBUTORS under Law VI, to 30th September 1940

Professor F. O. Bower, F.R.S.	£1 1 0
Dr G. S. Brock	1 1 0
Professor E. G. Coker, F.R.S.	1 1 0
Professor J. N. Collie, F.R.S.	1 1 0
W. B. Coutts, Esq.	1 1 0
A. Cowan, Esq.	1 0 0
L. M. Douglas, Esq.	1 1 0
Dr E. O. Fergus	1 1 0
J. S. Ford, Esq.	1 1 0
Alex. Fraser, Esq.	1 1 0
Wm. Fraser, Esq.	1 1 0
Sir Alexander Gibb, F.R.S.	1 1 0
Sir J. Graham Kerr, F.R.S.	1 1 0
Dr W. F. Hume	1 1 0
Dr G. R. Jeffrey	1 1 0
Professor T. J. Jehu	1 1 0
A. King, Esq.	1 1 0
F. H. Lightbody, Esq.	1 1 0
Dr P. McBride (1938–39)	1 1 0
	£19 18 0
Carried forward	£19 18 0
Dr P. McBride (1939–40)	1 1 0
Dr G. McGowan	1 1 0
Dr Arch. McKendrick	1 1 0
Professor J. MacKinnon	1 1 0
Dr D. J. Mackintosh (1939–40)	1 1 0
Dr D. J. Mackintosh (1940–41)	1 1 0
Dr H. R. Mill	1 1 0
A. G. Ramage, Esq.	1 1 0
J. R. Ratcliffe, Esq.	1 1 0
Col. T. E. Salvesen	1 1 0
C. A. Stevenson, Esq.	1 1 0
Dr H. F. Stockdale	1 1 0
G. S. Thomson, Esq.	1 1 0
R. T. Thomson, Esq.	1 1 0
Dr G. C. Trotter	1 1 0
Very Rev. L. MacLean Watt	1 1 0
Professor E. T. Whittaker (1938–39)	1 1 0
Professor E. T. Whittaker (1939–40)	1 1 0
W. Williamson, Esq.	1 1 0
Total	£39 17 0

THE COUNCIL OF THE SOCIETY

28th October 1940

President.

Professor E. T. WHITTAKER, M.A., Hon. Sc.D., D.Sc., LL.D., F.R.S.

Vice-Presidents.

LEONARD DOBBIN, Ph.D.
JOHN ALEXANDER INGLIS, K.C., M.A., LL.B.
Emeritus Professor R. STOCKMAN, M.D., LL.D., F.R.C.P.E.
Professor JAMES RITCHIE, M.A., D.Sc.
G. W. TYRRELL, A.R.C.S., D.Sc., F.G.S.
Emeritus Professor C. T. R. WILSON, C.H., M.A., D.Sc., LL.D., F.R.S.

General Secretary.

Professor JAMES P. KENDALL, M.A., D.Sc., F.R.S.

Secretaries to Ordinary Meetings.

Professor R. J. D. GRAHAM, M.A., D.Sc.
Professor W. M. H. GREAVES, M.A., F.R.A.S.

Treasurer.

E. MACLAGAN WEDDERBURN, O.B.E., LL.D., D.K.S.

Curator of Library and Museum.

JOHN E. MACKENZIE, D.Sc.

Councillors.

Professor R. C. GARRY, M.B., Ch.B., D.Sc.

Professor D. MURRAY LYON, M.D., F.R.C.P.E., D.P.H., D.Sc.

J. E. RICHEY, M.C., B.A., Sc.D., F.R.S., F.G.S.

The Hon. LORD ROBERTSON.

A. GRAHAM DONALD, M.A., F.F.A., F.S.A.Scot.

ALAN W. GREENWOOD, D.Sc., Ph.D.

Emeritus Professor T. H. MILROY, M.D., LL.D.

W. P. D. WIGHTMAN, Ph.D., M.Sc.

Professor EDWARD HINDLE, M.A., Sc.D., Ph.D., A.R.C.S.

Professor J. R. MATTHEWS, M.A., F.L.S.

Colonel Sir ARTHUR OLVER, C.B., C.M.G., F.R.C.V.S.

DAVID RUSSELL, LL.D.

Office Staff.

Assistant Secretary and Librarian, G. A. STEWART.
Assistant Librarian, R. J. B. MUNRO.
Junior Assistant Librarian, W. H. RUTHERFORD.
Housekeeper, W. BRYCE.
Tel. No. 22881.

PATRON

HIS MOST EXCELLENT MAJESTY THE KING

FELLOWS OF THE SOCIETY,

Corrected to 28th October 1940.

N.B.—*Those marked* * *are Annual Contributors.*
„ „ † *have commuted Voluntary Contribution* (*see 3rd Paragraph, Law VI*).

M-B. prefixed to a name indicates that the Fellow has received a Makdougall-Brisbane Medal.
K. „ „ „ „ Keith Medal.
N. „ „ „ „ Neill Medal.
V. J. „ „ „ „ the Gunning Victoria Jubilee Prize.
B. „ „ „ „ Bruce Medal.
B-P. „ „ „ „ Bruce-Preller Lectureship.
C. „ „ „ has contributed one or more Communications to the Society's Transactions or Proceedings.

Date of Election			Service on Council, etc.
1925	M-B. C.	* Aitken, Alexander Craig, M.A., D.Sc., F.R.S., Lecturer in Actuarial Science, University of Edinburgh (Drummond Street). 54 Braid Road, Edinburgh 10	1934–36. Sec. 1936–40
1889		† Alison, John, M.A., LL.D., formerly Head Master, George Watson's College. 126 Craiglea Drive, Edinburgh 10	
1927	C.	* Allan, Douglas Alexander, D.Sc., Director, City of Liverpool Public Museums, William Brown Street, Liverpool	
1920	C. M-B.	* Allen, Herbert Stanley, M.A. (Cantab.), D.Sc. (Lond.), F.R.S., Professor of Natural Philosophy, University of St Andrews	1921–24.
1939		* Amies, Arthur Barton Pilgrim, D.D.Sc. (Melb.), L.D.S. (Vict.), L.R.C.P.E., L.R.C.S.E., D.L.O. (Melb.), F.R.A.C.S., Professor of Dental Science, Australian College of Dentistry. 193 Spring Street, Melbourne	
1938		* Anderson, Charles Henry William Gatacre, B.Sc. (Edin.), Headmaster and Superintendent, Royal Blind School, Edinburgh. 12 West Savile Road, Edinburgh 9	
1920	C. M-B.	* Anderson, Ernest Masson, M.A., D.Sc., F.G.S., 62 Greenbank Crescent, Edinburgh 10	
1905		Anderson, William, M.A., formerly Head Science Master, George Watson's College, Edinburgh. 6 Lockharton Crescent, Edinburgh 11	
1905		Andrew, George, C.B.E., M.A., B.A., H.M.I.S. (retired). Hamewith, Kilmacolm, Renfrewshire	
1930		* Annan, William, M.A., C.A., Professor of Accounting and Business Method, University of Edinburgh (South Bridge). Tofthill, Ferry Road West, Edinburgh 5	
1939		* Annan, William Gillies, M.D. (Edin.), F.R.C.S.E., Clinical Medical Officer, Durham County Council. 41 Cleveland Avenue, Darlington	
1915		Anthony, Charles, M.Inst.C.E., M.Am.Soc.C.E., F.R.San.I., F.R.Met.S., F.R.A.S., F.C.S., Springcroft, Les Croutes, St Peter Port, Guernsey, Channel Islands	
1906		Appleton, Colonel Arthur Frederick, F.R.C.V.S. (*no permanent address until further notice*)	
1910	C.	Archibald, E. H., B.Sc., Professor of Chemistry, University of British Columbia, Vancouver, Canada	
1933		* Arnot, Frederick Latham, Sc.D., Ph.D. (Cantab.), B.Sc. (Sydney), Lecturer in Physics, University of Sydney, N.S.W. (*Died October* 1940)	
1921		* Arthur, William, M.A., Lecturer in Mathematics, University of Glasgow. 148 Carmunnock Road, Cathcart, Glasgow	
1938	C.	* Bacsich, Paul, M.D. (Szeged, Hungary), Lecturer in Human Embryology, University of Glasgow. 34 Victoria Crescent Road, Dowanhill, Glasgow, W. 2	
1920	C. N.	* Bailey, Edward Battersby, M.C., M.A., D.Sc. (Harvard and Birmingham), F.R.S., F.G.S., Director, Geological Survey of Great Britain and Museum of Practical Geology, Exhibition Road, London, S.W. 7	1932–35. V-P 1935–37.

Date of Election			Service on Council, etc.
1896	C.	† Baily, Francis Gibson, M.A., M.Inst.E.E., Emeritus Professor of Electrical Engineering, Heriot-Watt College, Edinburgh. Newbury, Juniper Green, Midlothian	1909–12, 1920–23. V-P 1929–32.
1934		* Bain, David, M.Sc. (Manch.), D.Sc. (Edin.), Lecturer in Technical Chemistry, University of Edinburgh (West Mains Road). 87 Cluny Gardens, Edinburgh 10	
1931	C.	* Bain, William Alexander, Ph.D., Reader in Pharmacology, School of Medicine, University of Leeds. 26 Weetwood Road, Headingley, Leeds 6	
1931		* Baird, Sir William Macdonald, Kt., J.P., Fellow and Past President of the Faculty of Surveyors of Scotland, F.S.A.Scot. Dalveen, Barnton Avenue, Davidson's Mains, Edinburgh 4	
1921		* Baker, Bevan Braithwaite, M.A., D.Sc., Professor of Mathematics, Royal Holloway College, Englefield Green, London	
1928	C.	* Baker, Edwin Arthur, D.Sc. (Edin.), Assistant at the Royal Observatory, Edinburgh. 17 Ladysmith Road, Edinburgh 9	
1905	C.	Balfour-Browne, William Alexander Francis, M.A., F.Z.S., F.L.S., F.R.E.S., F.R.M.S., Barrister-at-Law, formerly Professor of Entomology, Imperial College of Science, London. Brae, Crocketford, Kirkcudbrightshire	
1940		* Ball, Edmund Bruce, M.I.Mech.E., Managing Director of Messrs Glenfield & Kennedy, Kilmarnock. Eldo House, Monkton, Prestwick	
1933		* Banerjee, Prabodh Chandra, L.R.C.P.E., L.R.C.S.E., F.R.F.P.S.G., F.A.C.S., Lt.-Col., I.M.S. C/o Lloyds Bank, Ltd., 101–1 Clive Street, Calcutta, India	
1938		* Bannerman, David Armitage, M.B.E., Sc.D., M.A. (Cantab.), Supernumerary Staff, Department of Zoology, British Museum (Nat. Hist.). 7 Pembroke Gardens, Kensington, London, W. 8	
1928		Barbour, George Brown, M.A. (Edin.), M.A. (Cantab.), Ph.D., F.G.S., Department of Geology, University of Cincinnati, Ohio, U.S.A.	
1886		Barclay, A. J. Gunion, M.A., 44A Belsize Park Gardens, Hampstead, London, N.W. 3	
1903		Bardswell, Noël Dean, M.V.O., M.D., F.R.C.P. (Lond.), M.R.C.P.E. (*no permanent address until further notice*)	
1929		* Barker, Sydney George, O.B.E., D.I.C., F.Inst.P., Scientific Advisor, Indian Jute Mills Association. 191 Coombe Lane, Wimbledon, London, S.W. 20	
1914	C.	Barkla, Charles Glover, M.A., D.Sc., F.R.S., Professor of Natural Philosophy, University of Edinburgh (Drummond Street), Nobel Laureate, Physics, 1917. Braidwood, Corrennie Gardens, Edinburgh 10	1915–18, 1924–27.
1937		* Barnett, Adam John Guilbert, B.Sc., Ph.D. (Edin.), Lecturer in Chemistry, Education Department, Lagos, Nigeria, B.W.A.	
1927		* Barnett, John, F.F.A., C.A., Scottish Widows' Fund Life Assurance Society, 9 St Andrew Square, Edinburgh 2	
1921		* Bartholomew, John, M.C., M.A., F.R.G.S., Geographical Institute, Duncan Street, Edinburgh. The Manor House, Inveresk	1925–28.
1927		* Bastow, Stephen Everard, M.Inst.E.E., M.Inst.Min.E. Northwood, Russell Place, Trinity, Edinburgh 5	
1929		* Bath, Frederick, Ph.D., Lecturer in Mathematics, University of Edinburgh (Drummond Street). 6 Forrester Road, Edinburgh 12	
1939		* Baxter, Rev. James Houston, D.D. (Glas.), D.Litt. (St Andrews), Regius Professor of Ecclesiastical History, University of St Andrews, St Mary's College. 71 South Street, St Andrews	
1936		* Bayliss, Leonard Ernest, B.A., Ph.D., Hooles' Farm, Vines Cross, E. Sussex	
1913		† Beard, Joseph, F.R.C.S.E., M.R.C.S., L.R.C.P. (Lond.), D.P.H. (Cantab.), formerly Medical Officer of Health, City of Carlisle. 8 Carlton Gardens, Carlisle	
1897	C.	Beattie, Sir John Carruthers, K.B., D.Sc., LL.D., formerly Vice-Chancellor and Principal, The University, Cape Town	
1893	C. M-B.	Becker, Ludwig, Ph.D., Emeritus Regius Professor of Astronomy, University of Glasgow. The Observatory, University Gardens, Glasgow	
1939		* Beevers, Cecil Arnold, D.Sc. (Liverpool), F.Inst.P., Dewar Fellow in Crystallography, University of Edinburgh (Chemistry Department, West Mains Road)	
1933		* Begg, James Livingstone, F.G.S. (Treasurer, Geological Society of Glasgow). Elms, Mount Vernon, Glasgow	
1916		Bell, Robert John Tainsh, M.A., D.Sc., LL.D. (Glas.), Professor of Mathematics, University of Otago, Dunedin, New Zealand	

Date of Election			Service on Council, etc.
1929		* **Bennet**, George, A.H.-W.C., B.Sc., A.M.I.Mech.E., Lecturer in Mechanical Engineering, Heriot-Watt College. 68 Arden Street, Edinburgh 10	
1936		**Berry**, John, M.A. (Cantab.), Ph.D. (St Andrews), Superintendent, River Forss Salmon Research, Caithness (Fisheries Division, Scottish Home Department). Tayfield, Newport, Fife	
1897	C.	**Berry**, Richard J. A., M.D., F.R.C.S.E., Director of Medical Services, Stoke Park Colony, Stapleton, Bristol. Rufford, Canford Lane, Westbury-on-Trym, Bristol	
1938		* **Beveridge**, Alexander William Morton, formerly Treasurer of the Bank of Scotland, and *ex officio* Chairman of the Managers of the Banks in Scotland. 44 Inverleith Place, Edinburgh 4	
1932		**Bhatia**, Sohan Lal, M.C., M.A., M.D. (Cantab.), F.R.C.P. (Lond.), Major, I.M.S., Professor of Physiology and Dean, Grant Medical College, Bombay. Two Gables, Mount Pleasant Road, Malabar Hill, Bombay, India	
1937		* **Biswas**, Kalipada, M.A., D.Sc.(Edin.), Superintendent Royal Botanic Garden, Sibpur, Calcutta, India	
1937		* **Blackie**, Joseph John, Ph.D. (Edin.), F.C.S., F.I.C., a Partner in Messrs Duncan, Flockhart & Co., Chemical Manufacturers, 104 Holyrood Road, Edinburgh 8	
1936		* **Blair**, Duncan MacCallum, M.B., Ch.B. (Glas.), F.R.F.P.S.G., D.Sc. (Lond.), Regius Professor of Anatomy, University of Glasgow. 2 The University, Glasgow, W. 2	
1933		* **Bolam**, Thomas Robert, M.Sc. (Bristol), D.Sc. (Edin.), Lecturer in Chemistry, University of Edinburgh (West Mains Road). 8 Wilton Road, Edinburgh 9	
1915		**Boon**, Alfred Archibald, D.Sc., B.A., F.I.C., Emeritus Professor of Chemistry, Heriot-Watt College, Edinburgh. 87 Warrender Park Road, Edinburgh 9	
1937	C.	* **Born**, Max, M.A. (Cantab.), Hon. D.Sc. (Bristol), Dr.phil. (Göttingen), F.R.S., Tait Professor of Natural Philosophy, University of Edinburgh (Drummond Street). 84 Grange Loan, Edinburgh 9	
1925		* **Bose**, Sahay Ram, M.A., D.Sc., F.N.I. (India), Professor of Botany, Carmichael Medical College, Belgachia, Calcutta, India	
1886	C. N.	**Bower**, Frederick Orpen, M.A., D.Sc., LL.D., F.R.S., F.L.S., Emeritus Regius Professor of Botany, University of Glasgow. 2 The Crescent, Ripon, Yorks	1887–90, 1893–96, 1907–09, 1917–19. V-P 1910–16. P 1919–24.
1940		* **Bowman**, John, M.Inst.C.E., City Water Engineer, Edinburgh. Swanston Cottage, Edinburgh 10	
1926		* **Braid**, Kenneth William, M.A. (Cantab.), B.Sc., Professor of Botany, West of Scotland Agricultural College, 6 Blythswood Square, Glasgow	
1940		* **Bramwell**, Byrom Stanley, B.A. (Cantab.), LL.B. (Edin.), Advocate. 55 Hampstead Way, London, N.W. 11	
1907		† **Bramwell**, Edwin, M.D., LL.D., F.R.C.P. (Edin. and Lond.), formerly Professor of Clinical Medicine, University of Edinburgh. 23 Drumsheugh Gardens, Edinburgh 3	1935–38.
1932		* **Brash**, James Couper, M.C., M.A., M.B., Ch.B. (Edin.), M.D. (Birm.), Professor of Anatomy, University of Edinburgh (Teviot Place)	1932–35.
1918		* **Bremner**, Alexander, M.A., D.Sc., formerly Headmaster, Demonstration School, Training Centre, Aberdeen. 13 Belgrave Terrace, Aberdeen	
1893		**Brock**, G. Sandison, M.B.E., M.D., F.R.C.P.E., c/o Lloyd's Bank, 16 St James's Street, London, S.W. 1	
1938		* **Brook**, Geoffrey Bernard, D.Sc. (Vet. Sci.), F.R.C.V.S., Health of Animals Division, Ministry of Agriculture, 22–25 Soho Square, London, W.C. 1	
1937		* **Brook**, George Bernard, F.I.C., F.C.S., formerly Chief Chemist to the British Aluminium Co., Ltd., London. Erracht, Roslin, Midlothian	
1940		* **Brough**, James, D.Sc. (Manch.), Lecturer in the Zoology of Vertebrates, University of Edinburgh (West Mains Road). 7 Westgarth Avenue, Colinton	
1934		* **Brough**, Patrick, M.A., B.Sc. (Glas.), D.Sc. (Sydney), Lecturer in Botany, University of Sydney, N.S.W.	
1907		**Brown**, Alexander, M.A., B.Sc., Professor of Applied Mathematics, University, Cape Town	

Date of Election			Service on Council, etc.
1936		* Brown, Andrew Johnstone, F.R.C.S.E., L.R.C.P.E., L.D.S. (Edin.), Lecturer on Metallurgy, Tutor in charge of Anæsthetic Department and Demonstrator in Radiology, Edinburgh Dental Hospital and School. 51 Minto Street, Edinburgh 9	
1937		* Brown, Archibald Gray Robertson, F.F.A., Manager and Actuary of the Life Association of Scotland. Barnshalloch, 30 Gillespie Road, Colinton, Edinburgh 13	
1928		* Brown, Hugh Wylie, F.I.A., F.F.A., 1 Cobden Crescent, Edinburgh 9	
1924	C.	* Brown, Thomas Arnold, M.A., B.Sc., Professor of Mathematics, University College, Exeter	
1923		* Brown, Walter, M.A., B.Sc., Professor of Mathematics, University, Hong Kong, China	
1935		* Brownlie, James Law, M.D. (Glas.), D.P.H., M.R.C.P.E., formerly Chief Medical Officer, Department of Health for Scotland. C/o Mr Ross, Rannoch, 17 Ophir Road, Bournemouth, Hants	
1921		* Bruce, Alexander, B.Sc. (Edin.), Government Agricultural Analyst and City Analyst for Colombo, Kandy and Galle, The Laboratory, Green Path, Colpetty, Colombo, Ceylon	
1912		Bruce, Alexander Ninian, D.Sc., M.D., 8 Ainslie Place, Edinburgh 3	
1936		* Bruce, Sir Robert, Kt., D.L., J.P., LL.D., formerly Editor of the *Glasgow Herald*. Brisbane House, 9 Rowan Road, Glasgow, S. 1	
1898	C. K.	† Bryce, Thomas Hastie, M.A., M.D. (Edin.), LL.D., F.R.S., Emeritus Professor of Anatomy, University of Glasgow. The Loaning, Peebles	1911–14, 1922–25, 1935–38. V-P 1925–28.
1936		* Bryden, William, M.Sc., B.A., Ph.D. (Edin.), Knox Grammar School, Wahroonga, Sydney, N.S.W., Australia	
1940		* Buchan, Alexander Fairley, B.Sc., Ph.D. (Edin.), Teacher of Mathematics, James Gillespie's School, Edinburgh. 47 House o' Hill Road, Blackhall, Edinburgh 4	
1917		* Burnside, George Barnhill, M.Inst.Mech.E., A.F.R.Ae.S., Fairhill, Dullatur, Dumbartonshire	
1930	C.	* Burt, David Raitt Robertson, B.Sc. (St Andrews), F.L.S., Professor of Zoology, Ceylon University College, Colombo	
1896		Butters, John W., M.A., B.Sc., formerly Rector of Ardrossan Academy. 116 Comiston Drive, Edinburgh 10	
1940		* Caie, John Morrison, M.A., B.Sc., B.L. (Aberd.), Deputy Secretary, Department of Agriculture for Scotland. 2 Cobden Road, Edinburgh 9	
1929	C.	* Calder, Alexander, Ph.D., Chief Marketing Officer, Pig Marketing Board, Thames House, Millbank, London, S.W. 1	
1910		Calderwood, Rev. Robert Sibbald, D.D., formerly Minister of Cambuslang. 84 Findhorn Place, Edinburgh 9	
1893	C.	Calderwood, William Leadbetter, I.S.O., formerly Inspector of Salmon Fisheries of Scotland. Ardnacoille, Carr Bridge, Inverness-shire	1923–26.
1926	C. M-B.	* Cameron, Alfred Ernest, M.A., D.Sc. (Aberd.), Steven Lecturer in Agricultural and Forest Zoology, University of Edinburgh (10 George Square). 8 West Savile Road, Edinburgh 9	
1933		* Cameron, Finlay James, F.F.A., F.I.A., General Manager, Caledonian Insurance Company. Beech Knowe, Barnton, near Edinburgh	
1905	C.	Cameron, John, M.D., D.Sc., M.R.C.S. (Eng.), formerly Professor of Anatomy, Dalhousie University, Halifax, Nova Scotia. Balmashanner, Grove Road, East Cliff, Bournemouth	
1940		Cameron, John, F.C.S., F.R.G.S., formerly Head of the Pharmaceutical Department, Peiping Union Medical College, Peiping, China. 20 Kingslynn Drive, King's Park, Glasgow, S. 4	
1921		* Campbell, Andrew, Advisory Chemist, c/o Burmah Oil Co., Ltd., Research Laboratory, Fairlawn, Honor Oak Road, Forrest Hill, London, S.E. 30 Foxgrove Road, Beckenham, Kent	
1918		* Campbell, John Menzies, D.D.S. (Toronto), L.D.S. (Glas.), L.D.S. (Ontario), F.I.C.D., 14 Buckingham Terrace, Glasgow, W.	
1915	C. N.	Campbell, Robert, M.A., D.Sc., F.G.S., Reader in Petrology, University of Edinburgh (Grant Institute of Geology, West Mains Road). Maryton, Colinton	1920–23.

Date of Election			Service on Council, etc.
1927	C.	* Cannon, Herbert Graham, M.A., Sc.D. (Cantab.), D.Sc. (Lond.), M.Sc. (Manc.), F.R.S., F.L.S., Beyer Professor of Zoology, University of Manchester. *Hollin Knowle, Chapel-en-le-Frith, Derbyshire*	
1938		* Carlow, Charles Augustus, M.Inst.C.E., M.I.M.E., Managing Director, Fife Coal Co., Ltd. Linnwood Hall, Leven, Fife	
1910		Carnegie, Col. David, C.B.E., J.P., M.Inst.C.E., The Haven, Seasalter, Whitstable	
1931		* Carroll, John Anthony, M.A., Ph.D. (Cantab.), Professor of Natural Philosophy, University of Aberdeen. Marischal College, Aberdeen	
1905	C.	Carse, George Alexander, M.A., D.Sc., Reader in Natural Philosophy, University of Edinburgh (Drummond Street). 3 Middleby Street, Edinburgh 9	
1901		Carslaw, Horatio Scott, M.A., Sc.D. (Cantab.), D.Sc., LL.D. (Glas.), Emeritus Professor of Mathematics, University of Sydney, New South Wales. Burradoo, New South Wales	
1933		* Carswell, John Irvine, B.Sc., Ph.D., A.M.Inst.C.E., A.M.Inst.Mech.E., Lecturer in Engineering, University of Edinburgh (Sanderson Engineering Laboratory, Mayfield Road). 43 Mansionhouse Road, Edinburgh 9	
1925		* Carter, George Stuart, M.A., Ph.D., Corpus Christi College, Cambridge	
1932		* Cathcart, Edward Provan, C.B.E., M.D., D.Sc., LL.D., F.R.S., Professor of Physiology, University of Glasgow. 28 Hillhead Street, Glasgow	
1939		* Chakravarti, Amulyaratan, M.B., B.Sc. (Cal.), M.R.C.P.E., Research Fellow and Consulting Biochemist, B.I. Research Laboratories, Calcutta. 1 Furriapooker Street, Calcutta, India	
1932	C.	* Childe, Vere Gordon, B.A., B.Litt., D.Litt. (Harvard), Hon. D.Sc. (Pennsylvania), F.B.A., F.R.A.I., F.S.A., Professor of Prehistoric Archæology, University of Edinburgh	
1925	C.	* Chumley, James, M.A., Ph.D., formerly Lecturer in Oceanography, Department of Zoology, University of Glasgow. Thalassa, Milton Road East, Portobello, Midlothian	
1928	C.	* Clark, Alfred Joseph, M.C., B.A., M.D., F.R.S., Professor of Materia Medica, University of Edinburgh (Teviot Place). 67 Braid Avenue, Edinburgh 10	1932–35.
1933		* Clark, Arthur Melville, M.A. (Edin.), D. Phil. (Oxon.), Lecturer in English Literature, University of Edinburgh. 34 Bruntsfield Gardens, Edinburgh 10	
1891		Clark, John Brown, C.B.E., J.P., M.A., LL.D., formerly Head Master of George Heriot's School. Garleffin, 146 Craiglea Drive, Edinburgh 10	1928–1931. V-P 1931–34.
1935		* Clark, Robert Selbie, M.A., D.Sc. (Aberd.), Scientific Superintendent (Fisheries Division, Scottish Home Department). The Cottage, Murtle, Aberdeenshire	
1932		* Clark, Sir Thomas, Bart., Publisher, Head of T. & T. Clark, Ltd. 6 Wester Coates Road, Edinburgh 12	
1909		Clayton, Thomas Morrison, M.D., D.Hy., B.Sc., D.P.H., Medical Officer of Health, Greenesfield House, Gateshead-on-Tyne	
1932		* Clouston, David, C.I.E., M.A., B.Sc. (Agric.), D.Sc., F.R.S.G.S., formerly Director, Imperial Agricultural Research Institute, Pusa. Forthview, Boswall Road, Edinburgh 5	
1936	C.	* Cockburn, Alexander Murray, Ph.D. (Edin.), Assistant, Geological Department, University of Edinburgh (Grant Institute of Geology, West Mains Road). 53 Ladysmith Road, Edinburgh 9	
1904	C.	Coker, Ernest George, M.A. (Cantab.), D.Sc. (Edin.), Hon. D.Sc. (Sydney and Louvain), M.Sc. (McGill), F.R.S., M.Inst.C.E., M.I.Mech.E., Emeritus Professor of Civil and Mechanical Engineering, University of London. Engineering Laboratories, 3 Farnley Road, Chingford, London, E. 4	
1904		Coles, Alfred Charles, M.D., D.Sc., York House, Poole Road, Bournemouth, W.	
1888	C. V. J.	Collie, John Norman, Ph.D., D.Sc., LL.D., F.R.S., F.C.S., F.I.C., Emeritus Professor of Organic Chemistry, University College, Gower Street, London. 20 Gower Street, London, W.C. 1	
1938		* Colquhoun, Sir Iain, K.T., Baronet of Colquhoun and Luss, D.S.O., LL.D., Rector, University of Glasgow, 1934–37. Rossdu, Luss	
1909	C.	Comrie, Peter, M.A., B.Sc., LL.D., formerly Rector, Leith Academy. 19 Craighouse Terrace, Edinburgh 10	
1940		* Cook, James Wilfred, D.Sc., Ph.D. (Lond.), F.R.S., F.I.C., Professor of Chemistry, University of Glasgow	

Date of Election			Service on Council, etc.
1936		* Cooper, The Rt. Hon. Thomas Mackay, O.B.E., K.C., M.P., M.A., LL.B., *His Majesty's Advocate for Scotland.* 16 *Hermitage Drive, Edinburgh* 10	
1924	C.	* Copson, Edward Thomas, M.A. (Oxon.), D.Sc. (Edin.), Professor of Mathematics, University College, Dundee (University of St Andrews). 14 Balmyle Road, Broughty Ferry, Dundee	
1937		* Cousland, Charles Johnstone, A.R.P.S., formerly President, Royal Scottish Society of Arts. Achray, Kinnear Road, Edinburgh 4	
1928		* Coutie, Rev Alexander, B.Sc., Ph.D., Minister, Church of Scotland, Watten, Caithness	
1940		* Coutts, James, Ph.D. (Glas.), Ph.C., Lecturer in Pharmaceutics and Demonstrator in Practical Pharmacy, St Bartholomew's Hospital Medical College, and Pharmaceutist to the Hospital, St Bartholomew's Hospital, London, E.C. 1	
1914		Coutts, William Barron, M.A., B.Sc., F.Inst.P., Assistant Professor of Fire Control Instruments, Military College of Science, Woolwich, S.E. 18. Craigmillar, 70 Cambridge Drive, London, S.E. 12	
1911		Cowan, Alexander, M.A. (Cantab.), Papermaker, Valleyfield, Penicuik, Midlothian	
1931		* Cowan, John Macqueen, M.A., D.Sc. (Edin.), B.A. (Oxon.), F.L.S., Assistant Keeper, Royal Botanic Garden, Edinburgh. *17 Inverleith Place, Edinburgh* 4	
1916	C.	† Craig, E. H. Cunningham, B.A. (Cantab.), Geologist and Mining Engineer, The Dutch House, Beaconsfield	
1908		Craig, James Ireland, M.A., B.A., 88 Sharia Kasr el Eini, Cairo, Egypt	
1925	C. K.	* Craig, Robert Meldrum, M.A., D.Sc., F.G.S., Lecturer in Economic Geology, University of Edinburgh (Grant Institute of Geology, West Mains Road)	
1937		* Craig, William Stuart McRae, B.Sc. (Glas.), M.D. (Edin.), F.R.C.P.E., Medical Officer. *Temporary Address*, Ministry of Health, Regional Offices, The Chestnuts, Grove Hill Road, Tunbridge Wells, Kent. Mellendean, Harriots Lane, Ashtead, Surrey	
1933		* Craig-Bennett, Arthur Lancelot, M.A., Ph.D. (Cantab.), formerly Chief Fisheries Officer, Haifa, Palestine. 1 High Street, Musselburgh	
1903		† Crawford, Lawrence, M.A. (Cantab.), D.Sc. (Glas.), LL.D. (Wit.), formerly Professor of Pure Mathematics, University, Cape Town. 21 Pillans Road, Rosebank, Cape Town	
1922	C. K.	* Crew, Francis Albert Eley, M.D., D.Sc., Ph.D., F.R.S., Professor of Animal Genetics and Director of the Institute of Animal Genetics, University of Edinburgh (West Mains Road). 41 Mansionhouse Road, Edinburgh 9	1928–31. Sec. 1931–36. V-P 1936–39.
1931		* Crichton, John, M.A., B.Sc. (Edin.), The Observatory, Eskdalemuir, Langholm	
1936		* Cronshaw, Cecil John Turrell, B.Sc. (Vict.), Hon. D.Sc. (Leeds), F.I.C., Chairman, Dyestuffs Group of Imperial Chemical Industries, Ltd. (including Scottish Dyes, Ltd.), Hexagon House, Blackley, Manchester 9	
1929		* Cruickshank, Ernest William Henderson, M.D., D.Sc., Ph.D., Regius Professor of Physiology, University of Aberdeen	
1938		* Cruickshank, Martin Melvin, M.D., Ch.M. (Aberd.), F.R.C.S.E., D.O.M.S. (Lond.), Lieut.-Col. I.M.S. Professor of Surgery, Madras Medical College. Pantheon House, Pantheon Road, Madras	
1928		* Cumming, William Murdoch, D.Sc. (Glas.), F.I.C., M.Inst.Chem.E., "Young" Professor of Technical Chemistry, Royal Technical College, Glasgow. Bonnieblink, 4 Newlands Road, Newlands, Glasgow, S. 3	
1917		* Cunningham, Brysson, D.Sc., B.E., M.Inst.C.E., Editor *The Dock and Harbour Authority*, formerly Lecturer in Waterways, Harbours, and Docks, University College, London. 13 Jevington Gardens, Eastbourne, Sussex	
1930		* Cunningham, John, C.I.E., B.A., M.D., Lt.-Colonel, I.M.S. (retired). South Bank, Grange Loan, Edinburgh 10	
1938		* Cursiter, Stanley, O.B.E., R.S.W., R.S.A., Director of National Galleries of Scotland, Brunstane House, Portobello, Midlothian	
1934		* Daly, Ivan de Burgh, M.A., M.D., B.Ch., Professor of Physiology, University of Edinburgh (Teviot Place). Cooliney, Spylaw Avenue, Edinburgh	1935–38.
1934		* Darling, Frank Fraser, Ph.D. (Edin.), N.D.A., Isle of Tanera, Achiltibuie, Garve, Ross and Cromarty	

Date of Election			Service on Council, etc.
1924	V. J.	* Darwin, Charles Galton, M.C., M.A., Sc.D., LL.D., F.R.S., formerly Tait Professor of Natural Philosophy, University of Edinburgh, and Master of Christ's College, Cambridge. Director, National Physical Laboratory, Teddington, Middlesex. Bushy House, Teddington, Middlesex	1925–28. Sec. 1928–33. V-P 1933–36.
1940		* Datta, Satchidinanda, B.Sc. (Cal.), D.Sc. (Edin.), M.R.C.V.S., D.T.V.M. (Edin.), Veterinary Research Officer, Imperial Veterinary Research Institute, Mukteswar, 98 Beltala Road, Kalighat, Calcutta, India	
1935	C.	* Davidson, Charles Findlay, B.Sc. (St Andrews), F.G.S., Geologist, H.M. Geological Survey of Great Britain, Geological Survey and Museum, Exhibition Road, South Kensington, London, S.W. 7	
1939		* Davidson, James, F.S.A.Scot., Treasurer of the Carnegie Trust for the Universities of Scotland. 59 Morningside Park, Edinburgh 10	
1932		* Davidson, Leybourne Stanley Patrick, B.A. (Cantab.), M.D. (Edin.), F.R.C.P.E., Professor of Medicine, University of Edinburgh (Teviot Place)	
1935		* Davidson, Maxwell, B.Sc. Eng., Ph.D., M.I.Mech.E., Lecturer in Heat Engines and Thermodynamics, University of Edinburgh (Sanderson Engineering Laboratory, Mayfield Road). 47 Liberton Drive, Edinburgh 9	
1939		* Davies, Francis, M.D., D.Sc. (Lond.), B.S., M.R.C.S. (Eng.), L.R.C.P. (Lond.), Arthur Jackson Professor of Anatomy, University of Sheffield. 5 Tapton House Road, Sheffield, 10	
1930	C.	* Davies, Lewis Merson, M.A., Ph.D., F.G.S., F.R.A.I., Lt.-Colonel, Royal Artillery (retired). 8 Garscube Terrace, Murrayfield, Edinburgh 12	
1928		Dawson, Warren Royal, F.R.S.L., F.S.A., F.S.A.Scot., Honorary Librarian of Lloyd's, London, E.C. 3. Simpson House, Simpson, Bletchley, Bucks	
1923		* Deane, Arthur, M.R.I.A., Curator, Public Art Gallery and Museum, Belfast. Beach Road, Whitehead, Belfast	
1935	C.	* Desai, Bhimbhai Nichhabhai, B.A., M.Sc. (Bombay), D.Sc., Ph.D. (Edin.), Assistant Meteorologist, Government of India, Meteorological Office, Poona 5, India	
1938		* Dewar, John Michael, M.D. (Edin.), F.R.C.P.E. 5 Chalmers Street, Edinburgh 3	
1937		* Dhar, Sasindra Chandra, D.Sc. (Edin.), Head of the Department of Mathematics, College of Science, Nagpur, C.P., India	
1940		* Dickson, Thomas Elder, M.A. (Glas.), Ph.D. (Edin.), Art Master, George Watson's Ladies' College. 12 Corstorphine Bank Terrace, Edinburgh 12	
1924		* Dinham, C. H., B.A., H.M. Geological Survey. Edgemoor, 19 Highfield Road, Northwood, Middlesex	
1923		* Dixon, Ronald Audley Martineau, of Thearne, F.G.S., F.S.A.Scot., F.R.G.S., Thearne Hall, near Beverley, Yorks	
1881	C.	Dobbin, Leonard, Ph.D. (VICE-PRESIDENT), formerly Reader in Chemistry, University of Edinburgh. Faladam, Blackshiels, Midlothian	1904–07, 1913–16. Curator 1934–39. V-P 1939–
1918		* Dodd, Alexander Scott, Ph.D., F.I.C., F.C.S., City Analyst for Edinburgh. 20 Stafford Street, Edinburgh 3	
1940		* Dollar, Archibald Thomas John, B.Sc. (Lond.), Ph.D. (Cantab.), F.G.S., Assistant in the Department of Geology, University of Glasgow. 22 Southpark Avenue, Glasgow, W. 2	
1902		Dollar, John A. W., F.R.C.V.S., 72 Maida Vale, London, W. 9	
1925		* Donald, Alexander Graham, M.A., F.F.A., F.S.A.Scot., Secretary of the Scottish Provident Institution, Edinburgh. 18 Carlton Terrace, Edinburgh 7	1939–
1937	C.	* Donald, Hugh Paterson, M.Agr.Sc. (N.Z.), D.Sc. (Edin.), Lecturer, Institute of Animal Genetics, University of Edinburgh (West Mains Road). Hill Cottage, West Mains Road, Edinburgh 9	
1882	C.	Dott, David B., F.I.C., Memb. Pharm. Soc., Ravenslea, Musselburgh	
1938		* Dott, Norman McOmish, F.R.C.S.E., Neurological Surgeon to the Royal Infirmary and Lecturer on Neurological Surgery, University of Edinburgh. 3 Chalmers Crescent, Edinburgh 9	
1921	C. M-B.	* Dougall, John, M.A., D.Sc., 47 Airthrey Avenue, Glasgow, W. 4	
1910		Douglas, Loudon MacQueen, Newpark, Mid-Calder, Midlothian	

Date of Election			Service on Council, etc.
1934		* Dow, David Rutherford, M.D., D.P.H., F.R.C.P.E., Professor of Anatomy, University of St Andrews (University College, Dundee). 13 Marketgate, Crail	
1932		* Drennan, Alexander Murray, M.D. (Edin.), F.R.C.P.E., Professor of Pathology, University of Edinburgh (Teviot Place)	
1940	C.	* Drever, Harold Irving, B.Sc. (Edin.), Ph.D. (Cantab.), Assistant, Department of Geology, University of St Andrews	
1923	C.	* Drever, James, M.A., B.Sc., D.Phil., Professor of Psychology, University of Edinburgh (South Bridge). Ivybank, Wardie Road, Edinburgh 5	1929–32.
1923		* Drummond, J. Montagu F., M.A. (Cantab.), Harrison Professor of Botany, University of Manchester	1928–31.
1925		* Dryerre, Henry, M.R.C.S. (Eng.), Ph.D., Professor of Physiology, Royal (Dick) Veterinary College; Physiological Biochemist, Animal Diseases Research Association. Kenmore, Lasswade	
1921		* Drysdale, Charles Vickery, C.B., O.B.E., D.Sc. (Lond.), M.I.E.E., F.Inst.P., formerly Director of Scientific Research to the Admiralty. 151 Wick Hall, Furge Hill, Hove, Sussex	
1937		* Dunlop, Derrick Melville, B.A. (Oxon.), M.D. (Edin.), F.R.C.P.E., Christison Professor of Therapeutics, University of Edinburgh (Teviot Place). 14 Moray Place, Edinburgh 3	
1904		Dunlop, William Brown, M.A., 4A St Andrew Square, Edinburgh 2. Seton Castle, Longniddry, E. Lothian	
1933		* Dymond, Edmund Gilbert, M.A., Lecturer in Natural Philosophy, University of Edinburgh (Drummond Street), Kew Observatory, Richmond, Surrey. 7 Greenhill Gardens, Edinburgh 10	
1934		* Edge, William Leonard, M.A., Sc.D. (Cantab.), Lecturer in Mathematics, University of Edinburgh (Drummond Street)	
1931		* Eggleton, Philip, D.Sc., Lecturer in Biochemistry, Department of Physiology, University of Edinburgh (Teviot Place)	
1924		* Elliot, Right Hon. Walter Elliot, P.C., M.C., M.B., Ch.B., D.Sc., LL.D., M.P., F.R.S., formerly Minister of Health. 60 Eaton Square, London, S.W. 1	
1938		* Elphinstone, The Right Hon. Lord, K.T., LL.D., *formerly President*, Royal Scottish Geographical Society. Carberry Tower, Musselburgh, Midlothian	
1933		* Erskine, John Maxwell, General Manager of the Commercial Bank of Scotland, Ltd. Hazlieburn, West Linton	
1934	C.	* Etherington, Ivor Malcolm Haddon, B.A. (Oxon.), Ph.D. (Edin.), Lecturer in Technical Mathematics, University of Edinburgh (Drummond Street). 41 Scotland Street, Edinburgh 3	
1924		* Evans, Arthur Humble, M.A., Sc.D., Lecturer in English History. Cheviot House, Crowthorne, Berks	
1924		Evans, William Edgar, B.Sc., Assistant in charge of Herbarium, Royal Botanic Garden, Edinburgh 4. 38 Morningside Park, Edinburgh 10	
1902		Ewen, John Taylor, O.B.E., J.P., B.Sc., M.I.Mech.E., H.M. Inspector of Schools (Emeritus), Pitscandly, Forfar	
1939		* Eyles, Victor Ambrose, B.Sc. (Bristol), Senior Geologist, H.M. Geological Survey (Scotland), 19 Grange Terrace, Edinburgh 9. 27 Mentone Terrace, Edinburgh 9	
1900	C.	Eyre, John William Henry, M.D., M.S. (Dunelm), D.P.H. (Cantab.), Emeritus Professor of Bacteriology, Guy's Hospital. 51 Portland Place, London, W.1	
1931		* Fairbairn, William Ronald Dodds, M.A., M.D., D.Psych. (Edin.), F.R.A.I., 18 Lansdowne Crescent, Edinburgh 12	
1936		* Fairweather, James Falconer, W.S., N.P., Fiscal of the Society of Writers to the Signet. 14 Henderland Road, Edinburgh 12	
1907	C.	Falconer, John Downie, M.A., D.Sc., F.G.S., formerly Director of the Geological Survey of Nigeria. The Cedars, Hatton Road, Feltham, Middlesex	
1928		* Fenton, Edward Wyllie, M.A., D.Sc. (Aberd.), F.L.S., Head of Botany Department, Edinburgh and East of Scotland College of Agriculture, 13 George Square, Edinburgh 8	
1907		Fergus, Edward Oswald, c/o Messrs M'Kay & Boyd, Solicitors, 50 Wellington Street, Glasgow	
1933		* Ferguson, Thomas, M.D., D.P.H. (Edin.), F.R.C.P.E., D.Sc., Deputy Chief Medical Officer, Department of Health for Scotland. 32 Learmonth Street, Falkirk	
1932		* Findlay, Sir John Edmund Ritchie, Bart., B.A. (Oxon.), M.P., Proprietor of the *Scotsman*. Aberlour House, Aberlour, Banffshire	

Date of Election			Service on Council, etc.
1927	C.	* Finlay, Thomas Matthew, M.A., D.Sc. (Edin.), Lecturer in Palaeontology, University of Edinburgh (Grant Institute of Geology, West Mains Road). 11 Dudley Terrace, Leith 6	
1911		† Fleming, John Arnold, F.C.S., Pottery Manufacturer, Locksley, Helensburgh, Dumbartonshire	
1906		Fleming, Robert Alexander, M.A., M.D., LL.D., F.R.C.P.E., Consulting Physician, Royal Infirmary. 10 Chester Street, Edinburgh 3	
1900	C. N.	Flett, Sir John S., K.B.E., M.A., D.Sc., LL.D., F.R.S., formerly Director of the Geological Survey of Great Britain and Museum of Practical Geology, Exhibition Road, South Kensington, London, S.W. 7	1916–19.
1892		Ford, John Simpson, F.I.C., 7 Corrennie Drive, Edinburgh 10	
1928	C.	* Forrest, James, M.A., B.Sc. (Glas.), D.Sc. (St Andrews), Lecturer in Physics, University College, Dundee. Cumbrae, Oxford Street, Blackness, Dundee	
1933		* Forrester, Charles, A.H.-W.C., Ph.D., F.I.C., F.Inst.F., Principal and Professor of Chemistry, Indian School of Mines, Dhanbad, India	
1920	C.	* Franklin, Thomas Bedford, M.A. (Cantab.), 28 Kingshill Drive, Kenton, Harrow, Middlesex	
1910		Fraser, Alexander, Actuary, 5 St Margaret's Road, Edinburgh 9	
1929		* Fraser, David Kennedy, M.A., B.Sc., Psychologist to Glasgow Education Authority. Edge o' the Moor, Milngavie, Dumbartonshire	
1934		* Fraser, George, Chartered Civil Engineer, M.Inst.C.E., M.I.Struct.E. 25 Murrayfield Gardens, Edinburgh 12	
1939		* Fraser, Ian, M.D., M.Ch., F.R.C.S. (Eng.), Consultant Surgeon. 33 Wellington Park, Belfast	
1928		* Fraser, Sir John, K.C.V.O., M.C., M.D., Ch.M., F.R.C.S.E., Regius Professor of Clinical Surgery, University of Edinburgh (Royal Infirmary). 20 Moray Place, Edinburgh 3	
1928		* Fraser, Kenneth, M.D. (Edin.), D.P.H. (Cantab.), D.T.M. (Edin.), County Medical Officer of Health, Cumberland. The Croft, Scotby, near Carlisle	
1914		Fraser, William, F.S.A.Scot., formerly Managing Director, Neill & Co., Ltd., Printers. Scottish Conservative Club, Edinburgh 2	
1907		Galbraith, Alexander M., 6 Broomhill Avenue, Glasgow, W. 1	
1933		* Galbraith, Augustus William de Rohan, M.Inst.C.E., M.Inst.C.E.I., F.S.E., Chairman of the Advisory Council of the New Zealand Standards Institute. The Spur, Clifton, Sumner, New Zealand	
1939		* Galloway, John Galloway, J.P., F.I.S.I., Master of the Edinburgh Merchant Company. 7 Stirling Road, Edinburgh 5	
1901		Ganguli, Sanjiban, M.A., formerly Principal, Maharaja's College, and Director of Public Instruction, Jaipur State, Hathroi-Garh Road, Jaipur (Rajputana), India	
1933		* Gardner, Alfred Charles, M.Inst.C.E., M.Inst.Mech.E., M.Inst.E.E., F.G.S., Chief Engineer, Clyde Navigation Trust, 16 Robertson Street, Glasgow. 117 Fotheringay Road, Glasgow, S. 1	
1926		* Gardner, John Davidson, B.Sc., A.M.Inst.C.E., Partner, Gardner & Oswald, Civil Engineers, 84 George Street, Edinburgh 2	
1937		* Garry, Robert Campbell, M.B., Ch.B., D.Sc. (Glas.), Professor of Physiology, University College (University of St Andrews), Dundee. 6 Windsor Street, Dundee	1938–
1930	C.	* Geddes, Alexander Ebenezer M'Lean, O.B.E., M.A., D.Sc., Lecturer in Natural Philosophy, University of Aberdeen. 12 Louisville Avenue, Aberdeen	
1909	C.	† Geddes, Rt. Hon. Sir Auckland Campbell, P.C., G.C.M.G., K.C.B., M.D., LL.D., Frensham, The Layne, Rolvenden, Kent	
1909		† Gentle, William, B.Sc., Head Master, George Heriot's School. 10 West Savile Road, Edinburgh 9	
1939		* Ghosh, Birendra Nath, M.B.E., F.R.F.P.S.G., L.M. (Dub.), Professor of Pharmacology, Carmichael Medical College, Calcutta. 9 Taltolla Lane, Calcutta, India	
1914		Gibb, Sir Alexander, G.B.E., C.B., LL.D., F.R.S., M.Inst.C.E., Queen Anne's Lodge, Westminster, London, S.W. 1	
1910	C.	Gibb, David, M.A., B.Sc., Reader in Mathematics, University of Edinburgh (Drummond Street). 45 Fountainhall Road, Edinburgh 9	
1936		* Gibbons, Sydney Guy, Ph.D. (Lond.), Naturalist, Fisheries Division, Scottish Home Department, Aberdeen. 56 Union Street, Aberdeen	

Date of Election			Service on Council, etc.
1917	C.	* Gibson, Alexander, M.B., Ch.B., F.R.C.S. (Eng.), 620 Medical Arts Building, Winnipeg, Canada	
1921		* Gibson, Walcot, D.Sc., F.R.S., F.G.S., formerly Assistant Director, H.M. Geological Survey (Scotland). Pathways, Fairlight Road, Hythe, Kent	
1911		Gidney, Sir Henry A. J., Kt., J.P., M.L.A., F.R.C.S.E., Lt.-Col., I.M.S. (retired), c/o The Allahabad Bank, Ltd., Calcutta, India	
1938		* Gifford, Charles Henry Pearson, M.A. (Cantab.), Partner in Baillie, Gifford & Co., 3 Glenfinlas Street, Edinburgh 3. 32 Stafford Street, Edinburgh 3	
1937		* Gilchrist, Andrew Rae, M.D. (Edin.), F.R.C.P.E., M.R.C.P. (Lond.), Assistant Physician, Royal Infirmary, Edinburgh, and Lecturer on Therapeutics, University of Edinburgh (Teviot Place). 6 Lansdowne Crescent, Edinburgh 12	
1933		* Gillespie, Robert Pollock, M.A., B.Sc., Ph.D. (Cantab.), Lecturer in Mathematics, University of Glasgow, 51 Clouston Street, Glasgow, N.W.	
1933		* Gillespie, Thomas Haining, Director-Secretary, Zoological Society of Scotland. Corstorphine Hill House, Murrayfield, Edinburgh 12	
1909		Gladstone, Hugh Steuart, M.A., M.B.O.U., F.Z.S., Capenoch, Penpont, Dumfriesshire	
1911		Gladstone, Reginald John, M.D., F.R.C.S. (Eng.), formerly Reader in Anatomy, King's College, University of London. 22 Court Lane Gardens, London, S.E. 21	
1934		* Glaister, John, M.D., D.Sc. (Glas.), Professor of Forensic Medicine, University of Glasgow. 43 Victoria Crescent Road, Glasgow, W. 2	
1925	C.	* Goldie, Archibald Hayman Robertson, M.A., B.A., D.Sc., Frogmarsh Lodge, Woodchester, Stroud, Glos	1929–32.
1901		Goodwillie, James, M.A., B.Sc., 239 Clifton Road, Aberdeen	
1938		* Gordon, Cecil, M.Sc. (Cape Town), Ph.D. (Lond.), Lecturer in Genetics, Department of Natural History, University of Aberdeen	
1938		* Gordon, William Smith, Ph.D., M.R.C.V.S., Chief Bacteriologist, Moredun Institute, Animal Diseases Research Association, Gilmerton. 2 Northfield, Liberton, Edinburgh	
1913	C. M-B.	Gordon, William Thomas, M.A., D.Sc. (Edin.), M.A. (Cantab.), Professor of Geology, University of London, King's College, Strand, W.C. 2	
1897		Gordon-Munn, John Gordon, M.D., Croys, Castle Douglas	
1934		* Gorrie, Robert Maclagan, D.Sc. (Edin.), Silviculturalist to Punjab Government. Jubbulpore, C.P., India	
1938		* Goyal, Ram Kumar, M.B., B.S. (Punjab), M.R.C.P. (Lond.), M.R.C.S. (Eng.), Ph.D. (Edin.), Central Research Institute, Kasauli, Punjab, India	
1923		* Grabham, George Walter, O.B.E., M.A. (Cantab.), F.G.S., formerly Government Geologist, Anglo-Egyptian Sudan. English Rooms, Funchal, Madeira, via Lisbon	
1924	C.	Graham, Robert James Douglas, M.A., D.Sc. (SECRETARY TO ORDINARY MEETINGS), Professor of Botany, University of St Andrews	1938–39. Sec. 1939–
1931		* Grant, Robert, J.P., Publisher (Oliver & Boyd), Edinburgh. 6 Kilgraston Road, Edinburgh 9	
1918		* Gray, William Forbes, F.S.A.Scot., 8 Mansionhouse Road, Edinburgh 9	
1939		* Greaves, William Michael Herbert, M.A. (Cantab.), F.R.A.S. (SECRETARY TO ORDINARY MEETINGS), Astronomer Royal for Scotland, Professor of Astronomy, University of Edinburgh. Royal Observatory, Blackford Hill, Edinburgh	Sec. 1940–
1937	C.	* Green, George, M.A., D.Sc. (Glas.), Lecturer in Applied Physics, University of Glasgow. 64 Partickhill Road, Glasgow, W. 1	
1938		* Greenlees, James Robertson Campbell, M.A., M.B., B.C. (Cantab.), formerly Headmaster of Loretto School, Musselburgh	
1927	C. K.	* Greenwood, Alan William, D.Sc. (Melb.), Ph.D. (Edin.), Lecturer in the Institute of Animal Genetics, University of Edinburgh (West Mains Road)	1939–
1922		* Greenwood, Rev. William Osborne, M.D. (Leeds), B.S. (Lond.), L.S.A., Clerk in Holy Orders, Kirklands, Slingsby Walk, Harrogate, Yorks	
1906		Greig, Edward David Wilson, C.I.E., M.D., D.Sc., F.R.C.P.E., Lt.-Col., I.M.S. (retired), 38 Coates Gardens, Edinburgh 12	
1931		* Greig, John Russell, Ph.D. (Edin.), Director, Moredun Institute, Animal Diseases Research Association. Wedderlie, Kirkbrae, Liberton 9	

Date of Election			Service on Council, etc
1905		† Greig, Sir Robert Blyth, M.C., LL.D., formerly Secretary to the Department of Agriculture for Scotland. The Shaws, Barnton, Midlothian	1921–24. V-P 1924–27.
1935		* Grierson, Alexander Millar Meck, M.D. (Edin.), D.P.H. (Edin. and Glas.), Senior Assistant to the Medical Officer of Health, Public Health Department. Box No. 399, Town Hall, Manchester 2	
1899		† Guest, Edward Graham, J.P., M.A., B.Sc., 5 Newbattle Terrace, Edinburgh 10	
1927		* Gulland, John Masson, M.A. (Oxon.), D.Sc. (Edin.), Ph.D. (St Andrews), Sir Jesse Boot Professor of Chemistry, University College, Nottingham	
1907		Gulliver, Gilbert Henry, D.Sc., A.M.I.Mech.E., 99 Southwark Street, London, S.E.	
1930		* Guthrie, Douglas, M.D., F.R.C.S.E., Lecturer in Diseases of the Ear, Nose, and Throat, School of Medicine of the Royal Colleges, Edinburgh. 21 Clarendon Crescent, Edinburgh 4	
1933	C.	* Guthrie, William Gilmour, M.A. (Edin.), B.A. (Cantab.), Ph.D., Professor of Mathematics and Natural Philosophy, Magee College, Londonderry, Northern Ireland	
1911		Guy, William, F.R.C.S.E., L.R.C.P.E., L.D.S.Ed., LL.D. (Penn.), Consulting Dental Surgeon, Edinburgh Royal Infirmary; Lecturer on Human and Comparative Dental Anatomy and Physiology. 11 Wemyss Place, Edinburgh 3	
1934		* Haldane, David, B.Sc. (Edin.), Senior Geologist, H.M. Geological Survey (Scotland), 19 Grange Terrace, Edinburgh 9. 6 Kilmaurs Road, Edinburgh 9	
1939		* Hall, William Robert, J.P., Manager, Union Bank of Scotland, Ltd. (Head Office), 64 George Street, Edinburgh 2. The Sheiling, Barnton, Midlothian	
1936	C. N.	* Hamilton, William James, D.Sc. (Glas.), M.D., B.Ch. (Belfast), Professor of Anatomy, St Bartholomew's Hospital Medical College, London, E.C. 1. *Temporary Address*, Anatomy Department, Downing Street, Cambridge	
1923		* Hanneford-Smith, William, A.M.Inst.C.E., Hon. A.R.I.B.A. 1 The Avenue, Gravesend, Kent	
1918		* Hardie, Patrick Sinclair, M.A., B.Sc., formerly Head of the Physics Department, Medical School, Cairo. 19 Ardmillan Terrace, Edinburgh 11	
1928		* Harding, William Gerald, F.R.Hist.S., F.S.A.Scot., F.R.E.S., Peckwater House, Charing, Kent	
1939		* Harper, Walter Fearn, M.D., Ph.D. (St Andrews), Reader in Anatomy, University of London, Department of Anatomy, London Hospital Medical College. St Catherine's College, Cambridge	
1923	C.	* Harris, Robert Graham, M.A., D.Sc. (Edin.), 44 Manor Road, Farnborough, Hants	
1914		Harrison, Edward Philip, Ph.D., F.Inst.P., c/o Messrs H. Hughes & Son, Ltd., Husun Works, Barkingside, Essex	
1934		Harrison, John Vernon, D.Sc.(Glas.), F.G.S., 34 Rowallan Gardens, Glasgow, W. 1	
1921		* Harrison, John William Heslop, D.Sc. (Durham), F.R.S., Professor of Botany and Reader in Genetics, King's College, Newcastle-upon-Tyne. The Avenue, Birtley, Co. Durham	
1940		* Harrowes, William McConnachie, M.D. (Glas.), Ch.B., D.P.M. (Lond.), Medical Director, New Saughton Hall Private Mental Hospital, Polton. Linden Lodge, Loanhead, Midlothian	
1940		* Hart, Thomas, J.P., C.A., Collector of the Trades House of Glasgow, a Senior Partner of Thomson, Jackson, Gourlay & Taylor, C.A., Glasgow, and of Jackson, Taylor, Abernethy & Co., C.A., London. Loanhead, Giffnock, Renfrewshire	
1926		* Harvey, William Frederick, C.I.E., M.A., M.B., C.M., D.P.H., Lieut.-Col., I.M.S. (retired), Histologist, Research Laboratory, Royal College of Physicians, Edinburgh. 56 Garscube Terrace, Edinburgh 12	
1936		* Henderson, David Kennedy, M.D., F.R.C.P.E., F.R.F.P.S.G., Professor of Psychiatry, University of Edinburgh. Tipperlinn House, Edinburgh 10	
1931		* Henderson, John, F.C.I.I., Manager and Secretary, Edinburgh Assurance Co., Ltd. Seaforth Cottage, York Road, Trinity, Edinburgh 5	
1936		* Henderson, Thomas, B.Sc. (Lond.), Secretary to the Educational Institute of Scotland. 2 Hillview Terrace, Edinburgh 12	
1929		* Henderson, Thomas, C.B.E., J.P., F.S.A.Scot., Actuary of the Savings Bank of Glasgow. 5 Belmont Crescent, Glasgow, W. 2	

Date of Election			Service on Council, etc.
1908		Henderson, William Dawson, M.A., B.Sc., Ph.D., Lecturer, Zoological Laboratories, University, Bristol. 77 Coldharbour Road, Bristol 6	
1937		* Hepburn, William Allan Forsyth, M.C., M.A., B.Ed. (Edin.), Director of Education, Ayrshire Education Authority. 2 St Leonard's Road, Ayr	
1925		* Heron, Alexander Macmillan, D.Sc. (Edin.), F.G.S., *formerly Director*, Geological Survey of India, Calcutta, India. Secunderabad Club, Secunderabad, Deccan, India	
1916		Herring, Percy Theodore, M.D., F.R.C.P.E., Professor of Physiology, University of St Andrews. Linton, St Andrews	1917–20, 1931–34. V-P 1934–37.
1936		* Hewat, Andrew Fergus, M.D., F.R.C.P.E., Secretary, Royal College of Physicians, Edinburgh. 14 Chester Street, Edinburgh 3	
1922		Hindle, Edward, M.A., Sc.D. (Cantab.), Ph.D., A.R.C.S., Regius Professor of Zoology, University of Glasgow	1940–
1928	C.	* Hobson, Alfred Dennis, M.A. (Cantab.), Professor of Zoology, King's College, Newcastle-upon-Tyne	
1928		* Hodge, William Vallance Douglas, M.A. (Edin.), M.A. (Cantab.), F.R.S., Lowndean Professor of Astronomy and Geometry, University of Cambridge. 28 Barrow Road, Cambridge	
1938		* Hogarth, George, Chairman of the Herring Industry Board, Fisheries Division, Scottish Home Department, St Andrew's House, Edinburgh. 40 Elliot Road, Edinburgh 11	
1923	C. K.	* Hogben, Lancelot Thomas, M.A., D.Sc., F.R.S., Professor of Natural History, University of Aberdeen	1937–40.
1927		Holden, Henry Smith, D.Sc., F.L.S., Director of the East Midlands Forensic Science Laboratory, Burton Street, Nottingham	
1930		* Holland, Sir Thomas Henry, K.C.S.I., K.C.I.E., D.L., Hon. D.Sc., LL.D., F.R.S., Vice-Chancellor and Principal of the University of Edinburgh. 31 South Oswald Road, Edinburgh 9	1931–32. V-P 1932–35.
1929	C.	* Hora, Sunder Lal, D.Sc. (Punjab et Edin.), F.L.S., F.Z.S., F.A.S.B., Senior Assistant Superintendent, Zoological Survey of India. Indian Museum, Calcutta	
1920	C.	* Horne, Alexander Robert, O.B.E., B.Sc., M.I.Mech.E., A.M.Inst.C.E., Professor of Mechanical Engineering, Heriot-Watt College, Edinburgh. 31 Queen's Crescent, Edinburgh 9	
1896		Horne, John Fletcher, M.D., F.R.C.S.E., Shelley Hall, Huddersfield	
1912	C. M-B.	† Houstoun, Robert Alexander, M.A., Ph.D., D.Sc., F.Inst.P., Lecturer in Physical Optics, University of Glasgow. 45 Kirklee Road, Glasgow, W.2	1929–32.
1893		Howden, Robert, M.A., M.B., C.M., D.Sc., LL.D., Emeritus Professor of Anatomy, University of Durham. Dalruadh, Victoria Terrace, Crieff. (*Died 7th December* 1940)	
1933		* Hume, Edgar Erskine, D.S.M., M.A., M.D., LL.D., Lieut.-Col., U.S. Army, Librarian of the Army Medical Library, Washington. The Magnolias, Frankfort, Kentucky. *Temporary Address*, Medical Field Service School, Carlisle Barracks, Pa., U.S.A.	
1910		Hume, William Fraser, D.Sc. (Lond.), Director, Geological Survey of Egypt, Helwân, Egypt. The Laurels, Rustington, Sussex	
1927		* Hunt, Owen Duke, B.Sc. (Manch.), Corrofell, Newton Ferrers, South Devon	
1923		* Hunter, Rev. Adam Mitchell, M.A., D.Litt., Librarian of New College, Edinburgh. 3 Suffolk Road, Edinburgh 9	
1928		* Hunter, Arthur, F.F.A., LL.D. (Edin.), Chevalier de la Légion d'Honneur, Vice-President and Chief Actuary of the New York Life Insurance Co. 124 Lloyd Road, Montclair, N.J., U.S.A.	
1916		Hunter, Charles Stewart, M.A., L.R.C.P.E., L.R.C.S.E., D.P.H., F.R.E.S., Cotswold, 36 Streatham Hill, London, S.W. 2	
1935		* Hutchinson, Arthur Cyril William, M.D.S. (Manch.), D.D.S. (Witwatersrand), Dean of the Edinburgh Dental Hospital and School. 12 Glencairn Crescent, Edinburgh 12	
1940		* Illingworth, Charles Frederick William, M.D., Ch.M. (Edin.), F.R.C.S.E., Professor of Surgery, University of Glasgow	
1923	C.	* Ince, Edward Lindsay, M.A. (Cantab.), D.Sc. (Edin.), Lecturer in Technical Mathematics, University of Edinburgh (Drummond Street)	
1927		* Inglis, John Alexander, of Auchindinny and Redhall, K.C., M.A. (Oxon.), LL.B. (Edin.), King's and Lord Treasurer's Remembrancer (VICE-PRESIDENT), Auchindinny House, Milton Bridge, Midlothian	1935–38 V-P 1939–

Date of Election			Service on Council, etc.
1912		Inglis, Robert John Mathieson, M.Inst.C.E., Chief Engineer, Southern Area, L.N.E.R. Dixton, Monken Hadley, Herts	
1936		* Innes, Donald Esme, M.C., M.A. (Oxon.), Professor of Geology, University of St Andrews. Cortina, St Andrews	
1917		* Irvine, Sir James Colquhoun, Kt., C.B.E., D.L., Ph.D. (Leipzig), D.Sc. (St Andrews), Hon. D.Sc. (Liverpool, Princeton), Hon. Sc.D. (Cantab., Yale, Pennsylvania), Hon. LL.D. (Glas., Aberd., Edin., and Toronto), Hon. D.C.L. (Durham), F.R.S., Hon. F.E.I.S., Vice-Chancellor and Principal of the University of St Andrews	1920–22. V-P 1922–25.
1930	C.	* Jack, David, M.A., B.Sc. (Edin.), Ph.D. (St Andrews), Lecturer in Natural Philosophy, United College, University of St Andrews. 22 Grange Road, St Andrews	
1923		* Jack, John Louttit, C.B.E., Solicitor, Deputy Secretary, Department of Health for Scotland, St Andrew's House, Edinburgh	
1938		* Jamieson, James Dalgleish Hamilton, H.D.D., L.D.S.Ed., Lecturer on Dental Diseases, University of Edinburgh. 29 and 58 George Square, Edinburgh 8	
1912	C.	Jeffrey, George Rutherford, M.D. (Glas.), F.R.C.P.E., 11 Langland Gardens, Hampstead, London, N.W. 3	
1934		* Jeffrey, Sir John, K.C.B., C.B.E., formerly Under-Secretary of State for Scotland. 9 Cluny Gardens, Edinburgh 10	
1906	C. K.	Jehu, Thomas John, M.A., M.D., F.G.S., Professor of Geology, University of Edinburgh (Grant Institute of Geology, West Mains Road)	1917–20, 1923–26. V-P 1929–32.
1900		† Jerdan, David Smiles, M.A., D.Sc., Ph.D., Avenel, Melrose	
1939		* Johnson, Norman Miller, B.Sc. (Manch.), F.E.I.S., F.R.S.G.S., Fellow Bot. Soc. Edin., Headmaster of the Commercial Public School, Dunfermline. 115 Victoria Terrace, Dunfermline	
1936		* Johnston, John McQueen, M.D. (Glas.), F.R.C.S.E., Pharmacologist, Department of Health for Scotland. 5 Cumin Place, Edinburgh 9	
1928		* Johnston-Saint, Percy Johnston, M.A. (Cantab.), Conservator, Wellcome Historical Medical Museum, 183–193 Euston Road, London, N.W. 1. 4 Wyndham Place, Bryanston Square, London, W. 1	
1928		* Johnstone, Robert William, C.B.E., M.A., M.D. (Edin.), F.R.C.S.E., M.R.C.P.E., Professor of Midwifery and Diseases of Women, University of Edinburgh. 26 Palmerston Place, Edinburgh 12	
1927		* Jones, Edward Taylor, D.Sc. (Lond.), Hon. D.Sc. (Wales), Professor of Natural Philosophy, University of Glasgow	1927–30.
1928	C.	* Jones, Tudor Jenkyn, Sc.D., M.D. (Glas.), Lecturer in Anatomy (Embryology), University of Liverpool. 49 Prince Aldred Road, Liverpool 15	
1922		* Juritz, Charles Frederick, M.A., D.Sc., F.I.C., Chief of the Union Department of Chemistry. Grenoble, Avenue Fresnaye, Sea Point, Cape Town, South Africa	
1936		* Kemball, Charles Henry, H.D.D., L.D.S., D.D.S. (Univ. Pennsylvania), Dental Surgeon, Lecturer on Orthodontics in Edinburgh Dental Hospital and School. 20 Ainslie Place, Edinburgh 3	
1925	C.	* Kemp, Charles Norman, B.Sc., Technical Radiologist, Secretary of the Royal Scottish Society of Arts. Ivy Lodge, Laverockbank Road, Edinburgh 5	
1929	C.	* Kendall, James Pickering, M.A., D.Sc., F.R.S. (GENERAL SECRETARY), Professor of Chemistry, University of Edinburgh (West Mains Road). 14 Mayfield Gardens, Edinburgh 9	1931–33. Sec. 1933–36. Gen. Sec. 1936–
1912		† Kennedy, Robert Foster, M.D. (Belfast), M.B., B.Ch. (R.U.I.), Associate Professor of Neurology, Cornell University, New York. 410 East 57 Street, New York City, U.S.A.	
1927		* Kennedy, Walter Phillips, Ph.D. (Edin.), L.R.C.P. and S.E., A.I.C., Medical Officer in the Ministry of Health, Whitehall, London, S.W. 1. 112 Beaufort Street, Chelsea, London, S.W. 3	
1935		* Kenneth, John Henry, M.A., Ph.D. (Edin.), Assistant, Imperial Bureau of Animal Genetics, University of Edinburgh (West Mains Road). University Union, Edinburgh 8	
1909		Kenwood, Henry Richard, C.M.G., M.B., C.M., Emeritus Chadwick Professor of Hygiene, University of London. 126 Queen's Drive, London, N. 4	

Date of Election			Service on Council, etc.
1925	C. M-B.	* Kermack, William Ogilvy, M.A., D.Sc., LL.D., Chemist, Research Laboratory of the Royal College of Physicians, 2 Forrest Road Edinburgh 1	
1903 & 1923	C. N.	Kerr, Sir John Graham, Kt., M.A. (Cantab.), LL.D., F.R.S., Honorary Fellow of Christ's College, M.P., Scottish Universities, Emeritus Regius Professor of Zoology, University of Glasgow. Dalny Veed, Barley, near Royston, Herts	1904–07, 1913–16, 1924–27. V.P 1928–31.
1926		* Khastgir, Satis Ranjan, M.Sc. (Calcutta), D.Sc. (Edin.), Physics Department, University, Dacca, India	
1907		King, Archibald, M.A., B.Sc., H.M. Inspector of Schools, The Cottage, Barassie, Ayrshire	
1925		* King, Leonard Augustus Lucas, M.A., Professor of Zoology, West of Scotland Agricultural College, Glasgow. 14 Bank Street, Glasgow, W. 2	
1918		* Kingon, Rev. John Robert Lewis, M.A., D.Sc., The Manse, Simonston, Cape Colony, South Africa	
1937		* Kirby, Percival Robson, M.A., D.Litt., F.R.C.M., Professor of Music and Musical History, University of the Witwatersrand, Johannesburg, South Africa	
1937	C.	* Koller, Peo Charles, Ph.D. (Budapest), D.Sc. (Edin.), Cytologist, Institute of Animal Genetics, University of Edinburgh (West Mains Road)	
1940		* Kyd, James Gray, F.F.A., Registrar-General for Scotland. 24 Mortonhall Road, Edinburgh 9	
1940		* Laird, Patrick Ramsay, C.B., Secretary to Department of Agriculture for Scotland, St Andrew's House, Edinburgh 1. Dorlin, 24 St John's Road, Corstorphine, Edinburgh 12	
1927		* Lambie, Charles George, M.C., M.D., F.R.C.P.E., Bosch Professor of Medicine, University of Sydney. Capri, 4 Wyuna Road, Point Piper, Sydney, N.S.W., Australia	
1920	C.	* Lamont, John Charles, Lieut.-Col., I.M.S. (retired), C.I.E., M.B., C.M. (Edin.), M.R.C.S. (Eng.), formerly Professor of Anatomy, Medical College, Lahore, India. 16 Rothesay Terrace, Edinburgh 3	
1939		* Landale, Stenard Ernest Andrew, Ph.D. (Cantab.), D.Phil. (Oxon.), Honours Diploma, Faraday House, London, Director of William Younger & Co. and of Scottish Brewers, Ltd. Ellersly, Murrayfield, Edinburgh 12, and Cranshaws, Duns, Berwickshire	
1925	C. N.	* Lang, William Henry, M.B., C.M., D.Sc., LL.D. (Glas.), F.R.S., formerly Barker Professor of Cryptogamic Botany, University of Manchester	
1931		* Langrishe, John du Plessis, D.S.O., M.B., B.Ch. (Dub.), D.P.H., Lt.-Col., R.A.M.C. (retired), Lecturer in Public Health, University of Edinburgh (Usher Institute of Public Health, Warrender Park Road). 2 South Gillsland Road, Edinburgh 10	
1910	C.	Lauder, Alexander, D.Sc., formerly Head of Chemistry Department, Edinburgh and East of Scotland College of Agriculture, and Lecturer in Agricultural Chemistry, University of Edinburgh. 78 Dalkeith Road, Edinburgh 9	1917–20. Sec. 1923–28.
1885	C.	Laurie, Arthur Pillans, M.A., D.Sc., LL.D., formerly Principal, Heriot-Watt College, Edinburgh. The Cottage, Kiln Way, Grayshott, Hindhead, Surrey	1908–11, 1913–16.
1905		Lawson, David, M.A., M.D., L.R.C.P. and S.E., Druimdarroch, Banchory, Kincardineshire	
1903		Leighton, Gerald Rowley, O.B.E., M.D., D.Sc., formerly Medical Officer (Foods), Department of Health for Scotland. Sharston, near Ramsey, Isle of Man	
1937		* Leitch, William Orr, M.Inst.C.E., formerly Chief Engineer and General Manager, Peking Mukden Railway. 1 Gordon Terrace, Edinburgh 9	
1910		Levie, Alexander, F.R.C.V.S., D.V.S.M., Balmae, Manor Road, Littleover, Derby	
1916	C.	Levy, Hyman, M.A., D.Sc., Professor of Mathematics, Imperial College of Science and Technology, London, S.W. 7	
1914	C. N.	Lewis, Francis John, D.Sc., F.L.S., Professor of Botany, Egyptian University, Abbissea, Cairo	
1918		* Lidstone, George James, F.F.A., F.I.A., LL.D., formerly Manager and Actuary, Scottish Widows' Fund Life Assurance Society. Hermiston House, Hermiston, Currie, Midlothian	1919–22.
1931		* Lightfoot, Nicholas Morpeth Hutchinson, M.A. (Cantab.), Lecturer in Mathematics, Heriot-Watt College, Edinburgh. 3 Park Gardens, Liberton, Edinburgh 9	

Date of Election			Service on Council, etc.
1923		* Lim, Robert Kho Seng, M.B., Ch.B., D.Sc., Peking Union Medical College, Department of Physiology, Peking, China	
1912		Lindsay, John George, M.A., B.Sc. (Edin.), Rector of Dunfermline High School	
1920	C.	* Lindsay, Thomas A., M.A., B.Sc., Head Master, Higher Grade School, Bucksburn, Aberdeenshire	
1912		Linlithgow, The Most Honourable the Marquis of, P.C., K.T., G.C.I.E., D.L., LL.D., Viceroy and Governor-General of India. Hopetoun House, South Queensferry	V-P 1934–37.
1903		† Liston, William Glen, C.I.E., M.D., Lt.-Col., I.M.S. (retired), Milburn Tower, Gogar, Corstorphine, Edinburgh 12	
1929		* Little, John Robert, F.C.I.I., F.C.I.S., formerly Manager and Secretary, Century Insurance Co., Ltd. 5 Dalrymple Crescent, Edinburgh 9	
1932		* Lockhart, James Balfour, M.A., B.Sc. (Edin.), Mathematical Master, Edinburgh Academy. Westering, Inverleith Grove, Edinburgh 4	
1926		* Lorraine, Norman Stanley Rees, M.D., D.P.H. (Edin. and Glas.), Medical Officer of Health, Benfleet Urban District. 1 Burlescoombe Leas, Burlescoombe Road, Thorpe Bay, Southend-on-Sea	
1930		* Low, James Wotherspoon, B.Sc., Ph.D., 1 Hamilton Park Avenue, Glasgow, W.2	
1934		* Low, R. Cranston, M.D., F.R.C.P.E., formerly Lecturer in Dermatology, University of Edinburgh. 37 Oxgangs Road, Fairmilehead, Edinburgh 10	
1939		* Lucas, Cyril Edward, B.Sc. (Lond.), Officer in Charge University College of Hull Oceanographical Laboratory, Sandport Street, Leith. University College, Hull	
1923	C.	* Ludlam, Ernest Bowman, M.A., D.Sc., Lecturer in Chemistry, University of Edinburgh (West Mains Road)	
1923		* Lyford-Pike, James, M.A., B.Sc., Lecturer in Forestry, University of Edinburgh. Rosetta, 56 Kirkbrae, Liberton, Edinburgh 9	
1924		* Lyon, David Murray, M.D., F.R.C.P.E., D.P.H., D.Sc., Professor of Clinical Medicine, University of Edinburgh (Royal Infirmary). Druim, Colinton, Edinburgh 11	1938–
1940		* McAdam, Robert, B.Sc., Ph.D. (Edin.), M.Inst.M.E., Lecturer in Mining and Surveying, Heriot-Watt College, Edinburgh. Allermuir, Captain's Road, Edinburgh 9	
1921		* M'Arthur, Neil, M.A., B.Sc., Lecturer in Mathematics, University of Glasgow. The Whins, Heathfield Drive, Milngavie	
1926		* M'Bride, James Alexander, B.A. (Roy. Univ., Ireland), B.Sc. (Lond.), formerly Rector of Queen's Park Secondary School, Glasgow. Scottish Liberal Club, Princes Street, Edinburgh 2. *Temporary Address*, Broughshane, Co. Antrim	
1883		M'Bride, Peter, M.D., F.R.C.P.E., 3 St Peter's Grove, York	
1931	C.	* McCallien, William John, D.Sc. (Glas.), Lecturer in Geology, University of Glasgow. 116 Henderland Road, Bearsden, Dumbartonshire	
1935		* MacCallum, Peter, M.Sc., M.A. (New Zealand), M.B., Ch.B., D.P.H. (Edin.), Professor of Pathology, University of Melbourne. Blackrock House, Blackrock, Victoria, Australia	
1923		* M'Cracken, William, J.P., F.S.I., Englesea House, Crewe	
1931	C.	* M'Crea, William Hunter, M.A., Ph.D. (Cantab.), B.Sc. (Lond.), F.R.A.S., Professor of Mathematics, The Queen's University, Belfast. 79 Osborne Park, Belfast	
1918		* M'Culloch, Rev. James David, 17 Eldon Street, Greenock	
1940		* Macdonald, James Alexander, B.Sc., Ph.D. (Edin.), Lecturer in Botany, University of St Andrews. East Scores, St Andrews	
1920		* M'Donald, Stuart, M.A., M.D., F.R.C.P.E., formerly Professor of Pathology, University of Durham. C/o British Linen Bank, 141 Princes Street, Edinburgh 2	
1928		* MacDonald, Thomas Logie, M.A., B.Sc. (Glas.), F.R.A.S., Director, British Astronomical Association (Lunar Section). 1 Graingerville North, Newcastle-upon-Tyne 4	
1886		Macdonald, William J., M.A., 15 Comiston Drive, Edinburgh 10	
1931		* M'Dougall, John Bowes, M.D. (Glas.), F.R.F.P.S.G., F.R.C.P.E., Medical Director, British Legion Village, Preston Hall, Kent. Preston Hall, Aylesford, Kent	
1901	C.	MacDougall, R. Stewart, M.A., D.Sc., LL.D. (Edin.), Emeritus Professor of Biology, Royal (Dick) Veterinary College, Edinburgh. Ivy Lodge, Gullane, East Lothian	1914–17.

Date of Election			Service on Council, etc.
1910		Macewen, Hugh Allen, O.B.E., M.B., Ch.B., D.P.H. (Lond. and Cantab.), Local Government Board, Ministry of Health, Whitehall, London, S.W.	
1888	C.	M'Fadyean, Sir John, Kt., M.B., B.Sc., LL.D., formerly Principal and Professor of Comparative Pathology, Royal Veterinary College, Camden Town, London. Highlands House, Leatherhead	
1939		* Macfarlane, James Wright, Ph.D.Eng. (Glas.), Designer of Electrical Machinery and Superintendent of Research and Special-purpose Design. Cartbank, Cathcart, Glasgow, S. 4	
1885	C.	† Macfarlane, John M., D.Sc., LL.D., Emeritus Professor of Botany. 427 West Hansberry Street, Germantown, Pa., U.S.A.	
1938		* MacGillivray, Allister Middleton, M.D. (St Andrews), Lecturer in Clinical Ophthalmology, University of St Andrews. 5 Clarendon Terrace, Dundee	
1897		MacGillivray of MacGillivray, Angus, M.D., C.M., D.Sc., LL.D., Chattan Croft, Crail, Fife	
1878		M'Gowan, George, F.I.C., Ph.D., c/o Capt. W. M'Gowan, Cutley, Kingston, Taunton	
1932		* MacGregor, Archibald Gordon, M.C., D.Sc., Geologist, H.M. Geological Survey (Scotland). 1 Greenbank Terrace, Edinburgh 10	
1939		* MacGregor, John Ian Graham, M.Inst.C.E., Assistant Engineer (Scottish Area) to the London & North Eastern Railway Co. The Hawthorns, 84 Colinton Road, Edinburgh 11	
1922	C.	* Macgregor, Murray, M.A., D.Sc., F.G.S., Assistant Director (Scotland), H.M. Geological Survey, 19 Grange Terrace, Edinburgh 9	1930–33.
1903		M'Intosh, Donald C., M.A., D.Sc., formerly Education Officer, Elgin. Glenavon, Boat of Garten	
1911		M'Intosh, John William, M.R.C.V.S., Cairngower, Crieff	
1927	C.	* M'Intyre, Donald, M.B.E., M.D. (Glas.), F.R.C.S.E., Royal Samaritan Lecturer in Gynæcology, University of Glasgow. 9 Park Circus, Glasgow, C. 3	
1912	C.	M'Kendrick, Anderson Gray, M.B., D.Sc., F.R.C.P.E., Lt.-Col., I.M.S. (retired), Superintendent, Research Laboratory, Royal College of Physicians, 2 Forrest Road, Edinburgh 1	1924–27. 1933–36. V-P 1936–39.
1914		M'Kendrick, Archibald, J.P., F.R.C.S.E., D.P.H., L.D.S., 12 Rothesay Place, Edinburgh 3	
1900	C.	M'Kendrick, John Soutar, M.D., F.R.F.P.S.G., 6 Buckingham Terrace, Hillhead, Glasgow	
1910	C.	Mackenzie, Alister, M.A., M.D., D.P.H., Principal Medical Officer and Lecturer in Hygiene, Training Centre, Jordanhill, Glasgow. 22 Queen's Gate, Dowanhill, Glasgow, W. 2	
1916	C.	Mackenzie, John E., D.Sc. (CURATOR), Emeritus Reader in Chemistry, University of Edinburgh. 2 Ramsay Garden, Edinburgh 1. (R.S.E. representative on Governing Body of Heriot-Watt College)	1936–39. Curator 1939–
1938		* Mackie, Alexander, B.Sc., Ph.D. (Edin.), Science Master, Trinity Academy. 2 St John's Terrace, Edinburgh 12	
1929	C.	* Mackie, John, M.A.,D.Sc., Rector, Leith Academy. 7 York Road, Edinburgh 5	
1928		* Mackie, Thomas Jones, M.D., M.R.C.P.E., Professor of Bacteriology, University of Edinburgh (Teviot Place). 22 Mortonhall Road, Edinburgh 9	
1936		* M'Kinlay, Peter Laird, M.D., D.P.H., Medical Officer (Statistics), Department *of Health for Scotland. 69 Montrose Street*, Clydebank	
1910		MacKinnon, James, M.A., Ph.D., LL.D., Emeritus Professor of Ecclesiastical History, University of Edinburgh. 12 Lygon Road, Edinburgh 9	1933–36.
1904		Mackintosh, Donald James, C.B., M.V.O., D.L., M.B., C.M., LL.D., Superintendent, Western Infirmary, Glasgow. 3 Kirklee Gardens, Glasgow, W. 2	
1939		* Mackintosh, James Macalister, M.A., M.D. (Glas.), D.P.H. (Lond.), Barrister-at-Law (Gray's Inn), Professor of Public Health, University of Glasgow	
1940		* McLachlan, Angus Elrick William, M.B., Ch.B., D.P.H. (Edin.), Clinical Medical Officer, Newcastle General Hospital. 201 Wingrove Road, Newcastle-upon-Tyne 4	
1899		Maclean, Sir Ewen John, J.P., D.L., M.D., D.Sc. (Hon.), LL.D., F.R.C.P. (Lond.), Emeritus Professor of Obstetrics and Gynæcology, Welsh National Medical School. 12 Park Place, Cardiff	
1933		* Macleod, James, F.I.C., Manager, Glasgow Corporation Chemical Works Department. 16 Colebrooke Street, Glasgow, W. 2	
1916	C.	M'Lintock, William Francis Porter, D.Sc. (Edin.), Geological Survey and Museum, Exhibition Road, South Kensington, London, S.W. 7	

Date of Election			Service on Council, etc.
1936		* M'Michael, John, M.D., Ch.B., M.R.C.P.E., British Postgraduate Medical School, Ducane Road, London, W.12	
1923		* Macmillan, Rt. Hon. Lord, P.C., G.C.V.O., LL.D., formerly Minister of Information, Moon Hall, Ewhurst, near Guildford, Surrey	
1938		* McMillan, William Hutchison, B.Sc. (Glas.), M.I.M.E., Hood Professor of Mining, University of Edinburgh, and Professor of Mining, Heriot-Watt College, Edinburgh. 5 Gordon Terrace, Edinburgh 9	
1939		* McNee, John William, D.S.O., M.D., D.Sc. (Glas.), F.R.C.P. (Lond.), Regius Professor of Practice of Medicine, University of Glasgow. Ledcameroch House, Bearsden, Glasgow	
1932		* M'Neil, Charles, M.A., M.D. (Edin.), F.R.C.P.E., Professor of Child Life and Health, University of Edinburgh (Royal Edinburgh Hospital for Sick Children). 44 Heriot Row, Edinburgh 3	
1940		* MacNiven, Angus, M.B., Ch.B. (Glas.), M.R.C.P.E., D.P.M. (Lond.), Physician Superintendent, Royal Mental Hospital, Glasgow. 2 Whittingehame Gardens, Glasgow, W. 2	
1917		* Macpherson, Rev. Hector Copland, M.A., Ph.D., F.R.A.S., Guthrie Memorial U.F. Church. 7 Wardie Crescent, Edinburgh 5	
1921		* M'Quistan, Dougald Black, M.A., B.Sc., Associate-Professor of Natural Philosophy, Royal Technical College, Glasgow. 29 Viewpark Drive, Rutherglen	
1936		* McRae, William, C.I.E., M.A., D.Sc. (Edin.), Late Agricultural Adviser to the Government of India, and Director, Agricultural Research Institute, Pusa. Daramona, Gamekeepers' Road, Barnton, Midlothian	
1921	C.	* MacRobert, Thomas Murray, M.A., D.Sc., Professor of Mathematics, University of Glasgow. 10 The University, Glasgow	1931–34.
1921	C.	* M'Whan, John, M.A. (Glas.), Ph.D. (Gött.), Lecturer in Mathematics, University of Glasgow. 37 Airthrey Avenue, Jordanhill, Glasgow, W. 4	
1927		* Madwar, Mohamed Reda, Ph.D. (Edin.), A.M.Inst.C.E., Director, Helwân Observatory, Egypt	
1898	C.	† Mahâlanobis, S. C., B.Sc. (Edin.), formerly Professor of Physiology, University of Calcutta. 90 Park Street, Calcutta	
1938		* Mainland, Donald, M.B., Ch.B., D.Sc. (Edin.), Professor of Anatomy, Forrest Building, Dalhousie University, Halifax, Nova Scotia	
1939		* Mair, William, F.C.S., F.R.G.S., Manufacturing Chemist (retired). 32 Braid Hills Road, Edinburgh 10	
1938		* Malcolm, Charles Alexander, M.A., Ph.D., Librarian, Signet Library. 21 Findhorn Place, Edinburgh 9	
1933		* Malcolm, John, M.D. (Edin.), Professor of Physiology, University of Dunedin, New Zealand, Medical School, King Street, Dunedin	
1908		Mallik, Devendranath, Sc.D., B.A., Principal, Carmichael College, Rungpur, Bengal, India	
1912		† Maloney, William Joseph, M.B.E., M.C., M.D. (Edin.), LL.D., formerly Professor of Neurology, Fordham University. Casa del Sale, Newport, Rhode Island, U.S.A.	
1913		Marchant, Rev. Sir James, K.B.E., LL.D., F.R.A.S., F.L.S., Director, National Council for Promotion of Race-Regeneration. Pinegarth, Buccleuch Road, Bournemouth	
1940		* Marrian, Guy Frederic, D.Sc. (Lond.), F.R.S.C., Professor of Chemistry in relation to Medicine, University of Edinburgh (Teviot Place). 32 Ormidale Terrace, Edinburgh 12	
1901	C.	Marshall, Francis Hugh Adam, C.B.E., Sc.D., LL.D., F.R.S., Reader in Agricultural Physiology, University of Cambridge. Christ's College, Cambridge	
1920	C.	* Marshall, John, M.A., D.Sc. (St Andrews), B.A. (Cantab.), University Reader in Mathematics, Bedford College, London. Logan House, 123 Torrington Park, London, N. 12	
1931		* Mason, John Huxley, F.R.C.V.S., Government Veterinary Laboratory, Onderstepoort, Pretoria, South Africa	
1913		Masson, George Henry, O.B.E., M.D., D.Sc., F.R.C.P.E., Carradale, Port of Spain, Trinidad, British West Indies	
1898	C. M-B.	Masterman, Arthur Thomas, M.A., D.Sc., F.R.S., formerly Superintending Inspector, H.M. Board of Agriculture and Fisheries. 3 Kedale Road, Seaford	1902–04.

Date of Election			Service on Council, etc.
1939		* Matheson, Donald Capell, F.R.C.V.S., D.V.S.M. (Vict.), Professor of Pathology, Bacteriology and Meat Inspection, Royal (Dick) Veterinary College, Edinburgh. 3 Lockharton Crescent, Edinburgh 11	
1911		† Mathews, Gregory Macalister, C.B.E., M.B.O.U., Meadway, St Cross, Winchester, Hants	
1921		* Mathieson, John, F.R.S.G.S., Division Superintendent, Ordnance Survey (retired), 42 East Claremont Street, Edinburgh 7	
1906		Mathieson, Robert, F.C.S., St Serf's, Innerleithen	
1928		* Matthai, George, M.A. (Cantab.), Sc.D., F.Z.S., F.L.S., Professor of Zoology, Government College, Lahore, India	
1924		* Matthews, James Robert, M.A., F.L.S., Regius Professor of Botany, University of Aberdeen, and Keeper of the Cruickshank Botanic Garden	1940–
1938		* Maxwell, Sir John Maxwell Stirling, K.T., of Pollok, Baronet, and of Corrour, D.L., LL.D. Pollok House, Glasgow, S. 3	
1932		* Maxwell, William, Managing Director of R. & R. Clark, Ltd. 14 South Inverleith Avenue, Edinburgh 4	
1917		* Maylard, A. Ernest, M.B., B.Sc. (Lond.), F.R.F.P.S.G., Kingsmuir, Peebles	
1922		* Meakins, Jonathan Campbell, M.D., LL.D., F.R.C.P.E., F.R.C.P. (Lond.), Professor of Medicine and Director of the Department of Medicine, McGill University, Montreal, Canada	
1931		* Mears, Frank Charles, A.R.S.A., F.R.I.B.A., 44 Queen Street, Edinburgh 2	
1937	C.	* Melville, Harry Work, Ph.D. (Edin., Cantab.), D.Sc. (Edin.), Fellow of Trinity College, Cambridge, Professor of Chemistry, University of Aberdeen	
1901	C.	Menzies, Alan W. C., M.A., Ph.D., F.C.S., Professor of Chemistry, Princeton University, Princeton, New Jersey, U.S.A.	
1927		* Menzies, Sir Frederick Norton Kay, K.B.E., M.D., LL.D. (Edin.), F.R.C.P.E., D.P.H. (Lond.), formerly Medical Officer of Health and School Medical Officer, Administrative County of London, c/o The Royal Bank of Scotland, Burlington Gardens, London, W. 1	
1933	C.	* Menzies, William John Milne, Inspector of Salmon Fisheries of Scotland (Fisheries Division, Scottish Home Department). Caledonian United Service Club, Edinburgh	
1929		* Mercer, Walter, M.B., Ch.B., F.R.C.S.E., Lecturer in Clinical Surgery, University of Edinburgh (Royal Infirmary). 12 Rothesay Terrace, Edinburgh 3	
1940		* Merewether, Edward Rowland Alworth, M.D. (Durh.), M.R.C.P. (Lond.), Barrister-at-Law (Gray's Inn), H.M. Medical Inspector of Factories. Artillery House, Artillery Row, London, S.W. 1	
1917		* Merson, George Fowlie, Manufacturing Technical Chemist, St John's Hill Works, Edinburgh 8. 7 Cumin Place, Edinburgh 9	
1902	C	† Metzler, William H., A.B., D.Sc., Ph.D., formerly Dean of the New York State College for Teachers, Albany, N.Y., U.S.A. 5003 South Salina Street, Syracuse, N.Y., U.S.A.	
1885	C. M-B.	Mill, Hugh Robert, D.Sc., LL.D., Hill Crest, Dormans Park, E. Grinstead	
1937		* Miller, James, R.S.A., F.R.I.B.A., Randolphfield, Stirling	
1910		Miller, John, M.A., D.Sc., formerly Professor of Mathematics, Royal Technical College. 5 Kelvinside Terrace West, Glasgow, N.W.	
1930		* Miller, William Christopher, M.R.C.V.S., Courtauld Professor of Animal Husbandry, Royal Veterinary College, Camden Town, London, N.W. 1. Blair Hyrne, Hadley Common, Barnet, Herts	
1905		Milne, Archibald, M.A., D.Sc., formerly Deputy Director of Studies, Edinburgh Provincial Training College. 38 Morningside Grove, Edinburgh 10	
1905		† Milne, Christian Hoyer Millar, M.A., D.Litt., formerly Head Master, Daniel Stewart's College. 19 Merchiston Gardens, Edinburgh 10	
1904	C.	Milne, James Robert, D.Sc., Lecturer in Natural Philosophy, University of Edinburgh (Drummond Street). 7 Grosvenor Crescent, Edinburgh 12	
1886		Milne, William, M.A., B.Sc., 70 Beechgrove Terrace, Aberdeen	
1933	C.	Milne-Thomson, Louis Melville, M.A. (Cantab.), F.R.A.S., Professor of Mathematics, Royal Naval College, Greenwich. Gothic House, Maze Hill, London, S.E. 10	
1899		Milroy, Thomas Hugh, M.D., B.Sc., LL.D., formerly Professor of Physiology, Queen's University, Belfast. Woodville, North Berwick	1939–

Date of Election			Service on Council, etc.
1889	C. K.	Mitchell, A. Crichton, D.Sc., Hon. Doc. Sc. (Genève), formerly Director of Public Instruction in Travancore, India. 246 Ferry Road, Edinburgh 5	1915–16, 1930–33. Cur. 1916–26. V-P 1926–29.
1897		† Mitchell, Sir George Arthur, Kt., D.L., J.P., M.A., LL.D., M.I.M.E., 9 Lowther Terrace, Kelvinside, Glasgow	
1900		Mitchell, James, M.A., B.Sc., Islay Lodge, Lochgilphead, Argyll	
1939		* Molony, John Barré de Winton, M.B., Ch.B., D.P.H. (Edin.), F.R.C.S.E., Lt.-Col., I.M.S. (retired), c/o Chartered Bank of India, Australia, and China, Bombay	
1938		* Monteath, Harry Henderson, W.S., B.A., LL.B., Professor of Conveyancing, University of Edinburgh (South Bridge). 16 Palmerston Place, Edinburgh 12	
1936		* More, Francis, Chartered Accountant, of the firm of Lindsay, Jamieson and Haldane. 12 Albert Terrace, Edinburgh 10	
1896		Morgan, Alexander, O.B.E., M.A., D.Sc., LL.D., formerly Principal, Edinburgh Provincial Training College. 1 Midmar Gardens, Edinburgh 10	
1936		* Morgan, Daniel Owen, M.Sc. (Wales), Ph.D. (Lond.), Lecturer in Helminthology, Department of Zoology, University of Edinburgh (West Mains Road). 15 Mentone Terrace, Edinburgh 9	
1937		* Morison, John, C.I.E., M.B., Ch.B. (Glas.), D.P.H. (Cantab.), Lt.-Col., I.M.S. (retired). Engaged in Research in the Usher Institute of Public Health. 13 Cluny Drive, Edinburgh 10	
1926		* Morris, James Archibald, R.S.A., F.S.A.Scot., Savoy Croft, Ayr	
1919		* Morris, Robert Owen, O.B.E., M.A., M.D., C.M. (Edin.), D.P.H. (Liverpool). King Edward VII Welsh National Memorial Association (Tuberculosis). Hafod-ar-For, Aberdovey, N. Wales	
1892	C.	Morrison, J. T., M.A., D.Sc., Emeritus Professor of Mathematical Physics, University, Stellenbosch, South Africa	
1930		* Morton, Sir James, Kt., LL.D., Governing Director, Scottish Dyes, Ltd. Dalston Hall, near Carlisle	
1901		Moses, O. St John, M.D., D.Sc., F.R.C.S.E., Lt.-Col., I.M.S. (retired), formerly Professor of Medical Jurisprudence, Medical College, Calcutta. 18 Manstone Road, Cricklewood, London, N.W. 2	
1934		* Mowat, Magnus, C.B.E., T.D., M.Inst.C.E., M.I.Mech.E., Brigadier-General, Ebor House, Sheen Gate Gardens, East Sheen, London, S.W. 14	
1935	C.	* Mozley, Walter Alan, B.Sc. (Manitoba), D.Sc. (Edin.), Bilharzia Investigator, Public Health Department, Salisbury, Southern Rhodesia	
1916		Muir, Sir Robert, Kt., M.A., M.D., F.R.C.P.E., Sc.D., LL.D., F.R.S., Emeritus Professor of Pathology, University of Glasgow. 30 Victoria Crescent Road, Glasgow, W. 2	
1935		* Munnoch, James, F.S.A.Scot., formerly Controller, General Post Office, Edinburgh. 15 Liberton Drive, Edinburgh 9	
1938	C.	* Munro, Sanford Sterling, B.S.A. (McGill), M.S. (Wisconsin), D.Sc. (Edin.), Poultry Geneticist, Dominion Department of Agriculture, Ottawa, Canada	
1939		* Murray, Alastair Campbell, F.F.A., Manager and Actuary. Scottish Equitable Life Assurance Society, and Director, Scottish Industrial Estates, 28 St Andrew Square, Edinburgh 2	
1933		* Murray, John, M.A. (Aberd.), Ph.D. (Edin.), Rector, The Academy, Annan. 9 Seaforth Avenue, Annan. (*Died 13th November* 1940)	
1934		* Murray, Walter George Robertson, A.I.C., Technical Assistant, Department of Chemistry, University of Edinburgh (West Mains Road). 20 Montpelier Park, Edinburgh 10	
1935		* Narayana, Basudeva, M.Sc., M.B. (Calcutta), Ph.D. (Edin.), Professor of Physiology, University of Patna. Department of Physiology, Medical College, Patna, India	
1937		* Nasmith, Charles Roy, B.A., M.A. (Honorary), Colgate University, U.S.A., United States Consul in Edinburgh. Abden House, 1 Marchhall Crescent, Edinburgh 9	
1940	C.	* Neill, Robert Macfarlane, M.C., M.A. (Glas.), Senior Lecturer in Zoology, University of Aberdeen	
1931		Nelson, Alexander, B.Sc. (Glas.), Ph.D. (Edin.), N.D.A., Lecturer in Plant Physiology and Agricultural Botany, University of Edinburgh. 14 Netherby Road, Edinburgh 5	

Date of Election			Service on Council, etc.
1924		* **Nelson**, Philip, M.A., M.D., Ph.D., F.S.A., Beechwood, Calderstones, Liverpool	
1898		**Newman**, Sir George, G.B.E., K.C.B., M.D., D.C.L., Hon. D.Sc., LL.D., F.R.C.S. (Eng.), F.R.C.P. (Lond.), formerly Chief Medical Officer, Ministry of Health and Board of Education. Grims Wood, Harrow Weald, Middlesex	
1928		* **Nichols**, James Edward, M.Sc. (Dunelm), Ph.D. (Edin.), Deputy Director, Imperial Bureau of Animal Genetics, University of Edinburgh (West Mains Road)	
1933	C.	* **Nicol**, Thomas, M.D., Ch.B., D.Sc. (Glas. and Lond.), F.R.C.S.E., Professor of Anatomy and Head of the Department of Anatomy, University of London, King's College, Strand, London, W.C. 2	
1940		* **Nisbet**, Hugh Bryan, D.Sc., Ph.D. (Edin.), F.I.C., Lecturer in Chemistry, Heriot-Watt College, Edinburgh. 30 Lockharton Avenue, Edinburgh 11	
1927		* **Noble**, Thomas Paterson, M.D. (Edin.), F.R.C.S. (Eng.). Hillside, Ebbw Vale, Mon., Wales	
1934	C.	* **Normand**, Alexander Robert, M.A., B.Sc., Ph.D. (Edin.), formerly Professor of Chemistry, Wilson College, Bombay. C/o Church of Scotland Foreign Missions Office, 121 George Street, Edinburgh 2	
1928	C. N.	* **O'Donoghue**, Charles Henry, D.Sc. (Lond.), Professor of Zoology, University of Reading	Sec. 1936–39.
1925		* **Ogg**, William Gammie, M.A., Ph.D., Director, Macaulay Institute for Soil Research, Craigiebuckler, Aberdeen; Research Lecturer in Soil Science in the University of Aberdeen	
1923	C.	* **Ogilvie**, Alan G., O.B.E., M.A., B.Sc. (Oxon.), Professor of Geography, University of Edinburgh (High School Yards). 40 Fountainhall Road, Edinburgh 9	1932–35.
1929		* **Ogilvie**, Frederick Wolff, M.A. (Oxon.), LL.D., Director-General of the British Broadcasting Corporation. Broadcasting House, London, W. 1	
1939		* **Oliphant**, William Douglas, B.Sc. Eng. (Edin.), A.M.I.E.E., Scientific Officer to the Air Ministry. St Baldred's Road, North Berwick	
1886		**Oliver**, James, M.D., F.L.S., Physician to the London Hospital for Women. 5 Tankerville Terrace, Newcastle-upon-Tyne	
1895	C.	† **Oliver**, Sir Thomas, Kt., D.L., M.D., LL.D., F.R.C.P. (Lond.), formerly Vice-Chancellor of the University of Durham. 7 Ellison Place, Newcastle-upon-Tyne	
1930		* **Oliver**, William, B.Sc., A.M.Inst.C.E., Professor of Organisation of Industry and Commerce, University of Edinburgh (South Bridge). 70 Netherby Road, Trinity, Edinburgh 5	
1939		* **Olver**, Colonel Sir Arthur, C.B., C.M.G., F.R.C.V.S., Principal, Royal (Dick) Veterinary College, Edinburgh. Prestonfield House, Edinburgh 9	1940–
1930		* **O'Riordan**, George Francis, B.Sc. Eng., M.I.Mech.E., Principal of Battersea Polytechnic, London, S.W. 7. Hesslewood, 39 Langley Avenue, Surbiton, Surrey	
1924		* **Orr**, Sir John Boyd, Kt., D.S.O., M.C., M.A., D.Sc., M.D., LL.D., F.R.S., Director of the Rowett Research Institute for Research in Animal Nutrition, Aberdeen	
1915		**Orr**, Lewis P., F.F.A., formerly General Manager of the Scottish Life Assurance Co. 3 Belgrave Place, Edinburgh 4	
1932		* **Orr**, Matthew Young, 38 Lennox Row, Edinburgh 5	
1908		**Page**, William Davidge, formerly of 13 Yarrell Mansions, Queen's Club Gardens, London, W. 14. (*Present address not known*)	
1934		* **Pal**, Rudrendra Kumar, D.Sc. (Edin.), M.R.C.P.E., F.R.C.S.E., formerly Professor of Physiology, Prince of Wales Medical College, Patna. 5/4 Ballygunge Place, Calcutta, India	
1905		**Pallin**, Colonel William Alfred, C.B.E., D.S.O., F.R.C.V.S., 5 Tower Gardens, Hythe, Kent	
1933		**Parsons**, Charles Wynford, M.A., Lecturer in Zoology, University of Glasgow. Mapledene, Ralston Road, Bearsden, Dumbartonshire	
1901		**Paterson**, David, Leewood, Rosslyn Castle, Midlothian	
1937		* **Paterson**, Thomas T., B.Sc. (Edin.), Fellow of Trinity College, Cambridge. University Museum of Archæology and of Ethnology, Downing Street, Cambridge	
1936		* **Paterson**, William George Rogerson, O.B.E., B.Sc., Principal of the West of Scotland Agricultural College. Buckrigg, Beattock, Dumfriesshire	
1927		* **Patterson**, Charles, A.M.I.Mar.E., Lecturer in Mechanical Engineering Design, University of Edinburgh (Sanderson Engineering Laboratories, Mayfield Road). 22 Dudley Terrace, Trinity, Edinburgh 6	

Date of Election			Service on Council, etc.
1926		* Patton, Donald, M.A., B.Sc., Ph.D., Lecturer in Science, Glasgow Provincial College for the Training of Teachers. 15 Jordanhill Drive, Glasgow, W. 3	
1923	C.	* Peacock, Alexander David, D.Sc., Professor of Zoology, University College, Dundee	1935–38.
1914		Pearson, Joseph, D.Sc., F.L.S., formerly Director of the Colombo Museum, and Marine Biologist to the Ceylon Government. Director of the Tasmanian Museum, Hobart, Tasmania	
1904		Peck, Sir James Wallace, Kt., C.B., M.A., formerly Secretary to the Scottish Education Department, Chief Divisional Food Officer for Scotland, 23 Ainslie Place, Edinburgh 3	1926–28.
1887	C. M-B.	Peddie, William, D.Sc., Professor of Natural Philosophy, University College, Dundee. The Weisha, Ninewells, Dundee	1904–07, 1908–11, 1933–36. V-P 1919–22.
1931	C.	* Phemister, James, M.A., D.Sc. (Glas.), Petrographer, H.M. Geological Survey and Museum, Exhibition Road, London, S.W. 7	
1938		* Phemister, Thomas Crawford, D.Sc. (Glas.), Ph.D. (Cantab.), F.G.S., Professor of Geology, University of Aberdeen	
1907	C.	† Phillips, Charles E. S., O.B.E., Castle House, Shooters Hill, Woolwich, S.E. 18	
1929	C.	* Phillips, John Frederick Vicars, D.Sc., F.L.S., Professor of Botany, University of the Witwatersrand, Johannesburg, Union of South Africa	
1935		* Pichamuthu, Charles Solomon, B.Sc. (Mysore), Ph.D. (Glas.), F.G.S., Registrar, University of Mysore, India	
1932		* Pickard, James Nichol, B.A. (Cantab.), Ph.D. (Edin.), Lynburn, Carlops, Penicuik, Midlothian	
1928		* Pilcher, Robert Stuart, General Manager, Manchester Corporation Tramways. 55 Piccadilly, Manchester	
1908	C.	† Pirie, James Hunter Harvey, B.Sc., M.D., F.R.C.P.E., formerly Acting Director, South African Institute for Medical Research, Johannesburg. 23 Victoria Street, Oaklands, Johannesburg, South Africa	
1911		Pirie, James Simpson, M.Inst.C.E., 25 Grange Road, Edinburgh 9	
1906		Pitchford, Herbert Watkins, C.M.G., F.R.C.V.S., Victoria Club, Pietermaritzburg, South Africa	
1934		* Plenderleith, Harold James, M.C., B.Sc., Ph.D. (St Andrews), F.C.S., Deputy Keeper in Charge, Research Laboratory of British Museum, London. 198 Willesden Lane, London, N.W. 6	
1937		* Pollock, Sir John Donald, Bart., O.B.E. (Mil.), D.L., M.D., Hon. D.Sc. (Oxon.), LL.D. (Edin.), Rector of the University of Edinburgh. Manor House, Boswall Road, Edinburgh 5	
1888		† Prain, Sir David, Kt., C.M.G., C.I.E., M.A., M.B., LL.D., F.R.S., F.L.S., Lt.-Col., I.M.S. (retired), formerly Director, Royal Botanic Gardens, Kew, Surrey. The Well Farm, Whyteleafe, Surrey	
1937		* Prasad, Badri Narayan, Ph.D. (Edin.), M.Sc., M.B., D.T.M. (Cal.), Lecturer in Pharmacology, P. W. Medical College, Bankipore, P.O., Bihar, India	
1932		* Prasad, Gorakh, D.Sc. (Edin.), Reader in Mathematics, University of Allahabad. Beli Road, Allahabad, India	
1926	C.	* Prashad, Baini, D.Sc., Superintendent, Zoological Survey of India, Indian Museum, Calcutta	
1933		* Preston, Frank Anderson Baillie, L.R.I.B.A., F.S.A.Scot., Lecturer in Municipal Engineering, Royal Technical College, Glasgow. Craigrownie, Briarwell Road, Milngavie	
1915		† Price, Frederick William, M.D. (Edin.), C.M., F.R.C.P. (Lond.), Consulting Physician to the Royal Northern Hospital, London; Senior Physician to the National Hospital for Diseases of the Heart. 133 Harley Street, London, W.	
1932		* Price, Thomas Slater, O.B.E., D.Sc. (Lond., Birm.), Ph.D. (Leip.), F.R.S., formerly Professor of Chemistry, Heriot-Watt College, Edinburgh. "Cernyw," Castle Street, Criccieth, N. Wales	
1932		* Pringle, John, Hon. D.Sc., F.G.S., formerly Palæontologist, Geological Survey of Great Britain. 23 Melville Terrace, Edinburgh 9	
1920	C.	* Purser, George Leslie, M.A. (Cantab.), F.Z.S., Lecturer in Embryology, University of Aberdeen	

Date of Election			Service on Council, etc.
1898		**Purves, John Archibald**, D.Sc., Chilliswood, Trull, Taunton	
1936	C.	* **Raitt, Douglas Stewart**, D.Sc., Ph.D. (Aberd.), F.L.S., 65 Culduthel Road, Inverness	
1899	C.	**Ramage, Alexander G.**, Lochcote, Linlithgowshire	
1938		* **Ramsay, Andrew Maitland**, M.D., LL.D., F.R.F.P.S.G., formerly Lecturer in Ophthalmology, University of Glasgow. The Castle House, St Andrews	
1904		**Ratcliffe, Joseph Riley**, M.B., C.M., c/o The Librarian, University, Birmingham	
1937		**Rawlins, Francis Ian Gregory**, M.Sc. (Cantab.), F.Inst.P., Scientific Adviser to the Trustees of the National Gallery, London. 5 The Mount Square, Hampstead, London, N.W. 3	
1927	C.	* **Read, Herbert Harold**, D.Sc. (Lond.), A.R.C.S., F.R.S., F.G.S., formerly George Herdman Professor of Geology, University of Liverpool; Professor of Geology, University of London Imperial College of Science and Technology, London, S.W. 7	
1929		* **Read, Selwyn**, B.A., Schoolmaster, Edinburgh Academy. Mackenzie House, Kinnear Road, Edinburgh 4	
1902		**Rees-Roberts, John Vernon**, M.D., D.Sc., D.P.H., Frognal Dene, Hampstead, London, N.W. 3	
1940		* **Reid, James Scott Cumberland**, K.C., M.P., B.A., LL.B. (Cantab.), Advocate, Solicitor-General for Scotland since 1936. 46 Northumberland Street, Edinburgh 3	
1913		**Reid, Harry Avery**, O.B.E., F.R.C.V.S., D.V.H., Veterinary Officer to the New Zealand Government. C/o High Commissioner for New Zealand, 415 Strand, London, W.C. 2	
1936		* **Renouf, Louis Percy Watt**, B.A., Dip. Agric. (Cantab.), D.Sc. (Nat. Univ. Ireland), Professor of Zoology, University College, Cork, Director of Cork University Biological Station. St Philomena's, Tivoli, Cork, I.F.S.	
1914		**Renshaw, Graham**, M.D., M.R.C.S. (Eng.), L.R.C.P. (Lond.), L.S.A., F.Z.S., Lecturer in Zoology, Extramural Department, University of Manchester. Editor of *Natureland*, Sale Bridge House, Sale, Manchester	
1913		† **Richardson, Harry**, O.B.E., M.C., M.Inst.E.E., M.Inst.M.E., 72 Oakwood Court, London, W. 14	
1908		**Richardson, Linsdall**, P.A.Inst.W.E., F.G.S., 104 Greenfield Road, Harbourne, Birmingham	
1927		* **Richey, James Ernest**, M.C., B.A., B.A.I. (T.C.D.), Sc.D., F.R.S., F.G.S., District Geologist, H.M. Geological Survey (Scotland), 19 Grange Terrace, Edinburgh 9	1938–
1940		* **Riley, Harry**, Licentiate College of Preceptors, Founder and Headmaster of Strathallan School, Forgandenny, Perthshire	
1930		* **Ritchie, Allan Watt**, M.B.E., F.R.San.I., Chief Sanitary Inspector, City of Edinburgh, 8 Cumlodden Avenue, Edinburgh 12	
1916	C.	**Ritchie, James**, M.A., D.Sc. (VICE-PRESIDENT), Professor of Natural History, University of Edinburgh. 31 Mortonhall Road, Edinburgh 9	1921–24, 1926–28, 1937–40. Sec. 1928–31. V-P 1931–34, 1940–
1914	C.	**Ritchie, James Bonnyman**, D.Sc., Rector, The Academy, Ayr. 35 Carrick Road, Ayr	
1937		* **Ritchie, Mowbray**, Ph.D., D.Sc. (Edin.), Lecturer in Chemistry, University of Edinburgh (West Mains Road)	
1906	C.	**Ritchie, William Thomas**, O.B.E., M.D., F.R.C.P.E., formerly Professor of Medicine, University of Edinburgh (Teviot Place). Penvalla, Barnshot Road, Edinburgh 13	
1919		* **Ritchie-Scott, Alexander**, B.Sc. (Edin.), D.Sc. (Lond.), Allan House, Bridge of Earn Road, Dunning	
1936		* **Robb, James**, M.A., B.D., LL.B., LL.D. (St Andrews), Secretary to the Carnegie Trust for the Universities of Scotland. 26 Ormidale Terrace, Edinburgh 12	
1929	C.	* **Robb, Richard Alexander**, M.A., D.Sc., Lecturer in Mathematics, University of Glasgow. 27 Moor Road, Eaglesham, Renfrewshire	
1931		* **Robb, William**, N.D.A., Director of Research, Scottish Society for Research in Plant Breeding. Craigs House, Corstorphine, Edinburgh 12	

Date of Election			Service on Council, etc.
1919		* Roberts, Alfred Henry, O.B.E., M.Inst.C.E., formerly Superintendent and Engineer, Leith Docks. Dohnavur, Ravelston Dykes, Edinburgh 4	
1926		* Roberts, John Alexander Fraser, M.A. (Cantab.), M.B., D.Sc., Stoke Park Colony, Stapleton, Bristol	
1937		* Robertson, John Watson, M.A., B.Sc. (Aberd.), Headmaster, Central School, Aberdeen. 21 Belvedere Crescent, Aberdeen	
1937		* Robertson, Thomas Graham, The Hon. Lord Robertson, Senator of the College of Justice. 14 India Street, Edinburgh 3	1938–
1896	C.	Robertson, W. G. Aitchison, D.Sc., D.Litt., M.D., F.R.C.P.E., Barrister-at-Law, Lincoln's Inn. St Margarets, St Valerie Road, Bournemouth	
1932	C.	* Robson, John Michael, M.D., B.Sc., Lecturer in Materia Medica, University of Edinburgh (Teviot Place)	
1926		* Romanis, William Hugh Cowie, M.A., M.B., M.C. (Cantab.), F.R.C.S. (Eng.), Surgeon to St Thomas's Hospital, London. 120 Harley Street, London, W.1	
1916		Ronald, David, M.Inst.C.E., formerly Chief Engineer, Scottish Department of Health. 72A George Street, Edinburgh 2	
1938		* Rosebery, The Right Hon. The Earl of, D.S.O., M.C., Lord Lieutenant of Midlothian. Dalmeny House, Edinburgh	
1909	C.	Ross, Alexander David, M.A., D.Sc., F.Inst.P., F.R.A.S., Professor of Physics, University of Western Australia, Perth, Western Australia	
1921		* Ross, Edward Burns, M.A., formerly Professor of Mathematics, Madras Christian College, Madras. 41 Liberton Brae, Edinburgh 9	
1935		* Rowatt, Thomas, O.B.E., M.I.Mech.E., F.S.A.Scot., Director, Royal Scottish Museum. Spottiswoode, Spylaw Bank Road, Colinton, Edinburgh 13	
1931	C. K.	* Ruse, Harold Stanley, M.A. (Oxon.), D.Sc., Professor of Mathematics, University College, Southampton	
1906		Russell, Alexander Durie, B.Sc., F.R.A.S., formerly Head Mathematical Master, Falkirk High School. 12 Heugh Street, Falkirk	
1930		Russell, David, LL.D., Paper Manufacturer. Silverburn, Leven, Fife	1940–
1902	C. K.	Russell, James, 22 Glenorchy Terrace, Edinburgh 9	
1937		* Russell, William Ritchie, M.D. (Edin.), F.R.C.P.E., M.R.C.P. (Lond.), Assistant Physician to the Royal Infirmary and to the Deaconess Hospital, Edinburgh. 8 Randolph Cliff, Edinburgh 3. *Temporary Address*, 7 Northmoor Road, Oxford	
1934		* Rutherford, Daniel Edwin, M.A., B.Sc. (St Andrews), D.Math. (Amsterdam), Lecturer in Applied Mathematics, United College, University of St Andrews. Dunelm, Lawhead Road, St Andrews	
1925	C.	* Saddler, William, M.A., B.A., Professor of Mathematics, Canterbury College, Christchurch, N.Z.	
1906		Saleeby, Caleb Williams, M.D., 13 Greville Place, Hampstead, London, N.W. 6 (*Died 9th December* 1940)	
1916	C.	Salvesen, The Rt. Hon. Lord, P.C., K.C., LL.D., Judge of the Court of Session (retired), Dean Park House, Edinburgh 4	1920–22. V-P 1922–25.
1934		* Salvesen, Harold Keith, M.A. (Oxon. and Harvard), Captain (retired, I.A.), Shipowner, Inveralmond, Cramond, Midlothian	
1914		Salvesen, Colonel Theodore Émile, of Culrain, F.R.S.A., F.S.A.Scot., Chevalier de la Légion d'Honneur. 37 Inverleith Place, Edinburgh 4, and Carbisdale Castle, Ardgay, Ross-shire	
1927	C.	* Sandeman, Ian, M.A., D.Sc., Ph.D. (St Andrews), Acting Chief Inspector of Schools, Education Department, Colombo, Ceylon	
1938		* Sandilands, James, A.H.-W.C., F.I.C., Senior Lecturer in Chemistry, Heriot-Watt College. 102 Westholmes Gardens, Musselburgh	
1930		* Sansome, Frederick Whalley, Ph.D., F.L.S., Senior Lecturer in Horticulture, Botany Department, University of Manchester	
1922		* Sarkar, Bijali Behari, M.Sc., D.Sc. (Edin.), Lecturer in Physiology, University, Calcutta. 33/3 Lansdowne Road, Calcutta	
1903		Sarolea, Charles, Ph.D., D.Litt., LL.D. (Montreal), formerly Professor of French, University of Edinburgh. 21 Royal Terrace, Edinburgh 7	
1935		* Say, Maurice George, Ph.D., M.Sc. (Lond.), M.I.E.E., Professor of Electrical Engineering, Heriot-Watt College, Edinburgh. Dreghorn Loan, Colinton, Edinburgh	
1927	C.	* Schlapp, Robert, M.A. (Edin.), Ph.D. (Cantab.), Lecturer in Applied Mathematics, University of Edinburgh (Drummond Street). 40A Morningside Park, Edinburgh 10	

Date of Election			Service on Council, etc.
1885	C.	† Scott, Alexander, M.A., D.Sc., F.R.S., formerly Director of Scientific Research at the British Museum. 117 Hamilton Terrace, London, N.W. 8	
1919		* Scott, Alexander, M.A., D.Sc., 25 Glebe Street, Stoke-on-Trent	
1917		* Scott, Henry Harold, C.M.G., M.D., F.R.C.P. (Lond.), M.R.C.S. (Eng.), D.P.H., Director, Bureau of Hygiene and Tropical Diseases, Keppel Street, Gower Street, London, W.C. 1	
1928		* Senior-White, Ronald, F.R.E.S., Malariologist, Bengal-Nagpur Railway, Kidderpore, P.O., Calcutta, India	
1930		* Shankland, Ernest Claud, F.R.Met.S., River Superintendent, Port of London Authority. Mariners, Balfour Gardens, Folkestone	
1927		* Sharpley, Forbes Wilmot, B.Sc. Eng. (Lond.), Ph.D., M.Inst.E.E., Professor of Electrical and Mechanical Engineering, Indian School of Mines, Dhanbad, India	
1927		* Shearer, Ernest, M.A., B.Sc. (Edin.), Professor of Agriculture and Rural Economy, University of Edinburgh, and Principal, Edinburgh and East of Scotland College of Agriculture, 13 George Square, Edinburgh 8	
1931		* Shearer, James Fleming, M.A., B.Sc., Ph.D., Lecturer in Natural Philosophy, University of Glasgow. 149 Queen's Drive, Wavertree, Liverpool 15	
1932		* Simpson, Alexander Rudolf Barbour, B.Sc. (Edin.), M.A. (Cantab.), F.R.G.S., Hillstone School, Malvern, Worcestershire	
1908		Simpson, George Freeland Barbour, J.P., M.D., F.R.C.P.E., F.R.C.S.E., 43 Manor Place, Edinburgh 3	
1932	C.	* Simpson, John Baird, D.Sc. (Aberd.), Senior Geologist, H.M. Geological Survey (Scotland), 19 Grange Terrace, Edinburgh 9	
1938		* Sinclair, Arthur Henry Havens, M.D., F.R.C.S.E., Hon. Surgeon-Oculist to H.M. The King in Scotland, Consulting Ophthalmic Surgeon, Royal Infirmary, Edinburgh. 6 Charlotte Square, Edinburgh 2	
1939		* Sinclair, William Angus, M.A. (Edin.), Lecturer in Philosophy, University of Edinburgh (South Bridge). 5 Great Stuart Street, Edinburgh 3	
1900		Sinhjee, Sir Bhagvat, G.C.I.E., M.D., LL.D. (Edin.), H.H. the Thakur Sahib of Gondal, Kathiawar, Bombay, India	
1903		† Skinner, Robert Taylor, J.P., M.A., F.S.A.Scot., formerly House-Governor, Donaldson's Hospital. 35 Campbell Road, Edinburgh 12	
1937		* Slater, James Kirkwood, M.B. (Edin.), F.R.C.P.E., Assistant Physician, Royal Infirmary, Edinburgh, and Physician to the Deaconess Hospital, Edinburgh. 7 Walker Street, Edinburgh 3	
1930	C.	* Slater, Robert Henry, D.Sc., Ph.D. (Edin.), F.I.C., *Department of Chemical Pathology, St Mary's Hospital, London, W. 2*	
1929		* Smail, James Cameron, O.B.E., Companion Inst.E.E., Principal, Heriot-Watt College, Edinburgh. 1 Grange Terrace, Edinburgh 9	1934–37. V-P 1937–40.
1926	C.	* Small, James, D.Sc., Professor of Botany, Queen's University, Belfast. 5 Eden-a-Grena, Randal Park, Belfast	
1920		* Smellie, William Robert, M.A., D.Sc., Geologist on the Staff of the Anglo-Persian Oil Company. Baron Cliff, Cove, Dumbartonshire	
1937		* Smith, Alexander Martin, Ph.D., D.Sc. (Edin.), A.I.C., Lecturer in Agricultural Chemistry, University of Edinburgh, 13 George Square. 1 Mortonhall Road, Edinburgh 9	
1928		Smith, Alick Drummond Buchanan, O.B.E. (Mil.), M.A., B.Sc. (Agric.) (Aberd.), D.Sc. (Edin.), M.S.A. (Iowa), *Lecturer, Institute of Animal Genetics, University of Edinburgh (West Mains Road)*	
1937		* Smith, Horace George, B.Sc. (Glas.), Ph.D. (Bristol), Assistant, Department of Natural History, Marischal College, University of Aberdeen	
1921		* Smith, Norman Kemp, M.A., D.Phil., D.Litt., LL.D., Professor of Logic and Metaphysics, University of Edinburgh (South Bridge). 14 Kilgraston Road, Edinburgh 9	
1923		* Smith, Percy James Lancelot, M.A. (Oxon.), F.I.C., F.C.S., Science Master, Loretto School. 47 Dalrymple Loan, Musselburgh	
1911		Smith, Stephen, B.Sc., 34 Craigmillar Park, Edinburgh 9	
1929		* Smith, Sydney Alfred, M.D., F.R.C.P.E., D.P.H., Professor of Forensic Medicine, University of Edinburgh (Teviot Place). 10 Oswald Road, Edinburgh 9	1936–39.
1919		* Smith, Sir William Wright, Kt., M.A., D. ès Sc., Regius Professor of Botany, University of Edinburgh, Regius Keeper of the Royal Botanic Garden, and King's Botanist in Scotland. Inverleith House, Edinburgh 4	Sec. 1923–28. V-P 1928–31.

Date of Election			Service on Council, etc.
1932		* Sneeden, Jean-Baptiste Octave, B.Sc., Ph.D. (Glas.), Lecturer on Heat Engines Royal Technical College, Glasgow. 39 Kingshouse Avenue, Glasgow, S. 4	
1899		Snell, Ernest Hugh, M.D., B.Sc., D.P.H. (Cantab.), Barrister-at-Law, formerly Medical Officer of Health, Coventry. 3 Eaton Road, Coventry	
1933		* Somerville, John Livingston, C.A. Auditor, University of Edinburgh. 37 Melville Street, Edinburgh 3	
1929		* Southwell, Thomas, D.Sc., A.R.C.S., formerly Walter Myers Lecturer in Parasitology, School of Tropical Medicine, University of Liverpool. Stansfield Cottage, Todmorden, Lancashire	
1925		* Staig, Robert Arnot, M.A., Ph.D., Lecturer in Zoology, University of Glasgow. Glenlea, Lasswade, Midlothian	
1891		Stanfield, Richard, A.R.S.M., M.Inst.C.E., Emeritus Professor of Mechanics and Engineering, Heriot-Watt College, Edinburgh. 24 Mayfield Gardens, Edinburgh 9	1926–29.
1923		* Stebbing, Edward Percy, M.A., Professor of Forestry, University of Edinburgh (George Square)	
1923		* Stenhouse, Andrew G., F.G.S., 191 Newhaven Road, Edinburgh 6	
1929	C.	* Stephen, Alexander Charles, D.Sc., Keeper, Natural History Department, Royal Scottish Museum, Edinburgh. Eastcroft, Cramond Bridge, Edinburgh 4	
1931		* Steven, George Alexander, B.Sc. (Edin.), Assistant Naturalist, Marine Laboratory, Plymouth. 1 Seaview Villas, Pentyre Terrace, Plymouth, Devon	
1937	C.	* Steven, Henry Marshall, B.Sc., Ph.D. (Edin.), M.A., Hon. Causa (Oxon.), Professor of Forestry, University of Aberdeen. 75 Fountainhall Road, Aberdeen	
1886	C.	Stevenson, Charles A., B.Sc., M.Inst.C.E., Radella, North Berwick	
1919		* Stevenson, David Alan, B.Sc., M.Inst.C.E., 22 Glencairn Crescent, Edinburgh 12	
1935		* Stevenson, Eric, B.Sc. (Edin.), A.M.I.Mech.E., Lecturer in Engineering, University of Edinburgh (Sanderson Engineering Laboratory, Mayfield Road). 7 Beauchamp Road, Edinburgh 9	
1936		* Stewart, Alexander Dron, C.I.E., M.B., Ch.B. (Edin.), F.R.C.S.E., Lt.-Col., I.M.S. (retired), Superintendent of the Royal Infirmary, Edinburgh. Meadow Walk House, Edinburgh 3	
1939		* Stewart, Bernard Halley, M.A., M.D., B.Ch. (Cantab.), President of the Sir Halley Stewart Trust. Fairlight, Totteridge, London, N. 20	
1925		* Stewart, David Smith, Ph.D., M.Inst.C.E., Lecturer on Structural Engineering Drawing, University of Edinburgh (Sanderson Engineering Laboratory, Mayfield Road). 82 Lasswade Road, Edinburgh 9	
1938		* Stewart, James, M.A., D.Sc., Ph.D., Chief Biochemist, Moredun Institute, Animal Diseases Research Association, Gilmerton. 33 Hatton Place, Edinburgh 9	
1938		* Stewart, John Livingstone, B.Sc. (Edin.), M.R.C.V.S., Director of Veterinary Services, Gold Coast, Department of Animal Health, Pong-Tamale. P.O. Box No. 32, Tamale, Northern Territories, Gold Coast	
1924		* Stiles, Sir Harold Jalland, K.B.E., Kt., M.B., F.R.C.S.E., D.Sc. (Hon.), LL.D., Emeritus Professor of Clinical Surgery, University of Edinburgh. Whatton Lodge, Gullane, E. Lothian	1934–37.
1902		Stockdale, Herbert Fitton, LL.D., *formerly Director of the Royal Technical* College, Glasgow. Clairinch, Upper Helensburgh, Dumbartonshire	
1889	C.	Stockman, Ralph, M.D., LL.D., F.R.C.P.E., F.R.F.P.S.G. (VICE-PRESIDENT), Emeritus Professor of Materia Medica and Therapeutics, University of Glasgow. White Lodge, Barnton Avenue, Edinburgh 4	1903–05, 1936–39. V-P 1939–
1926		* Stokoe, William Norman, B.Sc., Ph.D. (Lond.), Chief Chemist, Craigmillar Creamery Co., Ltd. 8 Cobden Road, Edinburgh 9	
1906		Story, Fraser, O.B.E., formerly Professor of Forestry, University College, Bangor, North Wales. The Wall House, Yorke Road, Reigate, Surrey	
1907		Strong, John, C.B.E., M.A., LL.D., Emeritus Professor of Education, University of Leeds. C/o The Librarian, The University, Leeds	
1930	C.	* Stump, Claude Witherington, M.D., D.Sc., Professor of Embryology and Histology, University of Sydney	
1937		* Suffolk and Berkshire, The Rt. Hon. The Earl of, Charlton Park, Malmesbury, Wilts	

Date of Election			Service on Council, etc.
1935		* Sutherland, John Derg, B.Sc., B.Ed., Ph.D. (Edin.), M.B., Ch.B., Lecturer in Psychology, University of Edinburgh. 2 Windsor Street, Edinburgh 7	
1930		* Sutherland, Sir John Donald, C.B.E., LL.D., Legion of Honour (France), Order of Leopold (Belgium), formerly Forestry Commissioner, Scotland. 11 Inverleith Row, Edinburgh 4	
1925		Sutton, Richard L., M.D., D.Sc., LL.D., 1308 Bryant Building, 1102 Grand Avenue, Kansas City, Mo., U.S.A.	
1932		* Swinton, William Elgin, B.Sc., Ph.D. (Glas.), F.L.S., F.Z.S., F.G.S., Curator of Fossil Reptiles and Amphibia, British Museum (Natural History), South Kensington, London, S.W. 7	
1933		* Tait, John Barclay, D.Sc. (Edin.), Ph.D., A.H.-W.C., Senior Naturalist (Hydrographer), Marine Laboratory (Fisheries Division, Scottish Home Department), Aberdeen. 23 Cromwell Road, Aberdeen	
1937		* Tait, John Guthrie, Scholar of Peterhouse College, Cambridge, B.A. (Cantab.), Barrister-at-Law, Lincoln's Inn, Hon. Fellow, Madras University, formerly Principal, Central College, Bangalore, India. 38 George Square, Edinburgh 8	
1890	C.	† Tanakadate, Aikitu, Hon. Professor of Natural Philosophy, Imperial University of Japan. Koisikawa, Zôsigayamati, 144, Tokyo, Japan	
1933		* Taylor, George, D.Sc. (Edin.), F.L.S., Assistant Keeper, Department of Botany, British Museum. Ballochmyle, Loudwater, Rickmansworth, Herts	
1899		Taylor, James, M.A., formerly Mathematical Master, Edinburgh Academy. 18 Hillview, Blackhall, Edinburgh 4	
1885	C.	Thompson, Sir D'Arcy Wentworth, Kt., C.B., M.A., LL.D. (Aberd., Edin.), Hon. D.Sc. (Dublin, Witwatersrand), D.Litt., F.R.S., Professor of Natural History, University, St Andrews. 44 South Street, St Andrews	1892–95, 1896–99, 1907–10, 1912–15, 1922–25. V-P 1916–19. Curator 1926–34. P 1934–39.
1932		Thompson, Harold William, D.Litt. (Edin.), A.M., Ph.D. (Harvard), F.S.A. Scot., Goldwin Smith Hall, Cornell University, Ithaca, N.Y., U.S.A.	
1917	C. N.	* Thompson, John M'Lean, M.A., D.Sc., F.L.S., Professor of Botany, University of Liverpool	
1938		* Thomson, Arthur Landsborough, C.B., O.B.E., M.A., D.Sc. (Aberd.), Principal Assistant Secretary, Medical Research Council. 16 Tregunter Road, London, S.W. 10	
1896		Thomson, George Ritchie, C.M.G., M.B., C.M., formerly Professor of Surgery, University of the Witwatersrand, Johannesburg, Transvaal. Hordle Grange, Hordle, Hants	
1903		Thomson, George S., Kenmore Farm, Whelpley Hill, Chesham, Bucks	
1906		Thomson, Gilbert, M.A., M.Inst.C.E., 164 Bath Street, Glasgow, C. 2	
1926		* Thomson, Godfrey Hilton, D.Sc., Ph.D., Professor of Education, University of Edinburgh (Moray House)	1931–34.
1940		* Thomson, James, Distiller, Senior Partner of Messrs Charles Mackinlay & Co., Leith, Director in the firm of Messrs Mackinlay & Birnie, Inverness, 89 Constitution Street, Leith, Edinburgh 6	
1926	C.	* Thomson, John, M.A., Ph.D. (Glas.), Lecturer in Plant Physiology, University of Glasgow. 2 Chartwell Terrace, Bearsden, Glasgow	
1934		* Thomson, Matthew Sydney, M.A., M.D., B.Ch. (Cantab.), F.R.C.P. (Lond.), M.R.C.S. (Eng.), Physician for Diseases of the Skin, King's College Hospital, Belgrave Hospital for Children. 106 Harley Street, London, W. 1	
1912		Thomson, R. Tatlock, F.I.C., 156 Bath Street, Glasgow	
1938		* Thomson, Robert, B.Sc. Mech.Eng., A.M.I.Mech.E., Ph.D. (Edin.), Education Officer, Air Ministry. Moorfield, Terrick Road, Butler's Cross, Aylesbury, Bucks	
1882		Thomson, Sir William, Kt., M.A., B.Sc., LL.D., formerly Principal, University of the Witwatersrand. Dunedin, Glencairn, Simonstown, South Africa	
1917		* Thorneycroft, Wallace, J.P., Chalmington, Dorchester	

Date of Election			Service on Council, etc.
1933		* Timms, Geoffrey, Ph.D. (Cantab.), Lecturer in Mathematics, University of St Andrews. 8 St Mary's Street, St Andrews	
1937		* Tod, Henry, B.Sc., Ph.D. (Edin.), Biochemist, The Royal Edinburgh Hospital for Mental Disorders. 35 Oxgangs Road, Edinburgh 10	
1920		* Todd, John Barber, B.Sc., Ph.D., M.I.Mech.E., Reader in Engineering, University of Edinburgh. 4 Bright's Crescent, Edinburgh 9	
1938		* Topping, Andrew, T.D., M.A., M.D., D.P.H. (Aberd.), A Senior Medical Officer, Public Health Department, London County Council, County Hall, London, S.E. 1	
1940		* Traquair, Harry Moss, M.D., *F*.R.C.S.E., Lecturer on Diseases of the Eye, University of Edinburgh, Ophthalmic Surgeon, Royal Infirmary of Edinburgh. 16 Manor Place, Edinburgh 3	
1914		† Tredgold, Alfred Frank, M.D. (Durham), F.R.C.P. (Lond.), Lecturer on Mental Deficiency, University of London. St Martins, Guildford	
1915		Trotter, George Clark, M.D. (Edin.), D.P.H. (Aberd.), F.S.A.Scot., Medical Officer of Health, Metropolitan Borough, Islington. Braemar, 17 Haslemere Road, Crouch End, London, N. 8	
1938		* Trueman, Arthur Elijah, D.Sc., F.G.S., Professor of Geology, University of Glasgow. 20 Queensborough Gardens, Glasgow	
1922	C. K.	* Turnbull, Herbert Westren, M.A., F.R.S., Professor of Mathematics, University of St Andrews. Randa, Hepburn Gardens, St Andrews	1928–31.
1937		* Turnbull, Mathew McKerrow, M.A. (Edin.), Lecturer in Banking, University of Edinburgh (South Bridge). 14 Minto Street, Edinburgh 9	
1925		* Turner, Harry Morton Stanley, M.B.E., M.D., M.R.C.S. (Eng.), L.R.C.P. (Lond.), D.T.M. and H., Chevalier de l'Ordre Royale du Sauveur de Grèce, Wing Commander, R.A.F. (retired). The Haven, Brookwood, Surrey	
1918	C. N.	* Tyrrell, George Walter, A.R.C.S., D.Sc., F.G.S. (VICE-PRESIDENT), Lecturer in Petrology, Department of Geology, University of Glasgow	1926–29. 1937–40. V-P 1940–
1930	C.	* Voge, Cecil Innes Bothwell, Ph.D. (Edin.), 46 Roxborough Park, Harrow-on-the-Hill, London	
1932		* Wade, Henry, C.M.G., D.S.O., M.D., Senior Lecturer in Clinical Surgery, University of Edinburgh (Royal Infirmary). 6 Manor Place, Edinburgh 3	
1926		* Wakeley, Cecil Pembrey Grey, F.R.C.S. (Eng.), L.R.C.P. (Lond.), Surgeon Rear-Admiral, Surgeon to King's College Hospital, London, Lecturer in Anatomy, King's College, London. 14 Devonshire Street, Portland Place, London, W. 1	
1925	C.	* Walker, Frederick, M.A., Ph.D., D.Sc., Professor of Mineralogy and Geology, The University, Cape Town	
1938		* Walker, Oswald James, B.Sc., Ph.D. (Edin.), Lecturer in Chemistry, University College, Gower Street, London, W.C. 1. 41 Maresfield Gardens, London, N.W. 3	
1931		* Walker, William James Stirling, Ph.D. (Edin.), A.H.-W.C., F.I.C., F.I.L., F.S.A. Scot., 13 The Lawns, Lee Terrace, Blackheath, London, S.E. 3	
1898		Wallace, William, M.A., Campsie, Alta, Canada	
1920		* Walmsley, Thomas, M.D. (Glas.), Professor of Anatomy, Queen's University, Belfast	
1931	C.	* Walton, John, M.A. (Cantab.), D.Sc. (Manch.), Regius Professor of Botany, University of Glasgow. 23 Lilybank Gardens, Glasgow, W. 2	1934–37. V-P 1937–40.
1927	C.	* Wardlaw, Claude Wilson, D.Sc. (Glas.), Barker Professor of Cryptogamic Botany, University of Manchester	
1936		* Warr, The Very Rev. Charles Laing, C.V.O., M.A., D.D., LL.D., Hon. R.S.A., Dean of the Thistle and of the Chapel Royal in Scotland, Chaplain to H.M. The King, Minister of St Giles' Cathedral, Edinburgh. 63 Northumberland Street, Edinburgh 3	
1923		* Warren, John Alexander, M.Inst.C.E. 74 Balshagray Avenue, Partick, Glasgow	
1901	C.	Waterston, David, M.A., M.D., F.R.C.S.E., Professor of Anatomy, University of St Andrews	1916–19, 1925–28.
1927		* Watson, Charles Brodie Boog, F.S.A.Scot., 24 Garscube Terrace, Edinburgh 12	
1923		* Watson, H. Ferguson, M.D., F.R.F.P.S., Ph.D., D.P.H. (Glas.), formerly H.M. Senior Deputy Commissioner, General Board of Control for Scotland. Tarbeg, Ochiltree, Ayrshire	

Date of Election			Service on Council, etc.
1923	C.	* Watson, William, M.A., B.Sc. (Edin.), Lecturer in Physics, Heriot-Watt College, Edinburgh. 17 Braidburn Crescent, Edinburgh 10	
1911		† Watt, James, W.S., F.F.A., LL.D, 7 Blackford Road, Edinburgh 9	1924–26. Treasurer 1926–37. V-P 1937–40.
1933		* Watt, John Mitchell, M.B., Ch.B., M.R.C.P.E., F.R.S.S.A., Professor of Pharmacology, University of Witwatersrand, Johannesburg, South Africa	
1911		Watt, Very Rev. Lauchlan MacLean, M.A., D.D., LL.D., Kinloch, Lochcarron, Ross-shire	
1928		* Watters, Alexander Marshall, M.A., B.Sc. (Glas.), Rector of Hawick High School. High School House, Hawick	
1896		† Webster, John Clarence, C.M.G., M.D., D.Sc., LL.D., F.R.C.P.E., formerly Professor of Obstetrics and Gynæcology, Rush Medical College, Shediac, N.B., Canada	
1907	C. M-B.	† Wedderburn, Ernest Maclagan, O.B.E., M.A., D.Sc., LL.D., Deputy Keeper of the Signet (TREASURER). 6 Succoth Gardens, Edinburgh 12	1913–16, 1921–24, 1932–35, 1936–37. Treasurer 1937–
1903	C. M-B.	† Wedderburn, J. H. Maclagan, M.A., D.Sc., F.R.S., Professor of Mathematics, Princeton University. Fine Hall, Princeton, N.J., U.S.A.	
1934	C.	* Weir, John, M.A., Ph.D., D.Sc. (Glas.), Lecturer in Palæontology, University of Glasgow. 18 Botanic Crescent, Glasgow, N.W.	
1930		* White, Adam Cairns, M.D. (Edin.), Ph.D., Assistant Pharmacologist, Wellcome Physiological Research Laboratory, Beckenham, Kent	
1933		* Whitley, William Frederic James, M.D. (Edin.), D.P.H. (Oxon.), Medical Officer of Health, Northumberland County Council. C/o Cheltenham Rural District Council, 14 Imperial Square, Cheltenham	
1931		* Whitson, Sir Thomas Barnby, D.L., LL.D., C.A., Lord Provost of the City of Edinburgh (1929–32), and President, Society of Accountants in Edinburgh (1937–38). 27 Eglinton Crescent, Edinburgh 12	
1911		Whittaker, Charles Richard, F.R.C.S.E., F.S.A.Scot., Lynwood, Hatton Place, Edinburgh 9	
1912	C. V. J. B-P.	Whittaker, Edmund Taylor, M.A., Hon. Sc.D. (Dubl.), Hon. LL.D. (St Andrews and California), Hon. D.Sc. (Nat. Univ. Ireland), F.R.S., Foreign Member of the R. Accademia dei Lincei, Rome, Member of the Pontifical Academy of Sciences, Corresponding Member of the R. Society of Naples, Professor of Mathematics, University of Edinburgh (Drummond Street) (PRESIDENT). 48 George Square, Edinburgh 8	1912–15, 1922–25. Sec. 1916–22. V-P 1925–28, 1937–39. P. 1939–
1928	C.	* Whittaker, John Macnaghten, M.A. (Edin.), M.A. (Cantab.), D.Sc., Professor of Pure Mathematics, University of Liverpool	
1936		* Whyte, Andrew, A.C.A., F.R.G.S., Chartered Accountant. Green Gables, Carmel Road South, Darlington	
1918		* Whyte, Rev. Charles, M.A., LL.D., F.R.A.S., U.F. Church Manse, Kingswells, Aberdeen	
1935		* Whyte, Sir William Edward, Kt., O.B.E., J.P., Solicitor. Balgay, Uddingston	
1934		* Whyte, William, Cashier and General Manager of the Royal Bank of Scotland. Baberton House, Juniper Green, Edinburgh	
1929	C.	* Wiesner, Bertold Paul, D.Sc., formerly Macaulay Lecturer, Institute of Animal Genetics, University of Edinburgh. Brooklands, Woodlane, Halewood, Lancs.	
1918		* Wight, John Thomas, M.I.Mech.E., M.I.Mar.E., Joint Managing Director, Messrs MacTaggart, Scott & Co., Ltd., Loanhead. Calderwood Villa, Lasswade	
1934		* Wightman, William Persehouse Delisle, Ph.D., M.Sc. (Lond.), Science Master, Edinburgh Academy. 36 Coltbridge Terrace, Edinburgh 12	1939-

Date of Election			Service on Council, etc.
1926	C. N.	* Williams, Samuel, Ph.D., D.Sc., Lecturer in Plant Morphology, University of Glasgow. 27 Lindsay Place, Kelvindale, Glasgow	
1908		Williamson, Henry Charles, M.A., D.Sc., formerly Naturalist to the Fishery Board for Scotland. 13 Windsor Street, Dundee	
1928	C.	* Williamson, John, M.A. (Edin.), Ph.D. (Chicago), Associate Professor of Mathematics, Johns Hopkins University, Baltimore, U.S.A.	
1910	C.	Williamson, William, F.L.S., F.R.M.S., 47 St Alban's Road, Edinburgh 9. *Temporary Address*, Sunnythwaite, West Linton	
1927	C.	* Williamson, William Turner Horace, B.Sc. (Aberd.), Ph.D. (Edin.), Chief Chemist, Egyptian Ministry of Agriculture, Cotton Research Board, Giza, Egypt. *Temporary Address*, Ferney Cottage Hotel, Dursley, Glos	
1911		Wilson, Andrew, O.B.E., D.L., M.Inst.C.E. 51 Queen Street, Edinburgh 2	
1902	V. J.	† Wilson, Charles Thomson Rees, C.H., M.A., LL.D., D.Sc., F.R.S., Nobel Prize, Physics, 1927 (VICE-PRESIDENT), Emeritus Professor of Natural Philosophy, University of Cambridge. 196 Grange Loan, Edinburgh 9	1937–40. V-P 1940–
1922		* Wilson, John, F.R.I.B.A., Chief Architect, Scottish Department of Health. 20 Lomond Road, Edinburgh 5	
1920	C.	* Wilson, Malcolm, D.Sc. (Lond.), A.R.C.S., F.L.S., Reader in Mycology and Bacteriology, University of Edinburgh (Royal Botanic Garden). Brent Knoll, Kinnear Road, Edinburgh 4	1931–34.
1938		* Wilson, Robert, Master Printer, Partner of Messrs H. and J. Pillans & Wilson, Edinburgh. 13 Corrennie Drive, Edinburgh 10	
1924		* Wilson, William, M.A., LL.B., Advocate, Regius Professor of Public Law, University of Edinburgh (South Bridge). 38 Moray Place, Edinburgh 3	
1895		Wilson-Barker, Sir David, Kt., R.D., R.N.R., F.R.G.S., formerly Captain-Superintendent, Thames Nautical Training College, H.M.S. "Worcester." 12 Bolan Street, London, S.W. 11	
1939		* Wishart, George Macfeat, B.Sc., M.D. (Glas.), Professor of Physiological Chemistry, University of Glasgow. 5 Hillhead Street, Glasgow, W. 2	
1931	C.	* Wishart, John, M.A., B.Sc. (Edin.), M.A. (Cantab.), D.Sc. (Lond.), Reader in Statistics, University of Cambridge, School of Agriculture, Cambridge. Croft Lodge, Barton Road, Cambridge	
1922	C. B.	* Wordie, James Mann, M.A. (Cantab.), B.Sc. (Glas.), St John's College, Cambridge	
1937		* Wright, Edward Maitland, B.A. (Lond.), M.A., D.Phil. (Oxon.), Professor of Mathematics, University of Aberdeen. 52 College Bounds, Aberdeen	
1933	C. N.	* Wright, James, F.G.S., Balado, 212 Colinton Road, Edinburgh 11	
1911	C.	Wrigley, Ruric Whitehead, M.A. (Cantab.), Assistant Astronomer, Royal Observatory, Edinburgh	
1938	C.	* Wyburn, George McCreath, M.B., Ch.B. (Glas.), F.R.F.S.P.G., Lecturer in Anatomy, University of Glasgow	
1939		* Yarrow, Sir Harold Edgar, Bart., C.B.E., Chairman of Yarrow & Co., Ltd., Shipbuilders, etc. Craigend Castle, Milngavie, Stirlingshire	
1937	C.	* Young, Andrew White, M.A., B.Sc., LL.B. (Edin.), W.S. 15 Great Stuart Street, Edinburgh 3	
1882		Young, Frank W., C.B.E., F.C.S., H.M. Inspector of Schools (Emeritus). Panera, Shortheath, Farnham, Surrey	
1904		Young, Robert B., M.A., D.Sc., F.G.S., formerly Professor of Geology, University of the Witwatersrand (South African School of Mines and Technology), Johannesburg, Transvaal. St Patrick's Road, Houghton, Johannesburg, South Africa	

Number of Fellows, 762.

HONORARY FELLOWS OF THE SOCIETY

(*At* 28*th October* 1940.)

1920 HIS ROYAL HIGHNESS THE DUKE OF WINDSOR, K.G.

FOREIGN MEMBERS (LIMITED TO FORTY-FOUR BY LAW I).

Elected

1936 Leo Hendrik Baekeland, Professor (Honorary) of Chemical Engineering, Columbia University, New York. Bakelite Corporation, 247 Park Avenue, New York City.

1923 Henri Bergson, Honorary Professor, College of France, Paris.

1930 Vilhelm Frimann Koren Bjerknes, Professor of Physics, University of Oslo.

1937 Marston Taylor Bogert, Emeritus Professor of Organic Chemistry, Columbia University, N.Y.

1927 Niels Bohr, Nobel Laureate, Physics, 1922, Professor of Physics, University of Copenhagen.

1927 Jules Bordet, Nobel Laureate, Medicine, 1919, Professor of Bacteriology, University of Brussels.

1933 Filippo Bottazzi, Professor of Experimental Physiology, Royal Institute of Physiology, S. Andrea delle Dame, 21, Naples.

1923 Marcellin Boule, Director of the Institute of Human Palæontology, 1, Rue René-Panhard, Paris, XIII^{e}.

1916 Douglas Houghton Campbell, Emeritus Professor of Botany, Stanford University, California.

1930 Walter Bradford Cannon, Professor of Physiology, Harvard Medical School, Boston, Mass.

1930 Maurice Caullery, Professor of Zoology in the University of Paris. Laboratoire d'Évolution des Êtres organisés, 105 Boulevard Raspail, Paris, VI^{e}.

1933 Edwin Grant Conklin, formerly Professor of Biology, Princeton University, N.J.

1921 Reginald Aldworth Daly, Professor of Geology, Harvard University, Cambridge, Mass.

1927 Albert Einstein, Nobel Laureate, Physics, 1921, Princeton University, N.J.

1938 Federigo Enriques, Professor of Mathematics, Royal University, Rome.

1934 Björn Helland-Hansen, Geophysical Institute, Bergen.

1921 Johan Hjort, Professor of Marine Biology, University, Oslo.

1923 Arnold Frederik Holleman, Emeritus Professor of Organic Chemistry, University, Amsterdam. Boekensteijn Parkweg 7, Bloemendaal.

1934 Bernardo Houssay, Professor of Physiology, National University of Buenos Aires.

1937 C. U. Ariëns Kappers, Director of the Central Institute of Brain Research, Amsterdam, and Professor of Comparative Neurology, University, Amsterdam.

1933 Nikolaj Konstantinovic Koltzoff, formerly Professor of Zoology, State University, Moscow; Director of the Research Institute of Experimental Biology. Moscow 64, Voronzovo Pole 6.

1923 Tullio Levi-Civita, Professor of Mathematics, Regia Università, Rome.

1934 Frank Rattray Lillie, Emeritus Professor of Zoology and Embryology, University of Chicago, and President, National Academy of Sciences, Washington, D.C.

1939 Otto Loewi, lately Professor of Pharmacology, University of Graz, Austria. Nobel Laureate, Medicine, 1936, Department of Pharmacology, College of Medicine, New York City, N.Y., U.S.A.

1936 Maurice Lugeon, Professor of Geology, University, Lausanne.

1939 Bernard Lyot, For. Assoc. Roy. Astron. Soc., l'Observatoire, Meudon (S. et O.), France.

1927 Hans Horst Meyer, Emeritus Professor of Pharmacology, University of Vienna.

1934 Thomas Hunt Morgan, Nobel Laureate, Medicine, 1933, Professor of Biology, California Institute of Technology, Pasadena.

1933 Albrecht Penck, Geheimrat, Emeritus Professor of Geography, Friedrich-Wilhelms-Universität, Berlin. Meierotto-Strasse 5^{II}, Berlin, W. 15.

1920 Charles Émile Picard, Perpetual Secretary, Academy of Sciences, Paris.

1937 Max Planck, Nobel Laureate, Physics, 1918, Professor Ordinarius Emeritus of Theoretical Physics, Director of the Institute for Theoretical Physics, University of Berlin.

Elected

1938 Henry Norris Russell, Chairman of the Department of Astronomy and Director of the Observatory, Princeton University, U.S.A.

1934 Paul Sabatier, Nobel Laureate, Chemistry, 1912, Professor of Chemistry, University of Toulouse.

1936 George Sarton, Editor of *Isis* and *Osiris*, Harvard Library, 185, Cambridge, Mass.

1930 Erik Helge Oswald Stensiö, Professor of Palæontology and Historical Geology, Royal University of Upsala.

1936 George Linius Streeter, Director, Department of Embryology, Carnegie Institution of Washington, Corner Wolfe and Madison Streets, Baltimore. 3707 St Paul Street, Baltimore, Md.

1938 Karl Freiherr von Tubeuf, Professor of Botany, University of Munich.

1936 Nikolai Ivanovic Vavilov, Director of the Institute of Plant Industry, Academy of Sciences, Leningrad, U.S.S.R.

1927 Richard Willstätter, Nobel Laureate, Chemistry, 1915. Muralto-Locarno (Switzerland), Villa L'Gremitaggio.

1933 Pieter Zeeman, Nobel Laureate, Physics, 1902, Emeritus Professor of Physics, University, Amsterdam. Stadhouderskade 158, Amsterdam.

Total, 40.

British Subjects (limited to twenty-two by Law I).

1937 John Logie Baird, Inventor of the Televisor. 3 Crescent Wood Road, Sydenham, London, S.E.

1936 Sir Charles Vernon Boys, Kt., A.R.S.M., LL.D., F.R.S., St Marybourne, Andover.

1927 Sir William Henry Bragg, O.M., K.B.E., M.A., Hon. Sc.D. (Cantab.), Hon. D.Sc., Hon. D.C.L. (Durh.), LL.D., President R.S., Nobel Laureate, Physics, 1915, Fullerian Professor of Chemistry, Royal Institution, London.

1937 William Thomas Calman, C.B., D.Sc., LL.D. (St Andrews), F.R.S., F.L.S., formerly Keeper of Zoology, British Museum. Willowbrae, Tayport, Fife.

1936 Sir Henry Hallett Dale, Kt., C.B.E., M.A., M.D., LL.D., Hon. D.Sc., F.R.S., Nobel Laureate, Medicine, 1936, Director of the National Institute for Medical Research, Mount Vernon House, London, N.W. 3.

1936 Frederick George Donnan, C.B.E., M.A., Ph.D., D.Sc., LL.D., F.R.S., formerly Professor of Chemistry, University of London, and Director of Chemical Laboratories, University College. Hartlip House, near Sittingbourne, Kent

1930 Sir Arthur Stanley Eddington, Kt., O.M., M.A., Hon. D.Sc., LL.D., F.R.S., Plumian Professor of Astronomy and Experimental Philosophy, University of Cambridge.

1927 Sir John Bretland Farmer, Kt., M.A., D.Sc., LL.D., F.R.S., Emeritus Professor of Botany, Imperial College of Science and Technology, London. St Leonard's, Weston Road, Bath.

1900 Andrew Russell Forsyth, M.A., Sc.D., LL.D., Hon. Math.D., F.R.S., Emeritus Professor of Mathematics, Imperial College of Science and Technology, London; formerly Sadleirian Professor of Pure Mathematics, University of Cambridge. Bailey's Hotel, London, S.W. 7.

1910 Sir James George Frazer, Kt., O.M., D.C.L., LL.D., Litt.D., F.B.A., F.R.S., Commandeur de la Légion d'Honneur. Trinity College, Cambridge.

1927 Sir Frederick Gowland Hopkins, Kt., O.M., M.A., M.B., Hon. D.Sc., D.C.L. (Durh.), LL.D., Past President R.S., Nobel Laureate, Medicine, 1929, Sir William Dunn Professor of Biochemistry, University of Cambridge. 71 Grange Road, Cambridge.

1930 Sir Arthur Keith, Kt., M.D., LL.D., D.Sc., F.R.S., Master, Buckston Browne Research Farm, Downe, Farnborough, Kent.

1910 Sir Joseph Larmor, Kt., M.A., D.Sc., LL.D., D.C.L. (Durh.), F.R.S., formerly Lucasian Professor of Mathematics, University of Cambridge. St John's College, Cambridge.

1938 Sir Thomas Lewis, Kt., C.B.E., D.Sc., M.D., F.R.C.P. (Lond.), Hon. F.R.C.P.E., LL.D., F.R.S., Consulting Physician, Ministry of Pensions, Physician-in-Charge of Department of Clinical Research, University College Hospital, London.

1934 Sir Edward Bagnall Poulton, Kt., M.A., D.Sc., LL.D., Hon. D.Sc., F.R.S., formerly Hope Professor of Zoology, University of Oxford. Wykeham House, Banbury Road, Oxford.

1930 Sir Robert Robinson, Kt., M.A., D.Sc., Hon. D.Sc., LL.D., F.R.S., Waynflete Professor of Chemistry, University of Oxford.

1933 Sir William Napier Shaw, Kt., Sc.D. (Cantab.), LL.D., Hon. Sc.D., Hon. D.Sc., F.R.S., formerly Director Meteorological Office. 171 Old Brompton Road, London, S.W. 5.

Elected

1908 **Sir Charles Scott Sherrington**, O.M., G.B.E., M.A., D.Sc., M.D., LL.D., Past President R.S., Nobel Laureate, Medicine, 1932, formerly Waynflete Professor of Physiology, University of Oxford. Broomside, Valley Road, Ipswich.

1938 **Geoffrey Ingram Taylor**, M.A., Hon. D.Sc., D.C.L. (Brit. Columbia), F.R.S., Yarrow Research Professor of the Royal Society, Fellow of Trinity College, Cambridge.

1934 **William Whitehead Watts**, Sc.D., M.Sc., LL.D., F.R.S., Emeritus Professor of Geology, Imperial College of Science and Technology, London. Hazler Cottage, Hazler Crescent, Church Stretton, Shropshire.

Total, 20.

CHANGES IN FELLOWSHIP DURING SESSION 1939–1940

FELLOWS OF THE SOCIETY ELECTED

EDMUND BRUCE BALL
JOHN BOWMAN
BYROM STANLEY BRAMWELL
JAMES BROUGH
ALEXANDER FAIRLEY BUCHAN
JOHN MORRISON CAIE
JOHN CAMERON
JAMES WILFRED COOK
JAMES COUTTS
SATCHIDINANDA DATTA
THOMAS ELDER DICKSON
ARCHIBALD THOMAS JOHN DOLLAR
HAROLD IRVING DREVER
WILLIAM McCONNACHIE HARROWES
THOMAS HART
CHARLES FREDERICK WILLIAM ILLINGWORTH
JAMES GRAY KYD
PATRICK RAMSAY LAIRD
ROBERT McADAM
JAMES ALEXANDER MACDONALD
ANGUS ELRICK WILLIAM McLACHLAN
ANGUS MACNIVEN
GUY FREDERIC MARRIAN
EDWARD ROWLAND ALWORTH MEREWETHER
ROBERT MACFARLANE NEILL
HUGH BRYAN NISBET
JAMES SCOTT CUMBERLAND REID
HARRY RILEY
JAMES THOMSON
HARRY MOSS TRAQUAIR

FELLOWS DECEASED

SIR THOMAS HUDSON BEARE
SIR GEORGE A. BERRY
EDMUND W. W. CARLIER
ALEXANDER CHARLES CUMMING
CARSTAIRS CUMMING DOUGLAS
THOMAS WILLIAM DRINKWATER
WILLIAM MOSES FELDMAN
PERCY HALL GRIMSHAW
ROBERT KERR HANNAY
GILBERT MACINTYRE HUNTER
JOSHUA LAW KERR
FORREST HAY LIGHTBODY
VERY REV. JOHN HARRY MILLER
ROBERT COCKBURN MOSSMAN
BENJAMIN DAWSON PORRITT
NATHAN RAW
WILLIAM ALEXANDER ROBERTSON
RALPH ALLEN SAMPSON
JOHN JAMES McINTOSH SHAW
EDWARD SMART
SIR DONALD FRANCIS TOVEY
RICHARD TURNER
WILLIAM GIBSON WEDDERSPOON
(Date of death not known)

FOREIGN HONORARY FELLOWS DECEASED

CHARLES EUGÈNE BARROIS
WALDEMAR CHRISTOFER BRÖGGER
VITO VOLTERRA

BRITISH HONORARY FELLOWS DECEASED

SIR GEORGE MACDONALD
SIR JOSEPH JOHN THOMSON

RESIGNED

WILLIAM LEONARD FERRAR
WILLIAM KING GILLIES
RONALD GRANT
ANDREW HUNTER
PERCY SAMUEL LELEAN
DONALD NEIL McARTHUR
DAVID PENMAN
ARTHUR WINSTANLEY

FELLOWS REMOVED FROM ROLL

RICHARD SIDDOWAY BAGNALL
SAMUEL HUNTER COWAN

LAWS OF THE SOCIETY.

Adopted July 3, 1916; *amended December* 18, 1916.

LAW	I,	amended	February 5, 1934.	LAW	IX,	amended	May 3, 1920.
„	VI,	„	„ 7, 1921.	„	XIII,	„	May 3, 1920.
„	„	„	July 2, 1928.	„	XVI,	„	June 3, 1940.
„	VIII,	„	May 3, 1920.	„	XIX,	„	June 16, 1924.

I.

THE ROYAL SOCIETY OF EDINBURGH, which was instituted by Royal Charter in 1783 for the promotion of Science and Literature, shall consist of Ordinary Fellows (hereinafter to be termed Fellows) and Honorary Fellows. The number of Honorary Fellows shall not exceed sixty-six, of whom not more than twenty-two may be British subjects, and not more than forty-four subjects of Foreign States.

Fellows only shall be eligible to hold office or to vote at any Meeting of the Society.

ELECTION OF FELLOWS.

II.

Each Candidate for admission as a Fellow shall be proposed by at least four Fellows, two of whom must certify from personal knowledge. The Official Certificate shall specify the name, rank, profession, place of residence, and the qualifications of the Candidate. The Certificate shall be delivered to the General Secretary before the 30th of November, and, subject to the approval of the Council, shall be exhibited in the Society's House during the month of January following. All Certificates so exhibited shall be considered by the Council at its first meeting in February, and a list of the Candidates approved by the Council for election shall be issued to the Fellows not later than the 21st of February.

III.

The election of Fellows shall be by Ballot, and shall take place at the first Ordinary Meeting in March. Only Candidates approved by the Council shall be eligible for election. A Candidate shall be held not elected, unless he is supported by a majority of two-thirds of the Fellows present and voting.

IV.

On the day of election of Fellows two scrutineers, nominated by the President, shall examine the votes and hand their report to the President, who shall declare the result.

V.

Each Fellow, after his election, is expected to attend an Ordinary Meeting, and sign the Roll of Fellows, he having first made the payments required by Law VI. He shall be introduced to the President, who shall address him in these words:—

> *In the name and by the authority of THE ROYAL SOCIETY OF EDINBURGH, I admit you a Fellow thereof.*

PAYMENTS BY FELLOWS.

VI.

Each Fellow shall, before he is admitted to the privileges of Fellowship, pay an admission fee of three guineas, and a subscription of three guineas for the year of election. He shall continue to pay a subscription of three guineas at the beginning of each session so long as he remains a Fellow.

Each Fellow who was elected subsequent to December 1916 and previous to December 1920 shall also pay a subscription of three guineas at the beginning of each session so long as he remains a Fellow.

Each Fellow who was elected previous to December 1916, and who has not completed his twenty-five annual payments, shall, at the beginning of each session, pay three guineas until his twenty-five annual payments are made. Each Fellow who has completed or shall complete his payments shall be invited to contribute one guinea per annum or to pay a single sum of ten guineas.

A Fellow may compound for the annual subscriptions by a single payment of fifty guineas, or on such other terms as the Council may from time to time fix.

VII.

A Fellow who, after application made by the Treasurer, fails to pay any contribution due by him, shall be reported to the Council, and, if the Council see fit, shall be declared no longer a Fellow. Notwithstanding such declaration all arrears of contributions shall remain exigible.

ELECTION OF HONORARY FELLOWS.

VIII.

Honorary Fellows shall be persons eminently distinguished in Science or Literature. They shall not be liable to contribute to the Society's Funds. Personages of the Blood Royal may be elected Honorary Fellows at any time on the nomination of the Council, and without regard to the limitation of numbers specified in Law I.

IX.

Honorary Fellows shall be proposed by the Council. The nominations shall be announced from the Chair at the First Ordinary Meeting after their selection. The names shall be printed in the circular for the last Ordinary Meeting of the Session, when the election shall be by Ballot, after the manner prescribed in Laws III and IV for the Election of Fellows.

EXPULSION OF FELLOWS.

X.

If, in the opinion of the Council, the conduct of any Fellow is injurious to the character or interests of the Society, the Council may, by registered letter, request him to resign. If he fail to do so within one month of such request, the Council shall call a Special Meeting of the Society to consider the matter. If a majority consisting of not less than two-thirds of the Fellows present and voting decide for expulsion, he shall be expelled by declaration from the Chair, his name shall be erased from the Roll, and he shall forfeit all right or claim in or to the property of the Society.

XI.

It shall be competent for the Council to remove any person from the Roll of Honorary Fellows if, in their opinion, his remaining on the Roll would be injurious to the character or interests of the Society. Reasonable notice of such proposal shall be given to each member of the Council, and, if possible, to the Honorary Fellow himself. Thereafter the decision on the question shall not be taken until the matter has been discussed at two Meetings of Council, separated by an interval of not less than fourteen days. A majority of two-thirds of the members present and voting shall be required for such removal.

MEETINGS OF THE SOCIETY.

XII.

A Statutory Meeting for the election of Council and Office-Bearers, for the presentation of the Annual Reports, and for such other business as may be arranged by the Council, shall be held on the fourth Monday of October. Each Session of the Society shall begin at the date of the Statutory Meeting.

XIII.

Meetings for reading and discussing communications and for general business, herein termed Ordinary Meetings, shall be held, when convenient, on the first and third Mondays of each month from November to July inclusive, with the exception that in January the meetings shall be held on the second and fourth Mondays.

XIV.

A Special Meeting of the Society may be called at any time by direction of the Council, or on a requisition to the Council signed by not fewer than six Fellows. The date and hour of such Meeting shall be determined by the Council, who shall give not less than seven days' notice of such Meeting. The notice shall state the purpose for which the Special Meeting is summoned; no other business shall be transacted.

PUBLICATION OF PAPERS.

XV.

The Society shall publish Transactions and Proceedings. The consideration of the acceptance, reading, and publication of papers is vested in the Council, whose decision shall be final. Acceptance for reading shall not necessarily imply acceptance for publication.

DISTRIBUTION OF PUBLICATIONS.

XVI.

Fellows who are not in arrear with their Annual Subscriptions and all Honorary Fellows shall be entitled gratis to copies of the Parts of the Transactions and the Proceedings published subsequently to their admission. Copies shall be obtainable upon application, either personally or by an authorised agent, to the Librarian, provided the application is made within five years after the date of publication.

CONSTITUTION OF COUNCIL.

XVII.

The Council shall consist of a President, six Vice-Presidents, a Treasurer, a General Secretary, two Secretaries to the Ordinary Meetings (the one representing the Biological group and the other the Physical group of Sciences),* a Curator of the Library and Museum, and twelve ordinary members of Council.

ELECTION OF COUNCIL.

XVIII.

The election of the Council and Office-Bearers for the ensuing Session shall be held at the Statutory Meeting on the fourth Monday of October. The list of the names recommended by the Council shall be issued to the Fellows not less than one week before the Meeting. The election shall be by Ballot, and shall be determined by a majority of the Fellows present and voting. Scrutineers shall be nominated as in Law IV.

XIX.

The President may hold office for a period not exceeding five consecutive years; the Vice-Presidents, not exceeding three; the Secretaries to the Ordinary Meetings, not exceeding five; the General Secretary, the Treasurer, and the Curator of the Library and Museum, not exceeding ten; and ordinary members of Council, not exceeding three consecutive years; provided that the Treasurer may be re-elected for more than ten successive years in cases where the Council declares to the Society that an emergency exists.

XX.

In the event of a vacancy arising in the Council or in any of the Offices enumerated in Law XVII, the Council shall proceed, as soon as convenient, to elect a Fellow to fill such vacancy for the period up to the next Statutory Meeting.

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* The Biological group includes Anatomy, Anthropology, Botany, Geology, Pathology, Physiology, Zoology; the Physical group includes Astronomy, Chemistry, Mathematics, Metallurgy, Meteorology, Physics.

POWERS OF THE COUNCIL.

XXI.

The Council shall have the following powers:—(1) To manage all business concerning the affairs of the Society. (2) To decide what papers shall be accepted for communication to the Society, and what papers shall be printed in whole or in part in the Transactions and Proceedings. (3) To appoint Committees. (4) To appoint employees and determine their remuneration. (5) To award the various prizes vested in the Society, in accordance with the terms of the respective deeds of gift, provided that no member of the existing Council shall be eligible for any such award. (6) To make from time to time Standing Orders for the regulation of the affairs of the Society. (7) To control the investment or expenditure of the Funds of the Society.

At Meetings of the Council the President or Chairman shall have a casting as well as a deliberative vote.

DUTIES OF PRESIDENT AND VICE-PRESIDENTS.

XXII.

The President shall take the Chair at Meetings of Council and of the Fellows. It shall be his duty to see that the business is conducted in accordance with the Charter and Laws of the Society. When unable to be present at any Meetings or attend to current business, he shall give notice to the General Secretary, in order that his place may be supplied. In the absence of the President his duties shall be discharged by one of the Vice-Presidents.

DUTIES OF THE TREASURER.

XXIII.

The Treasurer shall receive the monies due to the Society and shall make payments authorised by the Council. He shall lay before the Council a list of arrears in accordance with Rule VII. He shall keep accounts of all receipts and payments, and at the Statutory Meeting shall present the accounts for the preceding Session, balanced to the 30th of September, and audited by a professional accountant appointed annually by the Society.

DUTIES OF THE GENERAL SECRETARY.

XXIV.

The General Secretary shall be responsible to the Council for the conduct of the Society's correspondence, publications, and all other business except that which relates to finance. He shall keep Minutes of the Statutory and Special Meetings of the Society and Minutes of the Meetings of Council. He shall superintend, with the aid of the Assistant Secretary, the publication of the Transactions and Proceedings. He shall supervise the employees in the discharge of their duties.

DUTIES OF SECRETARIES TO ORDINARY MEETINGS.

XXV.

The Secretaries to Ordinary Meetings shall keep Minutes of the Ordinary Meetings. They shall assist the General Secretary, when necessary, in superintending the publication of the Transactions and Proceedings. In his absence, one of them shall perform his duties.

DUTIES OF CURATOR OF LIBRARY AND MUSEUM.

XXVI.

The Curator of the Library and Museum shall have charge of the Books, Manuscripts, Maps, and other articles belonging to the Society. He shall keep the Card Catalogue up to date. He shall purchase Books sanctioned by the Council.

ASSISTANT SECRETARY AND LIBRARIAN.

XXVII.

The Council shall appoint an Assistant Secretary and Librarian, who shall hold office during the pleasure of the Council. He shall give all his time, during prescribed hours, to the work of the Society, and shall be paid according to the determination of the Council. When necessary he shall act under the Treasurer in receiving subscriptions, giving out receipts, and paying employees.

ALTERATION OF LAWS.

XXVIII.

Any proposed alteration in the Laws shall be considered by the Council, due notice having been given to each member of Council. Such alteration, if approved by the Council, shall be proposed from the Chair at the next Ordinary Meeting of the Society, and, in accordance with the Charter, shall be considered and voted upon at a Meeting held at least one month after that at which the motion for alteration shall have been proposed.

Additions to the Library—Presentations, etc.—1939-1940.

Donisthorpe, H. St J. K. British Ants: their Life-History and Classification. Second edition, revised and enlarged. 8vo. London, 1927.

—— The Guests of British Ants: their Habits and Life-Histories. 8vo. London, 1927.

Donovan, E. The Natural History of British Insects. Vols. I–X. 8vo. London, 1792–1801.

Francisci Redi. . . . Experimenta circa generationem Insectorum. . . . 12mo. Amstelodami, 1670, 1671.

Gosse, Philip Henry. Actinologia Britannica: a History of the British Sea-Anemones and Corals. 8vo. London, 1860.

Harris, Joseph. The Description and Use of the Globes, and the Orrery. . . . Ninth edition. 8vo. London, 1763.

Hieronymi Rorarii . . . Quòd Animalia bruta ratione utantur meliùs Homine, libri duo. 12mo. Amstelaedami, 1654.

Hogarth, A. Moore. British Mosquitoes and How to Eliminate Them. 8vo. London, no date.

Johnstone, William Grosart, and Croall, Alexander. The Nature-Printed British Sea-Weeds: A History accompanied by Figures and Dissections, of the Algae of the British Isles. Nature-Printed by Henry Bradbury. 4 vols. 8vo. London, 1859.

Keith, Sir Arthur. New Discoveries relating to the Antiquity of Man. 8vo. London, 1931.

Koch, C. L. Übersicht des Arachnidensystems. Hefte 1–5. 8vo. Nürnberg, 1837–1850.

(*Note.*—This vol. contains a photograph of Dr C. F. George, taken about 1900, to be found in the end papers.)

—— Deutschlands Crustaceen, Myriapoden und Arachniden. Hefte 1–40, in 8 cardboard boxes (unbound). Obl. 12mo. Regensberg, 1835–41.

Leakey, L. S. B. Adam's Ancestors: an Up-to-date Outline of what is known about the Origin of Man. 8vo. London, 1934.

Loring, F. H. Atomic Theories. 8vo. London, 1921.

Lowne, B. Thompson. The Anatomy, Physiology, Morphology, and Development of the Blow-Fly (*Calliphora erythrocephala*). A Study in the Comparative Anatomy and Morphology of Insects. Vols. I, II. 8vo. London, 1890–92 and 1893–95.

Macwatt, John. The Primulas of Europe. 8vo. London, 1923.

Miall, L. C., and Denny, Alfred. Studies in Comparative Anatomy—III: The Structure and Life-History of the Cockroach (*Periplaneta orientalis*). An Introduction to the Study of Insects. 8vo. London and Leeds, 1886.

Miall, L. C., and Hammond, A. R. The Structure and Life-History of the Harlequin Fly (*Chironomus*). 8vo. Oxford, 1900.

Morley, Claude. Ichneumonologia Britannica: The Ichneumons of Great Britain. A Descriptive Account of the Families, Genera and Species, indigenous to the British Islands, together with Notes as to Classification, Localities, Habitats, Hosts, etc. Vols. I–V. La. 8vo. Plymouth and London, 1903–1914.

Panzer, Georg Wolfgang Franz. Faunae Insectorum Germanicae initia; oder Deutschlands Insecten gesammelt und herausgegeben von D. G. W. F. Panzer. Obl. 12mo. Nürnberg. No dates.

Apidæ: A–T; 162 plates. Fossores: C–M and O–V; 149 plates. Tenthredinidæ: 116 plates. Ichneumonidæ: 137 plates.

(*Note.*—This set, which is without its title, consists of 564 plates. The complete set was published in 110 numbers, and has two editions. Each number consists of 24 coloured plates and 24 pages of text. The 1st edition was published in 1793–1813, each 12 numbers being followed by an Index arranged systematically, and bearing a date. The 2nd edition began in 1796, but after number 37, 1799, both editions appear to be identical. (Abstracted from SAUNDERS, Edward, Index to Panzer Fauna Insectorum Germaniae. London, 1888. In the Library of the University of Edinburgh.))

Piersig, R. Deutschlands Hydrachniden. (*Zoologica*, Heft 22.) 4to. Stuttgart, 1897–1900.

Piersig, R., and Lohmann, H. Hydrachnidae und Halacaridae. (*Das Tierreich*, Lief. 13. La. 8vo. Berlin, 1901.

Smith, G. Elliot. Human History. 8vo. London, 1930.

Sternberg, Charles H. The Life of a Fossil Hunter. 8vo. New York, 1909.

Thesing, Curt. Biologische Streifzüge: eine gemeinverstandliche Einführung in die allgemeine Biologie. 2. Auflage. 8vo. Esslingen u. München, 1908.

The Journal of the Postal Microscopical Society: a Miscellany of Natural and Microscopical Science. Edited by Alfred Allen. 16 vols. 8vo. London, 1882–1897.

(*Note.*—This journal was also known as the *Journal of Microscopy and Natural Science*, and the *International Journal of Microscopy and Natural Science*. It had three series in all.)

(*The above volumes were presented by Mr William Williamson, F.R.S.E., F.L.S.*)

The Advancement of Science. The Report of the British Association for the Advancement of Science. (New Quarterly Series No. 1.) October 1939. 8vo. London, 1939. (*Presented by Dr James Watt.*)

Anais da Primeira Reuniao Sul-Americana de Botânica. 1° Volume. (Published by Instituto de Biologia Vegetal.) 8vo. Rio de Janeiro, 1938. (*Exchange.*)

Anales del Instituto de Neurologia. Vol. II, 1939. 4to. Montevideo, 1939. (*Presented.*) (Vol. I was published in 1928.)

Anderson, J. G. C. The Granites of Scotland. (Special Reports on the Mineral Resources of Great Britain. Vol. XXXII. Geological Survey of Great Britain.) 8vo. Edinburgh, 1939. (*Exchange.*)

Annals of Mathematical Statistics. Vol. IX→. (Published by the Institute of Mathematical Statistics.) La. 8vo. Ann Arbor, 1938. (*Exchange.*)

Balfour-Browne, Frank. British Water Beetles. Vol. I. (Ray Society Publication.) 8vo. London, 1940. (*Purchased.*)

Briquet, John. Biographies des Botanistes a Genève de 1500 à 1931. Being Vol. L (a) of the Bulletin (Berichte) de la Société Botanique Suisse, 1940. (*Exchange.*)

Brough, James. The Triassic Fishes of Besano, Lombardy. 4to. London, 1939. (British Museum Publication.) (*Presented.*)

Carnegie Institution, Washington:—

No. 303*b*. Tables, Factors, and Formulas for Computing Respiratory Exchange and Biological Transformations of Energy. 3rd Ed. Prepared by Thorne M. Carpenter. 8vo. 1939.

No. 493. Studies on the Ice Age in India and Associated Human Cultures. By H. de Terra and T. T. Paterson. 4to. 1939.

No. 512. The Volcanic-Seismic Crisis at Montserrat, 1933–1937. By Frank A. Perret. 4to. 1939.

No. 513. Studies on the Physiology, Genetics, and Evolution of some Cladocera. By A. M. Banta, and others. (Paper No. 39: Department of Genetics.) 4to. 1939.

No. 515. Madison Catalogue of 2786 Stars for the Epoch 1910 from Meridian Observations by Albert S. Flint. Reductions by Arthur J. Roy. 4to. 1939.

No. 516. A Miocene Flora from the Western Border of the Mohave Desert. By Daniel I. Axelrod. (Contributions to Paleontology.) 8vo. 1939.

No. 519. Archæological Studies in the La Plata District: Southwestern Colorado and Northwestern New Mexico. By Earl H. Morris. With appendix: Technology of La Plata Pottery. By Anna O. Shepard. 4to. 1939.

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